FASCICULI MATHEMATICI

Nr 31 2001

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OSCILLATION AND NONOSCILLATION CRITERIA FOR SECOND ORDER LINEAR DIFFERENCE EQUATIONS

ABSTRACT: New oscillation and nonoscillation criteria for the second order linear difference equation

$$\Delta^2 u_k + p_k u_{k+1} = 0$$

are established via the Riccati technique

KEY WORDS: linear difference equation, discrete oscillation and nonoscillation criteria, Riccati difference equation.

1. INTRODUCTION

Consider the second order linear difference equation

$$\Delta^2 u_k + p_k u_{k+1} = 0, \tag{1}$$

where $\Delta^2 u_k = \Delta(\Delta u_k)$, $\Delta u_k = u_{k+1} - u_k$ is the usual forward difference operator and p_k, $k \in N$, is a real-valued sequence. The purpose of this contribution is to investigate oscillatory properties of (1), in particular, we look for new conditions, which guarantee that this equation is oscillatory (nonoscillatory).

The paper is organized as follows. In this introductory section we give some definitions and fix the notations used throughout the paper. We mention the motivation and (also as a consequence of this) the connection with differential equations. Section 2 is devoted to the oscillation criteria for equation (1) and some remarks. The nonoscillation criteria are contained in Section 3. The last section is devoted to the related comments and the comments concerning possible extensions of the results presented in this paper. Some examples are also given.

First, recall the concept of oscillation of equation (1).

DEFINITION 1.

- *A discrete interval* $(m, m+1$ *is said to contain a* ***generalized zero*** *of the solution* y *of equation* (1), *if* $y_m \neq 0$ *and* $y_m y_{m+1} \leq 0$.
- *A nontrivial solution of* (1) *is called* ***oscillatory*** *if it has infinitely many generalized zeros in a given (infinite) discrete interval. In the opposite case* (1) *is said to be* ***nonoscillatory***. *In view of the Sturm separation theorem which holds for* (1), *we have the following equivalence: Any*

solution of (1) *is oscillatory if and only if every solution of* (1) *is oscillatory. Hence we can speak about* ***oscillation or nonoscillation of equation*** (1).

In the last years, quite a lot of works dealing with the oscillation (or nonoscillation) of equation (1) have appeared. Mostly, the discrete versions of oscillation criteria for differential equation

$$u'' + p(t)u = 0 \tag{2}$$

are presented in these works, from which we select [2, 3, 4, 5, 8, 10]. To introduce some of them here, we need the following notation: Denote a finite limit

$$\tilde{c} = \lim_{k\to\infty} c_k, \tag{3}$$

where

$$c_k = \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j} p_i.$$

Proposition 1 ([3]). *A sufficient condition for equation* (1) *to be oscillatory is that either*

$$\lim_{k\to\infty} c_k = \infty,$$

or that

$$-\infty < \liminf_{k\to\infty} c_k < \limsup_{k\to\infty} c_k. \tag{4}$$

Note that the continuous (and original) version of the above criterion is due to A. Wintner [12] and P. Hartman [6], respectively. The assumption of the nonexistence of $\lim_{k\to\infty} c_k$ as a finite number is a sufficient condition for oscillation of (1). From this point of view, the case when $\tilde{c}$ exists finite, seems to be interesting for the examination. We will show, among others, that (1) remains oscillatory if (3) holds, but the sequence c_k does not tend to $\tilde{c}$ too quickly.

However, in contrast to the differential equation (2), the following criterion holds for difference equation (1), regardless whether or not (3) holds.

Proposition 2 ([3]). *A sufficient condition for equation* (1) *to be oscillatory is that either*

$$\lim_{k\to\infty}\sum_{j=1}^{k} p_j = \infty,$$

or that

$$\liminf_{k\to\infty}\sum_{j=1}^{k} p_j < \limsup_{k\to\infty}\sum_{j=1}^{k} p_j .$$

Note that criterion (5) has no continuous analogy, since here it is allowed

$$\liminf_{k\to\infty}\sum_{j=1}^{k} p_j = -\infty.$$

Moreover, Proposition 2 is better than Proposition 1 in a certain sence, since the existence of a finite limit

$$\sum_{j=1}^{\infty} p_j < \lim_{k\to\infty}\sum_{j=1}^{k} p_j$$

implies (3) and hence (4) implies (5). These discrepancies between the continuous and the discrete cases are due to some specific properties of difference calculus.

From the above observation we can see that it is interesting to examine the following two cases:

- Assumption (6) holds (and then (3) holds). What are additional conditions (e.g. how fast c_k converges to $\widetilde{c}$), which imply oscillation of (1)? This problem is solved in this paper.
- Assumption $\lim_{k\to\infty}\sum_{j=1}^{k} c_k = -\infty$ holds. In this case both, oscillation and

nonoscillation of (1) are possible, see [3], and hence it makes a sense to look for additional condition implying (non)oscillation of (1). This problem is a subject of the present investigation.

Hence, below, we shall always assume that (6) holds.

Now we are going to formulate the statement playing an important role in the proof of the results, which are included in Section 2.

