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SOME REMARKS ON SPACES OF ALMOST PERIODIC FUNCTIONS

ABSTRACT: In this note we present remarks on completeness of spaces of almost periodic functions of various types and some theorems on connection between almost periodicity of the derivative and the indefinite integral of the function and almost periodicity of the function.

KEY WORDS: almost periodic function, complete space, derivative, indefinite inetragl.

1. PRELIMINARIES

1.1. A continuous function $f:\Re\to\Re$ is called uniformly almost periodic or Bohr almost periodic fynction (*B*-a.p.) iff for each $\varepsilon>0$ the set

$$E\{\varepsilon, f\}=\{\tau\in R:\sup\{|f(u)-f_\tau(u)|:u\in\Re\}\le\varepsilon\},$$

where $f_\tau(u)\equiv f(u+\tau)$, is relatively dense, i.e. there is a number $\ell>0$ such that

$$E\{\varepsilon; f\}\cap(\alpha,\alpha+\ell)\neq 0$$

for every $\alpha\in\Re$. For example, the function of the from

(1) $$f(u)=\sin u+\sin(\sqrt{2}u) \quad \text{for} \quad u\in\Re$$

is *B*-a.p. and *f* is not periodic.

Let $\widetilde{B}$ be the space of uniformly a.p. functions. It is known (see [3]), that every *B*-a.p. function is bounded and if a sequence (f_n) of *B*-a.p. functions is uniformly convergent to a function *f*, the *f* is *B*-a.p.

1.2. Let L^p_{loc}, where $p\ge1$, denote the space of functions $f:\Re\to\Re$ measurable in the sense of Lebesgue for which the *p*-th power of modulus is locally integrable.

A function $f\in L^p_{loc}$ is called Stepanov almost periodic $(S^p-\text{a.p.})$ iff for each $\varepsilon>0$ the set

$$E_p\{\varepsilon; f\}=\left\{\tau\in\Re: D_S p(f,f_\tau)=\sup\left\{\left(\int_t^{t+1}|f(s)-f_\tau(s)|^p\,ds\right)^{1/p}:t\in\Re\right\}\le\varepsilon\right\}$$

is relatively dense. For example the continuous function

$$f(u) = \sin\left(\frac{1}{2+\cos u+\cos(\sqrt{2}u)}\right) \quad \text{for} \quad u\in\Re$$

is S-a.p., i.e. f is S^1 – a.p., and f is not B-a.p.

Let $\widetilde{S}^p$ be the space of S^p-a.p. functions. Every S^p-a.p. function f is S^p-bounded, i.e. $D_S p(f) = D_S p(f,0) < \infty$. If a sequence (f_n), where $f_n \in S^p$ for $n=1,2,\ldots$, is S^p-convergent to $f \in L^p_{loc}$, i.e. $D_S p(f_n, f) \to 0$, then f is S^p-a.p. (see [3]).

We say that $f \in L^p_{loc}$ is S^p-continuous if for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h\in\Re$, $|h| < \delta$, we have $D_S p(f, f_h) \le \varepsilon$, where $f_h(x) \equiv f(x+h)$. For $p=1$ we obtain an S-continuous function.

1.3. Let $v(t; f)$ denote the Jordan variation of the function $f:\Re\to\Re$ on the interval $[t-1, t+1]$ for an arbitrary $t\in\Re$ and let

$$V(f) = \sup\{|f(t)| + V(t;f) : t\in\Re\}.$$

Let us put

$$X_0 = \{f:\Re\to\Re : f \text{ is continuous and } V(t;f)<\infty \text{ for every } t\in\Re\}.$$

A function $f\in X_0$ is called almost periodic in variation (V-a.p.) iff for each $\varepsilon > 0$ the set

$$E_v\{\varepsilon; f\} = \{\tau\in\Re : V(f - f_\tau) \le \varepsilon\}$$

is relatively dense. Although the Bohr's almost periodic function f given by formula (1) happens to be V-a.p., in general this implication fails to hold. To see that, write $f(u) = f_1(u) + f_2(u)$ for $u\in\Re$, where

$$f_1(u) = \begin{cases} (u-k)\sin(\pi/u-k) & \text{for} \quad u\in(k,k+1), \\ 0 \quad \text{for} \quad u=k, \end{cases} \qquad k = 0,\pm1,\pm2,\ldots,$$

$$f_2(u) = \sin(\sqrt{2}\pi u) \qquad \text{for} \quad u\in\Re.$$

Then f is the Bohr's a.p. function and f is not V-a.p.

Let $\widetilde{V}$ denote the spave V-a.p. functions. Every V-a.p. function f is V-bounded, i.e. $V(f) < \infty$. It is known (see [9]) that if a sequence (f_n) of V-a.p.

functions is V-convergent to a function f for which $V(t;f)<\infty$ for every $t\in\Re$, i.e. $V(f_n-f)\to 0$, then f is V-a.p.

