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Nr 31 2001

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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

ABSTRACT: This paper consists of three theorems. For the nonlinear difference equation

$$\text{(E)} \qquad \Delta^2 x_n = \sum_{i=0}^{k} a_n^i f(x_{n+i}) + b_n, \qquad n, k \in N$$

sufficient conditions for the existence of the asymptotically constant solutions are given in Th. 1. In Th. 2 conditions under which there exists a solution (x_n) of Eq. (E) such that $x_n = cn + o(1)$, are given. In Th. 3 conditions under which every solution (x_n) of Eq. (E) possesses property: the sequence (x_n/n) is convergent in R, are presented.

KEY WORDS: difference equations, asymptotic behavior.

In this paper we are concerned with Eq. (E), where (a_n^i) are sequences of real numbers, f is a real function. Here by N we denote the set of positive integers, R the set of real numbers. For a function $x : N \to R$ the difference operator Δ is defined as follows:

$$\Delta x_n = x_{n+1} - x_n, \quad n \in N,$$

$$\Delta^i x_n = \Delta(\Delta^{i-1} x_n) \quad \text{for} \quad i > 1, \; n \in N.$$

We use the convention

$$\sum_{j=n}^{n-k} x_j = 0 \quad \text{for any} \quad k, n \in N.$$

Instead of $\lim_{n\to\infty} x_n = c$ we shall write $x_n = c + o(1)$.

Next lemma are given in [1].

LEMMA 1. *Assume the series* $\sum_{n=1}^{\infty} n|a_n|$ *is convergent and* $r_n = \sum_{j=n}^{\infty} a_j$. *Then the series* $\sum_{n=1}^{\infty} r_n$ *is absolutely convergent and*

$$\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} n a_n.$$

THEOREM 1. *Let* $a^i, b : N \to R$,

(1) $$\sum_{n=1}^{\infty} n|a_n^i| < \infty \qquad \text{for} \quad i = 0,1,\dots,k$$

(2) $$\sum_{n=1}^{\infty} n|b_n| < \infty$$

and $f : R \to R$ *is a continuous function such that*

(*) *for every* $\alpha, x \in R$ *there exists a constant* $t \in R$ *such that* $t + \alpha f(t) = x$.

Then for any arbitrary constant $c \in R$ *there exists a solution* (x_n) *of Eq.* (E) *such that*

(3) $$x_n = c + o(1).$$

PROOF. Let $c \in R$, and let us choose a real number $a > 0$. Then there exists a constant $M > 0$ such that

(4) $$|f(t)| \le M \qquad \text{for every} \quad t \in [c-a, c+a].$$

Let us denote

(5) $$r_n^i = \sum_{j=n}^{\infty} |a_j^i|, \quad \beta_n = \sum_{j=n}^{\infty} b_j \qquad \text{for} \quad n \in N, \; i = 0,1,\dots,k.$$

Using Lemma 1 one can see that the series $\sum_{n=1}^{\infty} r_n^i$, $\sum_{n=1}^{\infty} \beta_n$ are convergent.

Let us denote

(6) $$\rho_n = \sum_{i=0}^{k} \sum_{j=n}^{\infty} r_j^i, \qquad \eta_n = \sum_{j=n}^{\infty} |\beta_j|, \quad n \in N.$$

There exists an index $m \in N$ such that $M\rho_n + \eta_n < a$ for every $n \ge m$.

Let l_∞ denotes the Banach space of all real bounded sequences, equipped with "sup" norm. Let

$$T = \{x \in \ell_\infty : x_1 = \dots = x_{m-1} = c \text{ and } |x_n - c| \le M\rho_n + \eta_n \text{ for } n \ge m\}.$$

Obviously, T is a convex subset of the space l_∞. Let $\varepsilon > 0$. It is easy to construct a finite $\varepsilon - net$ for the set T. Hence T is a compact set.