LEMMA 1 ([3]). *Assume that* (6) *holds,* (1) *is nonoscillatory and u is any of its solution such that* $u_k u_{k+1} > 0$ *for* $k \ge m$. *Then*

$$\sum_{j=m}^{\infty} R_k < \infty, \tag{7}$$

where

$$w_k = \frac{\Delta u_k}{u_k} \quad \text{and} \quad R_k = \frac{w_k^2}{1 + w_k}. \tag{8}$$

In the continuous case this lemma comes from P. Hartman, see [7]. The proofs of these auxiliary statements (both continous and discrete), and also of all hereafter mentioned oscillation criteria are based on the ***Riccati technique***. This method uses the following idea: Suppose (by contradiction) that equation (1) is nonoscillatory. Then there exists $m \in N$ such that we can introduce the Riccati type transformation, i.e., w_k is defined by (8), $k \geq m$. Since $+w_k > 0$ for $k \geq m$, this leads to the Riccati difference equation

$$\Delta w_k + p_k + R_k = 0, \tag{9}$$

where R_k is also defined by (8). In the proofs of our nonoscillation criteria we use an equivalence between the nonoscillation of (1) and the existence of solution of the Riccati difference inequality, see Lemma 4.

NOTATION 1. We set

$$Q_k = (k+1)\left(\tilde{c} - \sum_{j=1}^{k-1} p_j\right) = (k+1)\sum_{j=k}^{\infty} p_j, \qquad H_k = \frac{1}{k+1}\sum_{j=k}^{k-1} (j+1)^2 p_j,$$

$$Q_* = \liminf_{k\to\infty} Q_k, \qquad Q^* = \limsup_{k\to\infty} Q_k,$$

$$H_* = \liminf_{k\to\infty} H_k, \qquad H^* = \limsup_{k\to\infty} H_k.$$

2. OSCILLATION CRITERIA

In this section we give oscillation criteria for (1) in the case when (6) holds.

THEOREM 1. *Let* $Q_* > -\infty$ *and*

$$\limsup_{k\to\infty} \left(\sum_{j=1}^{k+1} \frac{1}{j}\right)^{-1} \sum_{j=1}^{k-1} j p_{j+1} > \frac{1}{4}, \tag{10}$$

or (*equivalently in this case*)

$$\limsup_{k\to\infty} \frac{1}{\ln k} \sum_{j=1}^{k-1} j p_j > \frac{1}{4}. \tag{11}$$

Then equation (1) *is oscillatory.*

Proof. Suppose, by contradiction, that (1) is nonoscillatory, i.e., there exists $m \in N$ such that $u_k u_{k+1} > 0$ for $k \geq m$, where u is a solution of equation (1). Then one can define the Riccati equation (9), which we rewrite as $\Delta w_k = -p_k - R_k$. By summation of this equality from $k+1$, $k \geq m-1$, to l and letting $l \to \infty$ we get

$$w_{k+1} = \sum_{j=k+1}^{\infty} p_j + \sum_{j=k+1}^{\infty} R_j,$$

since (7) implies $w_l \to 0$ as $l \to \infty$. From here we obtain

$$w_{k+1} = \widetilde{c} - \sum_{j=1}^{k} p_j + \sum_{j=k+1}^{\infty} R_j. \tag{12}$$

The summation of (12) from m to k yields

$$\sum_{j=m}^{k} w_{j+1} = \widetilde{c}(k-m+1) - \sum_{j=m}^{k} \sum_{i=1}^{j} p_i + \sum_{j=m}^{k} \sum_{i=j+1}^{\infty} R_i.$$

Let us denote $C_{k,m} = k(\widetilde{c} - c_k) + \widetilde{c} - \widetilde{c}m$. Using summation by parts we have

$$\sum_{j=m}^{k} [w_{j+1} - (j+1)R_{j+1}] =$$

$$= C_{k,m} + \sum_{j=1}^{m-1} \sum_{i=1}^{j} p_i + (k+1) \sum_{j=k+2}^{\infty} R_j - m \sum_{j=m+1}^{\infty} R_j =$$

$$= C_{k,m} + (k+1) \sum_{j=k+2}^{\infty} R_j + m \sum_{j=1}^{m} p_j - p_1 - \sum_{j=1}^{m-1} (j+1) p_{j+1} - m \sum_{j=m+1}^{\infty} R_j =$$

$$= C_{k,m} + (k+1) \sum_{j=k+2}^{\infty} R_j + m \sum_{j=1}^{m} p_j - m \sum_{j=m+1}^{\infty} R_j - \sum_{j=1}^{\infty} j p_j =$$

$$= C_{k,m} + (k+1) \sum_{j=k+2}^{\infty} R_j - \sum_{j=1}^{m} j p_j + m \sum_{j=1}^{m-1} p_j - m \sum_{j=m}^{\infty} R_j - m \Delta w_m.$$

Thus we get

$$k(\widetilde{c} - c_k) = \sum_{j=m}^{k} [w_{j+1} - (j+1)R_{j+1}] - (k+1) \sum_{j=k+2}^{\infty} R_j + \widetilde{R},$$

where

$$\widetilde{R} = -\widetilde{c} + \sum_{j=1}^{m} jp_j + mw_{m+1}.$$

Now, let $\varepsilon > 0$ be arbitrary. Further, suppose that m is chosen such that $|w_k| \leq \varepsilon$ for $k \geq m$. Such m exists since (7) holds. Then

$$w_{j+1} - (j+1)R_{j+1} \leq \frac{1+\varepsilon}{4(1+j)},$$

since

$$\left(\frac{w_{j+1}}{1+\varepsilon}\sqrt{1+j} - \frac{1}{2\sqrt{1+j}} \right)^2 \geq 0.$$

Hence

$$k(\widetilde{c} - c_k) \leq \frac{1+\varepsilon}{4} \sum_{j=1}^{k+1} \frac{1}{j} + \widetilde{R}$$

and thus

$$\left(\sum_{j=1}^{k+1} \frac{1}{j} \right)^{-1} k(\widetilde{c} - c_k) \leq \frac{1}{4} + \frac{\varepsilon}{4} + \left(\sum_{j=1}^{k+1} \frac{1}{j} \right)^{-1} \widetilde{R}. \tag{13}$$

Now, we can rewrite the term $\widetilde{c} - c_k$ from the left side by this way

$$\begin{aligned} \widetilde{c} - c_k &= \sum_{j=1}^{\infty} p_j - \frac{1}{k}\left[(j-1)\sum_{i=1}^{j} p_i \right]_1^{k+1} + \frac{1}{k}\sum_{j=1}^{k} jp_{j+1} = \\ &= \sum_{j=k+1}^{\infty} p_j + \frac{1}{k}\sum_{j=1}^{k-1} jp_{j+1}. \end{aligned} \tag{14}$$

Using this in (13) we obtain

$$\begin{aligned} \left(\sum_{j=1}^{k+1} \frac{1}{j} \right)^{-1} \sum_{j=1}^{k-1} jp_{j+1} &\leq \frac{1}{4} + \frac{\varepsilon}{4} + \left(\sum_{j=1}^{k+1} \frac{1}{j} \right)^{-1} \widetilde{R} - \left(\sum_{j=1}^{k+1} \frac{1}{j} \right)^{-1} k \sum_{j=k+1}^{\infty} p_j = \\ &= \frac{1}{4} + \frac{\varepsilon}{4} + \left(\sum_{j=1}^{k+1} \frac{1}{j} \right)^{-1} \left(\widetilde{R} - Q_{k+1} - 2\sum_{j=k+1}^{\infty} p_j \right), \end{aligned}$$

which contradicts (10).

The condition (11) is also sufficient for equation (1) to be oscillatory, since

$$\sum_{j=1}^{k+1} \frac{1}{j} = c^{[E]} + \ln k + \widetilde{\varepsilon}_k ,$$

where $c^{[E]}$ is the Euler constant and $\lim\limits_{k\to\infty} \widetilde{\varepsilon}_k = 0$. Then we have

$$\frac{k}{\ln k}(\widetilde{c} - c_k) \le \frac{1}{4} + \frac{\varepsilon}{4} + \frac{1}{\ln k}\left(\frac{1+\varepsilon}{4} c^{[E]} + \frac{1+\varepsilon}{4}\widetilde{\varepsilon}_k + \widetilde{R}\right),$$

again a contradiction.

REMARK 1. From inequality (13) of the above proof follows that a sufficient condition for equation (1) to be oscillatory has also the from

$$\text{(15)} \qquad \limsup_{k\to\infty} \left(\sum_{j=1}^{k+1} \frac{1}{j}\right)^{-1} k(\widetilde{c} - c_k) > \frac{1}{4},$$

or (equivalenty in this case)

$$\text{(16)} \qquad \limsup_{k\to\infty} \frac{k}{\ln k}(\widetilde{c} - c_k) > \frac{1}{4}.$$

COROLLARY 1. *Let*

$$\text{(17)} \qquad \liminf_{k\to\infty} (Q_k + H_k) > \frac{1}{2}.$$

Then equation (1) *is oscillatory.*

PROOF. We start with the following calculation

$$\frac{2}{k+1}\sum_{j=1}^{k} Q_j = \frac{2}{k+1}\sum_{j=1}^{k} (j+1)\widetilde{c} - \frac{2}{k+1}\sum_{j=1}^{k}\left[(j+1)\sum_{i=1}^{j-1} p_i\right] =$$

$$= \frac{1}{k+1}\sum_{j=1}^{k} (2j+1)\widetilde{c} + \frac{1}{k+1}\sum_{j=1}^{k} \widetilde{c} -$$

$$- \frac{1}{k+1}\sum_{j=1}^{k}\left[(2j+1)\sum_{i=1}^{j-1} p_i\right] - \frac{1}{k+1}\sum_{j=1}^{k}\sum_{i=1}^{j-1} p_i =$$

$$= Q_k + H_k + \frac{k}{k+1}\widetilde{c} - \frac{\widetilde{c}}{k+1} - \frac{k}{k+1}c_k + \frac{1}{k+1}\sum_{j=1}^{k} p_j .$$

From here we get

$$Q_k + H_k = \frac{2}{k+1}\sum_{j=1}^{k} Q_j + \frac{k}{k+1}(c_k - \widetilde{c}) - \frac{1}{k+1}\sum_{j=1}^{k} p_j + \frac{\widetilde{c}}{k+1}.$$

Condition (17) implies an existence of $\varepsilon > 0$ such that

$$\sum_{j=1}^{k} \frac{1}{(j+1)(j+2)} \sum_{i=1}^{j} Q_i > \left(\frac{1}{4} + \varepsilon\right) \sum_{j=1}^{k+1} \frac{1}{j}$$

for k sufficiently large.

$$k(\widetilde{c} - c_k) = \sum_{j=1}^{k} \frac{Q_j}{j+1} - \sum_{j=1}^{k} p_j =$$

$$= \frac{k+1}{(k+1)(k+2)} \sum_{j=1}^{k} Q_j + \sum_{j=1}^{k} \frac{1}{(j+1)(j+2)} \sum_{i=1}^{j} Q_i - \sum_{j=1}^{k} p_j >$$

$$> \left(\frac{1}{4} + \varepsilon\right) \sum_{j=1}^{k+1} \frac{1}{j} + \frac{1}{k+2} \sum_{j=1}^{k} Q_j - \sum_{j=1}^{k} p_j.$$

Dividing it by $\sum_{j=1}^{k+1} 1/j$ we see that condition (15) holds and hence the assumption of Theorem 1 is fulfilled.