If $x \in T$ then $x_n \in [c-a, c+a]$ for each $n \in N$. Therefore, by (4)

(7) $$|f(x_n)| \le M \qquad \text{for every} \quad x \in T, \; n \in N.$$

Let $x \in T$. From (1), (7) the series

(8) $$\sum_{j=1}^{\infty} a_j^i f(x_{i+j}), \qquad i=0,1,\dots,k$$

are absolutely convergent.

Denoting

(9) $$u_n^i = \sum_{j=n}^{\infty} a_j^i f(x_{i+j}), \qquad i=0,1,\dots,k$$

by (8), (7) and (5) we have

(10) $$|u_n^i| \le \sum_{j=n}^{\infty} M|a_j^i| = Mr_n^i \quad \text{for all} \quad i=0,1,\dots,k.$$

Since the series $\sum_{j=1}^{\infty} r_j^i$, $i=0,1,\dots,k$ are convergent, the series $\sum_{j=1}^{\infty} |u_j^i|$ are convergent, too.

Now we define a sequence $A(x)$ by formula

$$A(x)(n) = \begin{cases} c & \text{for} \quad n<m, \\ c + \sum_{i=0}^{k} \sum_{j=n}^{\infty} u_j^i + \sum_{j=n}^{\infty} \beta_j & \text{for} \quad n \ge m. \end{cases}$$

If $n \ge m$ then

$$|A(x)(n) - c| \le \sum_{i=0}^{k} \sum_{j=n}^{\infty} |u_j^i| + \sum_{j=n}^{\infty} |\beta_j|.$$

By (5), (10) we have

$$|A(x)(n) - c| \le M \sum_{i=0}^{k} \sum_{j=n}^{\infty} r_j^i + \eta_n = M\rho_n + \eta_n.$$

Hence $A(x) \in T$ for every $x \in T$ and we get a map $A: T \to T$.

We prove that A is a continuous on T. Let $x \in T$, $\varepsilon > 0$. The function f is uniformly continuous on the interval $[c-a, c+a]$. Hence there exists a constant $\delta > 0$ such that

$$\text{if } t,s \in [c-a, c+a] \text{ and } |t-s| < \delta \text{ then } |f(t) - f(s)| < \varepsilon.$$

Let x, z be any two elements of the set T such that $\|x - z\| < \delta$. Then $|x_n - z_n| < \delta$ for every $n \in N$. Hence

(11) $$|f(x_n) - f(z_n)| < \varepsilon \quad \text{for all} \quad n \in N.$$

Let us denote

(12) $$v_n^i = \sum_{j=n}^{\infty} a_j^i f(z_{i+j}), \quad i = 0,1,\ldots,k, \; n \in N.$$

Using (9), (12) we get

$$\|A(x) - A(z)\| = \sup_{n \ge m} \left| \sum_{i=0}^{k} \sum_{j=n}^{\infty} u_j^i - \sum_{i=0}^{k} \sum_{j=n}^{\infty} v_j^i \right| \le \sum_{i=0}^{k} \sum_{j=m}^{\infty} |u_j^i - v_j^i|.$$

By (9), (11), (12), (5) one yields

(13) $$|u_j^i - v_j^i| = \left| \sum_{p=j}^{\infty} a_p^i f(x_{i+p}) - \sum_{p=j}^{\infty} a_p^i f(z_{i+p}) \right| \le$$
$$\le \sum_{p=j}^{\infty} |a_p^i \| f(x_{i+p}) - f(z_{i+p})| \le \varepsilon \sum_{p=j}^{\infty} |a_p^i| = \varepsilon r_j^i.$$

Hence, by (6)

$$\|A(x) - A(z)\| \le \sum_{i=0}^{k} \sum_{j=m}^{\infty} \varepsilon r_j^i = \varepsilon \rho_m.$$

This shows that A is a continuous map. So, by Schauder fixed point theorem there exists $z \in T$ such that $A(z) = z$. Then