THEOREM 2. *Let*

$$\limsup_{k \to \infty} \, (Q_k + H_k) > 1.$$

Then equalion (1) is oscillatory.

PROOF. Suppose, by contradiction, that equation (1) is nonoscillatory. Then one can define the Riccati equation (9), which we rewrite as $\Delta w_k = -p_k - R_k$. By multiplying both sides of this equality by $(k+1)^2$ and by summation from $n \geq m$ to $k-1$ we obtain

$$\sum_{j=n}^{k-1} (j+1)^2 \Delta w_j = -\sum_{j=n}^{k-1} (j+1)^2 p_j - \sum_{j=n}^{k-1} (j+1)^2 R_j.$$

From here we get

(19)
$$(k+1)w_k = -H_k + \frac{1}{k+1}\sum_{j=1}^{k-1}\left[(2j+1)w_j - (j+1)^2 R_j\right] + \frac{1}{k+1}\sum_{j=1}^{n-1}(j+1)^2 p_j + \frac{(n+1)^2}{k+1}w_n.$$

Clearly, $(2k+1)w_k - (k+1)^2 R_k \le 1$, since $(kw_k - 1)^2 \ge 0$. Hence

$$(k+1)w_k \le -H_k + 1 - \frac{n+1}{k+1} + \frac{1}{k+1}\sum_{j=1}^{n-1}(j+1)^2 p_j + \frac{(n+1)^2}{k+1}w_n.$$

Now, using (12), we have

$$Q_k + H_k \le 1 + \frac{1}{k+1}\sum_{j=1}^{n-1}(j+1)^2 p_j + \frac{(n+1)^2}{k+1}w_n - \frac{n+1}{k+1},$$

which contradicts with the assumption.

For proofs of the next oscillation criteria we need the following two lemmas.

LEMMA 2. *Suppose that equation* (1) *is nonoscillatory,* $0 \le Q_* \le 1/4$ *and* u *is a solution of* (1). *Then*

(20)
$$\liminf_{k\to\infty}\frac{(k+1)\Delta u_k}{u_k} \ge \tfrac{1}{2}(1-\sqrt{1-4Q_*}).$$

PROOF. Let u_k be nonoscillatory solution of (1), i.e., there exists $n \in N$ such that $u_k u_{k+1} > 0$ for $k \ge n$. According to Lemma 1 an equality (12) holds. We set

$$A = \liminf_{k\to\infty}\,(k+1)w_k,$$

where w is defined by (8). If $A = \infty$, then obviously there is nothing to prove. So, let $A < \infty$. If $Q_* = 0$, then (20) is trivial in view of (12) holds. Thus, let us suppose $Q_* > 0$. For arbitrary $\varepsilon \in (0, Q_*)$ there exists $k^{[\varepsilon]} > m$ such that

(21)
$$Q_k > Q_* - \varepsilon, \qquad k \ge k^{[\varepsilon]}.$$

From (12) we get

$$(k+1)w_k \ge A - \varepsilon, \quad k \ge k^{[\varepsilon]}.$$

Hence

$$A \geq Q_*.$$

Now, we choose $\widetilde{k}^{[\varepsilon]} > k^{[\varepsilon]}$ such that

$$(k+1)w_k \geq A - \varepsilon, \quad \text{and} \quad |w_k| \leq \varepsilon, \; k \geq \widetilde{k}^{[\varepsilon]}.$$

Taking this and inequality (21) into account, from (12) we have

$$(k+1)w_k \geq Q_* - \varepsilon + (k+1)\sum_{j=k}^{\infty} \frac{w_j^2}{1+w_j}, \qquad k \geq \widetilde{k}^{[\varepsilon]}.$$

Further,

$$(k+1)w_k \geq Q_* - \varepsilon + \frac{k+1}{\varepsilon+1}\sum_{j=k}^{\infty} \frac{(A-\varepsilon)^2}{(j+1)^2} \geq$$

$$\geq Q_* - \varepsilon + \frac{(k+1)(A-\varepsilon)^2}{\varepsilon+1}\sum_{j=k}^{\infty} \Delta\left(\frac{-1}{j+1}\right) =$$

$$= Q_* - \varepsilon + \frac{(A-\varepsilon)^2}{1+\varepsilon}, \qquad k \geq \widetilde{k}^{[\varepsilon]}.$$

Therefore

$$A \geq Q_* - \varepsilon + \frac{(A-\varepsilon)^2}{1+\varepsilon}$$

and, since $\varepsilon > 0$ was arbitrary, we have $A^2 - A + Q_* \leq 0$. Solving this quadratic inequality we observe that (20) holds.

LEMMA 3. *Suppose that equation* (1) *is nonoscillatory*, $0 \leq H_* \leq 1/4$ *and* u *is a solution of* (1). *Then*

$$\limsup_{k\to\infty} \frac{(k+1)\Delta u_k}{u_k} \leq \tfrac{1}{2}(1+\sqrt{1-4H_*}). \tag{22}$$

PROOF. Let u_k be nonoscillatory solution of (1), i.e., there exists $n \in N$ such that $u_k u_{k+1} > 0$ for $m \in N$ We set

$$B = \limsup_{k\to\infty} \,(k+1)w_k,$$

where w is defined by (8). If $B = -\infty$, then obviously there is nothing to prove. Inequality (22) is even valid for $B \leq 0$. Hence we shall be assumed that $B > 0$. Now, consider equality (19). From here

$$(k+1)w_k \le 1 - H_k + \widetilde{P}_{k,n},$$

where

$$\widetilde{P}_{k,n} = \frac{1}{k+1}\sum_{j=1}^{n-1}(j+1)^2 p_j + \frac{(n+1)^2}{k+1} w_n. \tag{23}$$

Hence $B \le 1 - H_*$. Thus, the estimate (22) obviously holds for $H_* = 0$.

We shall proceed with $H_* > 0$. The equality (19) can be rewritten by the following way

$$(k+1)w_k = -H_k + \frac{1}{k+1}\sum_{j=n}^{k-1}\left[jw_j(2 - jw_j) + \frac{w_j}{1+w_j}(jw_j - 1)^2 \right] + \widetilde{P}_{k,n}.$$

For arbitrary ε, $0 < \varepsilon < \min\{H_*, 1-B\}$, there exists $k^{[\varepsilon]}$ such that

$$kw_k < B + \varepsilon, \quad H_k > H_* - \varepsilon \quad \text{and} \quad \left| \frac{w_k}{1+w_k} \right| \le \varepsilon, \quad k \ge k^{[\varepsilon]}.$$

Now, for $n = k^{[\varepsilon]}$ we get

$$(k+1)w_k \le -H_* + \varepsilon + \frac{k - k^{[\varepsilon]}}{k+1}[(B+\varepsilon)(2 - B - \varepsilon)) + \varepsilon(B + \varepsilon - 1)^2] + \widetilde{P}_{k,k^{[\varepsilon]}},$$

since the behaviour of the function $f(x) = x(2-x) + \varepsilon(x-1)^2$ is "similar" to that of the function $x(2-x)$ for $\varepsilon > 0$ sufficiently small, where $x = B + \varepsilon$. Namely, $f(1) = 1$, $f'(1) = 0$ and $f''(x) = -2 + 2\varepsilon < 0$ for $\varepsilon < 1$. But this holds, since we assume $B \in (0,1)$. Hence

$$B \le -H_* + \varepsilon + (B+\varepsilon)(2 - B - \varepsilon) + \varepsilon(B + \varepsilon - 1)^2.$$

Therefore $B^2 - B + H_* \le 0$ and from here clearly (22) holds.

Remark 2. Corollary 1 and equality (18) imply that if $Q_* > 1/4$, then (1) is oscillatory Theorem 1 and equality

$$\sum_{j=1}^{k-1} \frac{1}{j} H_j + \frac{1}{k}\sum_{j=1}^{k-1}(j+1)^2 p_j = \sum_{j=1}^{k-1}(j+1)p_j,$$

which we can rewrite as

$$\frac{1}{\sum_{j=1}^{k}\frac{1}{j}}\sum_{j=1}^{k-2} jp_{j+1} = \frac{1}{\sum_{j=1}^{k}\frac{1}{j}}\sum_{j=1}^{k-1}\frac{1}{j}H_j + \frac{k+1}{k\sum_{j=1}^{k}\frac{1}{j}}H_k - \frac{2}{\sum_{j=1}^{k}\frac{1}{j}}\sum_{j=1}^{k-1} p_j,$$

follow that the assumption $H_* > 1/4$ is also sufficient for equation (1) to be oscillatory. Hence, it is purposeful to proceed with $Q_* \le 1/4$ and (or) $H_* \le 1/4$.

THEOREM 3. *A sufficient condition for* (1) *to be oscillatory is that either*

$$0 \le Q_* \le \tfrac{1}{4} \quad \textit{and} \quad H^* > \tfrac{1}{2}(1+\sqrt{1-4Q_*}), \tag{24}$$

or that

$$0 \le H_* \le \tfrac{1}{4} \quad \textit{and} \quad Q^* > \tfrac{1}{2}(1+\sqrt{1-4H_*}). \tag{25}$$

PROOF. Suppose, by contradicition, that equation (1) is nonoscillatory, i.e., there exists $m \in N$ such that $u_k u_{k+1} > 0$ for $k \ge m$, where u is a solution of equation (1). Then equalities (12) and (19) are valid. By Lemma 2 (in the case that (24) holds) and Lemma 3 (in the case that (25) holds) for arbitrary $\varepsilon > 0$ there exists $k^{[\varepsilon]}$ such that

$$(k+1)w_k > \widetilde{A} - \varepsilon \quad \text{and} \quad (k+1)w_k < \widetilde{B} + \varepsilon, \ k \ge k^{[\varepsilon]}, \tag{26}$$

where

$$\widetilde{A} = \tfrac{1}{2}(1-\sqrt{1-4Q_*}) \quad \text{and} \quad \widetilde{B} = \tfrac{1}{2}(1+\sqrt{1-4H_*}). \tag{27}$$

Hence, if (24) holds, then form (19) we get

$$\begin{aligned} H_k &\le -(k+1)w_k + 1 - \frac{k^{[\varepsilon]}+1}{k+1} + \widetilde{P}_{k,k^{[\varepsilon]}} \le \\ &\le -\widetilde{A} + \varepsilon + 1 - \frac{k^{[\varepsilon]}+1}{k+1} + \widetilde{P}_{k,k^{[\varepsilon]}} \le \\ &\le \tfrac{1}{2}(1+\sqrt{1-4Q_*}) + \varepsilon - \frac{k^{[\varepsilon]}+1}{k+1} + \widetilde{P}_{k,k^{[\varepsilon]}} \end{aligned}$$

for $k \ge k^{[\varepsilon]}$, where $\widetilde{P}$ is defined by (23), and if (25) holds, then from (12) we get

$$(k+1)w_k = Q_k + (k+1)\sum_{j=k}^{\infty} R_j.$$

Thus, $Q_k \le \widetilde{B} + \varepsilon$ for $k \ge k^{[\varepsilon]}$, which contradicts the assumptions and proves the theorem.