(14) $$z_n = c + \sum_{i=0}^{k} \sum_{j=n}^{\infty} v_j^i + \sum_{j=n}^{\infty} \beta_j \quad \text{for all} \quad n \ge m.$$

Applying operator Δ to (14) we obtain

$$\Delta z_n = -\sum_{i=0}^{k} v_n^i - \beta_n = -\sum_{i=0}^{k} v_n^i - \sum_{j=n}^{\infty} b_j \quad \text{for all} \quad n \ge m.$$

Hence

$$\Delta^2 z_n = -\sum_{i=0}^{k} \left[\sum_{j=n+1}^{\infty} a_j^i f(z_{i+j}) - \sum_{j=n}^{\infty} a_j^i f(z_{i+j}) \right] + b_n =$$
$$= \sum_{i=0}^{k} a_n^i f(z_{i+n}) + b_n \quad \text{for} \quad n \ge m.$$

This means that the sequence (z_n) satisgies Eq. (E) but for $n \ge m$. Eq. (E) can be transformed to the form

(15) $$x_n - a_n^0 f(x_n) = 2x_{n+1} + a_n^1 f(x_{n+1}) - x_{n+2} + \sum_{i=2}^{k} a_n^i f(x_{n+i}) + b_n.$$

Substituting in (15) $n = m-1$, $x_j = z_j$ for $j \ge m$ and using $(*)$ we obtain x_{m-1}. Analogously we can calculate $x_{m-2}, x_{m-3}, \ldots, x_1$. Consequently we get the sequence which satisfies Eq. (E) for all $n \in N$. Moreover this sequence coincides with the sequence z for $n \geq m$ and so possesses the asymptotic property (3) because by (14)

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left[c + \sum_{i=0}^{k}\sum_{j=n}^{\infty} v_j^i + \sum_{j=n}^{\infty}\beta_j\right] = c.$$

REMARK. It easy to show that if $f : R \to R$ is bounded function or if it is a polynomial of degree $2k+1$, $k \in N$ then f satisfies the condition $(*)$.

A similar property for linear difference equation

$$\Delta x_n = \sum_{i=0}^{r} a_n^i x_{n+i} + b_n, \qquad n \in N$$

can be found in J. Popenda and E. Schmeidel [2], and for the nonlinear equation

$$\Delta^2 x_n = a_n \varphi(x_{n+k})$$

in M. Migda, J. Migda [1].

THEOREM 2. *If the assumptions* (1) *and* (2) *of Th. 1 hold and* $f : R \to R$ *is a bounded and uniformly continuous function then for every* $c \in R$ *there exists a solution* (x_n) *of Eq.* (E) *which possesses the asymptotic behavior*

$$x_n = cn + o(1).$$

PROOF. Let $M > 0$ be a constant such that $|f(t)| \leq M$ for all $t \in R$. Similarly as in the proof of Th. 1 for $n \in N$ we define r_n^i, β_n, ρ_n, η_n by (5), (6).

Let ℓ be the space of all sequences $x : N \to R$ and let

$$T = \{x \in \ell_\infty : |x_n| \leq M\rho_n + \eta_n \quad \text{for all} \quad n \in N\}$$

$$S = \{x \in \ell : |x_n - nc| \leq M\rho_n + \eta_n \quad \text{for all} \quad n \in N\}.$$

Define now a map $F : T \to S$ by

$$F(x)(n) = nc + x_n.$$

Obviously, the following formula $d(x,z) = \sup\{|x_n - z_n| : n \in N\}$ defines a metric on the set S such that F is an isometry of the set T into S.

Similarly as in the proof of Th. 1 it can be shown that T is convex and compact subset of ℓ_∞. The space S is homeomorphic to T. Hence by Schauder theorem every continuous map $A : S \to S$ has a fixed point.