THEOREM 4. *Let*

$$0 \le Q_* \le \tfrac{1}{4} \quad and \quad 0 \le H_* \le \tfrac{1}{4}. \tag{28}$$

Then a sufficient condition for (1) *to be oscillatory is that either*

$$Q^* > Q_* + \tfrac{1}{2}(\sqrt{1-4Q_*} + \sqrt{1-4H_*}), \tag{29}$$

or that

$$H^* > H_* + \tfrac{1}{2}(\sqrt{1-4Q_*} + \sqrt{1-4H_*}). \tag{30}$$

PROOF. Let u_k, $u_k u_{k+1} > 0$ for $k \ge m$, is some solution of equation (1). Obviously, (19) again holds. Let us assume that (30) is fulfilled. We shall proceed with $H_* > 0$, since for $H_* = 0$ condition (30) is equivalent to condition (24) from Theorem 3. By Lemmas 2 and 3, for arbitrary $\varepsilon \in (0, 1-\widetilde{B})$ there exists $k^{[\varepsilon]} > m$ such that

$$(k+1)w_k > \widetilde{A} - \varepsilon, \quad kw_k < \widetilde{B} + \varepsilon \quad \text{and} \quad \frac{w_k}{1+w_k} \le \varepsilon, \quad k \ge k^{[\varepsilon]}, \tag{31}$$

where $\widetilde{A}$ and $\widetilde{B}$ are defined by (27). Since $\widetilde{B} + \varepsilon < 1$, we have

$$kw_k(2-kw_k) + \varepsilon(kw_k - 1)^2 < (\widetilde{B}+\varepsilon)(2-\widetilde{B}-\varepsilon) + \varepsilon(\widetilde{B}+\varepsilon-1)^2 \tag{32}$$

for $k \ge k^{[\varepsilon]}$. Now, using the above inequalities, form (19) we obtain

$$H_k \le -\widetilde{A} + \varepsilon + \frac{k-k^{[\varepsilon]}}{k+1}\left[(\widetilde{B}+\varepsilon)(2-\widetilde{B}-\varepsilon) + \varepsilon(\widetilde{B}+\varepsilon-1)^2\right] + \widetilde{P}_{k,k^{[\varepsilon]}}$$

for $k \ge k^{[\varepsilon]}$, where $\widetilde{P}$ is defined by (23). Hence $H^* \le -\widetilde{A} + \widetilde{B}(2-\widetilde{B})$, which contradicts to (30).

Assuming that the condition (29) is fulfilled, then from (12) we have

$$Q_k \le \widetilde{B} + \varepsilon - (k+1)\sum_{j=k}^{\infty} R_j \le \widetilde{B} + \varepsilon - \frac{(\widetilde{A}-\varepsilon)^2}{1+\varepsilon}, \quad k \ge k^{[\varepsilon]},$$

where $k^{[\varepsilon]}$ is chosen sufficiently large and $\varepsilon > 0$ is arbitrary. Therefore, $Q^* \le \widetilde{B} - \widetilde{A}^2$, which contradicts (29).

When condition (28) holds, then Theorem 2 can be formulated in a more precise way.

THEOREM 5. *Let* (28) *holds and*

$$\limsup_{k\to\infty}\,(Q_k+H_k)>H_*+Q_*+\tfrac{1}{2}(\sqrt{1-4Q_*}+\sqrt{1-4H_*}).$$

Then equation (1) *is oscillatory.*

PROOF. The technique is the same as in the proof of Theorem 2, with only the difference that we use inequalities (31) and (32). This manner, from (19) we get

$$Q_k+H_k\le(\widetilde{B}+\varepsilon)(2-\widetilde{B}-\varepsilon)+\varepsilon(\widetilde{B}+\varepsilon-1)^2-\frac{(\widetilde{A}-\varepsilon)^2}{1+\varepsilon}.$$

Hence

$$\limsup_{k\to\infty}\,(Q_k+H_k)\le 2\widetilde{B}-\widetilde{B}^2-\widetilde{A}^2,$$

which contradicts the assumption.

3. NONOSCILLATION CRITERIA

In this section we give some nonoscillation criteria for equation (1). For their proofs we need the following lemma, which shows that for nonoscillation of (1) it is sufficient to find certain solution of Riccati difference inequality.

LEMMA 4 ([3]). *Equation* (1) *is nonoscillatory if and only if there exists a sequence* w_k, $k\ge m$, $m\in N$, *with* $1+w_k>0$ *such that*

(33) $$\Delta w_k+p_k+R_k\le 0,$$

where R_k *is defined by* (8).

Recall that we still assume that validity of (6)

THEOREM 6. *A sufficient condition for equation* (1) *to be nonoscillatory is that either*

$$Q_*>-\tfrac{3}{4}\quad\textit{and}\quad Q^*<\tfrac{1}{4},$$

or

$$H_*>-\tfrac{3}{4}\quad\textit{and}\quad H^*<\tfrac{1}{4}.$$

PROOF. It is sufficient to show that the Riccati inequality (33) has a solution w with $1+w_k>0$ in a neighbourhood of infinity.

First, we will deal with the sequence Q_k. Set

$$w_k = \frac{C}{k+1} + \sum_{j=k}^{\infty} p_j,$$

where C is a suitable constant, which will be specified later. Then the following equivalences hold for $k \geq K$, where $K \in N$ is sufficiently large. At the same time we choose K such that $1 + w_k > 0$ for $k \geq K$. We have

$$\Delta w_k < -p_k - R_k \Leftrightarrow -\frac{C}{(k+1)(k+2)} - p_k < -p_k - \frac{\left(\frac{C}{k+1} + \sum_{j=k}^{\infty} p_j\right)^2}{1 + \frac{C}{k+1} + \sum_{j=k}^{\infty} p_j}$$

$$\Leftrightarrow \frac{(Q_k + C)^2}{\left(1 + \frac{C}{k+1} + \sum_{j=k}^{\infty} p_j\right)(k+1)^2} < \frac{C}{(k+1)^2}$$

$$\Leftrightarrow (Q_k + C)^2 < C$$

$$\Leftrightarrow -\sqrt{C} - C < Q_k < \sqrt{C} - C$$

$$\Leftrightarrow -\tfrac{3}{4} < Q_k < \tfrac{1}{4},$$

i.e., $C = 1/4$. But the last estimate follows from the assumptions.

Concerning the sequence H_k, we have the similar situation as for Q_k. Set

$$w_k = \frac{D}{k} - \frac{1}{k(k+1)} \sum_{j=1}^{k-1} (j+1)^2 p_j = \frac{D}{k} - \frac{H_k}{k}.$$

Again, D will be specified later. Then for $k \geq K$, where K is sufficiently large, we have

$$\Delta w_k < -p_k - R_k \Leftrightarrow \frac{D}{k(k+1)} - \frac{(k+1)p_k}{k+2} + \frac{\sum_{j=1}^{k-1} (j+1)^2 p_j}{(k+1)^2(k+2)} + \frac{H_k}{k(k+1)} <$$

$$< -p_k - \left(\frac{D}{k} - \frac{H_k}{k}\right)^2 \left(\frac{D}{k} - \frac{H_k}{k} + 1\right)^{-1}$$

$$\Leftrightarrow -\frac{D}{k^2} + \frac{2H_k}{k^2} < -\frac{(D - H_k)^2}{k^2}$$

$$\Leftrightarrow H_k^2 + (2-2D)H_k + D^2 - D < 0$$

$$\Leftrightarrow -\sqrt{1-D} - (1-D) < H_k < \sqrt{1-D} - (1-D)$$

$$\Leftrightarrow -\tfrac{3}{4} < H_k < \tfrac{1}{4},$$

i.e., $D = 3/4$ and theorem is proved.

In the case when $Q_* > -3/4$ $(H_* > -3/4)$ does not hold, we can use the following criterion.

THEOREM 7. *A sufficient condition for* (1) *to be nonoscillatory is that either*

(34) $$-\infty < Q_* \le -\tfrac{3}{4} \quad \text{and} \quad Q^* < Q_* - 1 + \sqrt{1-4Q_*},$$

or

(35) $$-\infty < H_* \le -\tfrac{3}{4} \quad \text{and} \quad H^* < H_* - 1 + \sqrt{1-4H_*},$$

PROOF. Assume that (34) (or (35)) holds. Now, we proceed by the same way as in the continuous case, see [1]. Hence we get

$$-\sqrt{C} - C < Q_k < \sqrt{C} - C, \qquad k \ge K$$

$$(-\sqrt{1-D} - (1-D) < H_k < \sqrt{1-D} - (1-D), \qquad k \ge K)$$

for $K \in N$ sufficiently large, where

$$C = \left[\frac{1}{2}\left(1 + \sqrt{1 - 4(Q^* + \varepsilon)}\right)\right]^2 \qquad \left(D = 1 - \frac{1}{4}\left(1 + \sqrt{1 - 4(H^* + \varepsilon)}\right)^2\right)$$

with suitable $\varepsilon > 0$. The assertion now follows from the proof of the previous theorem.

4. REMARKS AND EXAMPLES

In this last section some comments and two examples are given.

The first example gives an application of Remark 1 (and also of Proposition 2) and at the same time shows that (3) does not imply (6).

EXAMPLE 1. *Let* $\gamma, \lambda \in \Re$ *are arbitrary. Set*

$$p_k = 2\Delta g_k + (k+1)\Delta^2 g_k,$$

where

$$g_k = -\gamma \frac{\sum_{j=1}^{k-1} \frac{1}{j}}{k} + \lambda \frac{(-1)^k - 1}{k}.$$

Then

$$\sum_{j=1}^{k-1} p_j = g_k + k\Delta g_k + \frac{\gamma}{2}$$

and

$$\limsup_{k\to\infty} \sum_{j=1}^{k} p_j = \frac{\gamma}{2} + 2|\gamma|, \qquad \liminf_{k\to\infty} \sum_{j=1}^{k} p_j = \frac{\gamma}{2} - 2|\lambda|.$$

But $c_k = \frac{1}{k}\sum_{j=1}^{k} p_j + g_{k+1} + \frac{\gamma}{2}$, and thus $\tilde{c} = \frac{\gamma}{2}$. Hence (3) $\not\Rightarrow$ (6) for $\lambda \neq 0$. Further, we proceed with $\lambda = 0$, since in the case $\lambda \neq 0$ equation (1) is oscillatory by Proposition 2. So, if we suppose $\lambda = 0$, then for $\gamma > \frac{1}{4}$ equation (1) is oscillatory by Remark 1.

The following example shows that for any pair of numbers (X, Y), where $X \leq Y$, there exists a sequence p_k such that (6) holds and $Q_* = X$, $Q^* = Y$ $(H_* = X,\ H^* = Y)$. Therefore Theorems 3, 4, 5 and 7 are meaningful.