For $x \in S$, $n \in N$, $i = 0,1,\ldots,k$ we define u_n^i by (9) and an operator A by

$$A(x)(n)=nc+\sum_{i=0}^{k}\sum_{j=n}^{\infty}u_j^i+\sum_{j=n}^{\infty}\beta_j.$$

Then

$$|A(x)(n)-nc|\le\sum_{i=0}^{k}\sum_{j=n}^{\infty}|u_j^i|+\sum_{j=n}^{\infty}|\beta_j|\le M\rho_n+\eta_n,$$

for every $n\in N$. Hence $A(x)\in S$ and A maps S into S.

Let $x,z\in S$. We take $\varepsilon>0$. Since the function f is uniformly continuous there exists $\delta>0$ such that

$$\text{if}\quad |t-s|<\delta\quad\text{then}\quad |f(t)-f(s)|<\varepsilon.$$

If $d(x,z)<\delta$ then $|x_n-z_n|<\delta$ for every $n\in N$. Hence $|f(x_n)-f(z_n)|<\varepsilon$ for all $n\in N$.

Denoting u_n^i, v_n^i by (9), (12) one yields

$$d(A(x),A(z))=\sup\left|\sum_{i=0}^{k}\sum_{j=n}^{\infty}u_j^i-\sum_{i=0}^{k}\sum_{j=n}^{\infty}v_j^i\right|\le\sum_{i=0}^{k}\sum_{j=1}^{\infty}|u_j^i-v_j^i|.$$

By (13) it follows that

$$d(A(x),A(z))\le\sum_{i=0}^{k}\sum_{j=1}^{\infty}\varepsilon r_j^i=\varepsilon\rho_1.$$

This shows that A is a continuous map. Hence there exists a sequence $x\in S$ such that $A(x)=x$.

Then for every $n\in N$ we have

$$x_n=nc+\sum_{i=0}^{k}\sum_{j=n}^{\infty}u_j^i+\sum_{j=n}^{\infty}\beta_j.$$

Hence

$$\Delta^2x_n=\sum_{i=0}^{k}a_j^if(x_{n+1})+b_n\quad\text{for each}\quad n\in N.$$

Therefore x is a solution of Eq. (E). Since the series $\sum_{j=1}^{\infty}u_j^i$ $(i=0,1,\ldots,k)$ are convergent it follows that $x_n=cn+o(1)$.

THEOREM 3. *Suppose* $\sum_{n=1}^{\infty}|a_n^i|<\infty$, $i=0,1,\ldots,k$, $\sum_{n=1}^{\infty}|b_n|<\infty$ *and* $f:R\to R$ *is a bounded function. If* (x_n) *is a solution of Eq.* (E) *then the sequence* (x_n/n) *is convergent in R.*

PROOF. Assume that $|f(t)| \le M$ for every $t \in R$. If $m > n$ then

$$\Delta x_m - \Delta x_n = \sum_{j=n}^{m-1} \Delta^2 x_j = \sum_{j=n}^{m-1} \left(\sum_{i=0}^{k} a_j^i f(x_{i+j}) + b_j \right).$$

Hence

$$|\Delta x_m - \Delta x_n| \le \sum_{j=n}^{m-1} \left(M \sum_{i=0}^{k} |a_j^i| + |b_j| \right) \le M \sum_{i=0}^{k} \sum_{j=n}^{m-1} |a_j^i| + \sum_{j=n}^{m-1} |b_j|.$$

Therefore the sequence (Δx_n) is convergent.

By virtue of Stolz theorem

$$\lim_{n\to\infty} \frac{x_n}{n} = \lim_{n\to\infty} \Delta x_n.$$

EXAMPLE. *Let us consider the equation*

$$\Delta^2 x_n = \frac{1}{3^{4n}} x_n^3 + \frac{4}{3^{2n+3}} x_{n+1}^3 - \frac{1}{3^n}. \tag{16}$$

The sequence $x_n = 3^n$ *is a solution of Eq.* (15) *and possesses the property* $\lim_{n\to\infty} x_n/n = \infty$. *Hence we see the assumption of boundedness of the function f in Th. 3 cannot be omitted.*

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Received on 1998.02.06 and, in revised from, on 1999.11.10.