EXAMPLE 2. *Let $X, Y \in \Re$, $X \leq Y$, are arbitrary. Set*

$$\alpha = \frac{X+Y}{2}, \qquad \beta = \frac{Y-X}{2}, \qquad p_k = \Delta h_k,$$

where

$$h_k = -\frac{\alpha}{k} + \beta\frac{(-1)^k}{k} + \frac{\alpha+\beta}{k^2}.$$

Then

$$\sum_{j=1}^{k-1} p_j = h_k, \qquad \lim_{k\to\infty} \sum_{j=1}^{k} p_j = 0.$$

Hence

$$Q_k = (k+1)(-h_k), \quad Q_* = \alpha - \beta = X, \quad Q^* = \alpha + \beta = Y$$

and

$$H_k = \frac{1}{k+1}[j^2 h_j]_1^k - \frac{1}{k+1}\sum_{j=1}^{k-1} (2j+1)h_j =$$

$$= \frac{k^2}{k+1} h_k - \frac{2}{k+1} \sum_{j=1}^{k-1} \left(-\alpha + \beta(-1)^j + \frac{\alpha+\beta}{j} \right) - \frac{1}{k+1} \sum_{j=1}^{k-1} h_j,$$

thus

$$H_* = \alpha - \beta = X, \quad H^* = \alpha + \beta = Y.$$

REMARK 3. Throughout this paper we investigate oscillation properties of the second order equation in the from (1). Instead of this one can consider a more general equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0. \tag{36}$$

But, if $r_k \neq 0$ in the interval under consideration, then the transformation

$$y_k = h_k u_k \text{ with } h_0 = 1, \quad h_{k+1} = \frac{1}{h_k r_k} \tag{37}$$

transforms (36) into the equation of the form (1). Indeed, by a direct computation one can verifty the identity

$$h_{k+1}[\Delta(r_k \Delta y_k) + p_k y_{k+1}] = \Delta(r_k h_k h_{k+1} \Delta u_k) + h_{k+1}[\Delta(r_k \Delta h_k) + p_k h_{k+1}] u_{k+1},$$

which yields the required transformation. The definition of a generalized zero for equation in the form (36) has the same form as for equation (1) (see Definition 1), only with the difference that instead of the inequality $y_m y_{m+1} \leq 0$ we require $r_m y_m y_{m+1} \leq 0$. The transformation (37) preserves oscillation properties since we have

$$u_k u_{k+1} = r_k h_k h_{k+1} u_k u_{k+1} = r_k y_k y_{k+1}.$$

Hence $u_k u_{k+1} > 0$ if and only if $r_k y_k y_{k+1} > 0$.

REMARK 4. In the last years, many papers appeared showing that basic oscillation properties of the so called half-linear differential equation

$$(\Phi(y'))' + p(t)\Phi(y) = 0,$$

where $\Phi(y) := |y|^{\alpha-1} \operatorname{sgn} y$ with $\alpha > 1$, are essentially the same as those of the linear equation (2). Especially, we have the generalization (in the half-linear sense) of the so called Roundabout theorem in mind. In the discrete case we have the similar situation. The results presented in [1] was also extended to half-linear differential equations, see [9]. Therefore we would like to find a generalization of the above results for equation

$$\Delta(\Phi(\Delta y_k)) + p_k\Phi(y_{k+1}) = 0 \tag{38}$$

and this problem is a subject of the present investigation. Note that the basic oscillatory properties (especially, the half-linear discrete version of the Roundabout theorem) of equation (38) are described in [11].

ACKNOWLEDGEMENT: *I would like to thank Professors O. Došlý and A. Lomtatidze for helpful comments on an earlier version of this paper.*

REFERENCES

[1] T. Chantladze, N. Kandelaki, A. Lomtatidze, Oscillation and nonoscillation criteria of a second order linear equation, *Georgian Math. J.* 6(1999), 401-414.

[2] S. Chen, Disconjugacy, disfocality and oscillation of second order difference equations, *J. Differ. Equations* 107(1994), 383-394.

[3] S. Chen, L.H. Erbe, Riccati techniques and discrete oscillations, *J. Math. Anal. Appl.* 142(1989), 468-487.

[4] S.S. Cheng, T.C. Yan, H.J. Li, Oscillation criteria for second order difference equations, *Funkcialaj Ekvacioj* 34(1991), 223-239.

[5] L.H. Erbe, B.G. Zhang, Oscillation of second order difference equations, *Chinese J. Math.* 16(1988), 239-252.

[6] P. Hartman, On non-oscillatory linear differential equation of second order, *Amer. J. Math.* 74(1952), 389-400.

[7] P. Hartman, *Ordinary differential equations*, J. Wiley, New York, 1963.

[8] J.W. Hooker, M.K. Kwong, W.T. Patula, Oscillatory second order linear difference equations and Riccati equations, *SIAM J. Math. Anal.* 18(1987), 54-63.

[9] N. Kandelaki, A. Lomtatidze, D. Ugulava, Oscillation and nonoscillation criteria of a second order half-linear equation, *Georgian Math. J.* 7(2000), 1-19.

[10] M.K. Kwong, J.W. Hooker, W.T. Patula, Riccati type transformations for second-order linear difference equations, II, *J. Math. Anal. Appl.* 107(1985), 182-196.

[11] P. Řehák, Oscillatory properties of second order half-linear difference equations, *to appear in Czech. Math. J.*

[12] A. Wintner, Acriterion of oscillatory stability, *Quart. Appl. Math.* 7(1949), 115-117.

Supported by the Grants No 201/98/0677 and No 201/99/0295 of the Czech Grant Agenty (Prague).

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Received on 1999.04.20 and, in revised from, on 2000.05.31.