We say that $f:\Re\to\Re$ is a V-continuous function if for an arbitrary $\varepsilon>0$ there exists $\delta>0$ such that for every $h\in\Re$, $|h|<\delta$, we have $V(f-f_n)\le\varepsilon$. If f is absolutely continuous, then f *is* V-continuous. A linear combination of two V-continuous and V-a.p. functions is V-a.p. (see [9]).

1.4. For $f:\Re\to\Re$ put

$$L_\alpha(t,\delta;f)=\sup_{\substack{u,v\in[t-\delta,t+\delta]\\ u\neq v}}\frac{|f(u)-f(v)|}{|u-v|^\alpha}\quad\text{for }\alpha\in(0,1],\quad t\in\Re,$$

where $\delta>0$. We say that $t_0\in\Re$ is an α-singular point of f if $L_\alpha(t_0,\delta;f)=+\infty$ for every $\delta>0$. Let us write

$$X^\alpha=\{f:\Re\to\Re:\text{there are no }\alpha\text{ - singular points of }f\}.$$

For $f\in X^\alpha$ put

$$L_\alpha(f)=\sup\{|f(t)|+\lim_{\delta\to 0}L_\alpha(t,\delta;f):t\in\Re\}.$$

A function $f:\Re\to\Re$ for which the function $L_\alpha(\cdot,1;f)$ is locally bounded is called almost periodic in the sense of Hölder (L_α-a.p.) iff for each $\varepsilon>0$ the set

$$E_\alpha\{\varepsilon;f\}=\{\tau\in\Re:L_\alpha(f-f_\tau)\le\varepsilon\}$$

is relatively sense. For $\alpha=1$ we obtain an almost periodic function in the sense of Lipschitz (L-a.p.). Although the Bohr's almost periodic function f given by formula (1) happens to be L-a.p., in general this implication fails to hold. To see that, write $f(u)=f_1(u)+f_2(u)$ for $u\in\Re$, where

$$f_1(u)=\begin{cases}\arcsin(u-4k) & \text{for}\quad u\in[4k-1,4k+1),\\ \arcsin(-u+4k+2) & \text{for}\quad u\in[4k+1,4k+3),\end{cases}$$

$$f_2(u)=\begin{cases}\arcsin(\sqrt{2}u-4k) & \text{for}\quad u\in[(4k-1)/\sqrt{2},(4k+1)/\sqrt{2}),\\ \arcsin(-\sqrt{2}u+4k+2) & \text{for}\quad u\in[(4k+1)/\sqrt{2},(4k+3)/\sqrt{2}),\end{cases}$$

$k=0,\pm1,\pm2,\ldots$. The sum f_1+f_2 is V-a.p. and f is not L-a.p. (see [11]).

Because for every $t\in\Re$ we have $L_\alpha(t,\delta;f)\le L_\alpha(t,1;f)$ for $0<\delta\le 1$, so for $f\in X^\alpha$ we obtain

$$L_\alpha(f)\le L_\alpha^0(f)=\sup\{|f(t)|+L_\alpha(t,1;f):t\in\Re\}.$$

A function $f:\mathfrak{R}\to\mathfrak{R}$ for which the function $L_\alpha(\cdot,1;f)$ is locally bounded is called L^0_α-almost periodic (L^0_α-a.p.) iff for each $\varepsilon>0$ the set

$$E^0_\alpha\{\varepsilon;f\}=\{\tau\in\mathfrak{R}:\ L^0_\alpha(f-f_\tau)\le\varepsilon\}$$

is relatively dense.

Let $\widetilde{L}^0_\alpha$ be the set of L^0_α-a.p. functions. Every L^0_α-a.p. function *f* is L_α-a.p. and *f* is L^0_α-bounded, i.e. $L^0_\alpha(f)<\infty$. If for a sequence (f_n) of L^0_α-a.p. functions we have

$$L^0_\alpha(f_n-f)\to 0 \text{ for } n\to\infty,$$

where the function $L_\alpha(\cdot,1;f)$ is locally bounded, then *f* is L^0_α-a.p. (see [16]).

1.5. Let F_Δ denote the class of closed and bounded with respect to the *y* axis point sets on the $0xy$ plane, such that their respective projections on the *x* axis are identical with the interval Δ (finite or infinite), and such that the intersection of every straight line $x=x_0$, $x_0\in\Delta$, and $F\in F_\Delta$ is a closed interval. If $A,B\in F_\Delta$, then the Hausdorff distance between *A* and *B* is defined as the following number

$$r_\Delta(A,B) = \max\left(\sup_{X\in A}\inf_{Y\in B}\|X-Y\|_0,\ \sup_{X\in B}\inf_{Y\in A}\|X-Y\|_0\right),$$

where

$$\|X-Y\|_0=\|X(x_1,y_1)-Y(x_2,y_2)\|_0=\max(|x_1-x_2|,|y_1-y_2|).$$

LEMMA. *Suppose that* $A,B\in F_\Delta$. *In order that* $r(A,B)\le\delta$ *it is necessary and sufficient that:*

(a) *for an arbitrary* $X\in A$ *there exists* $Y\in B$ *such that* $\|X-T\|_0\le\delta$
and
(b) *for an arbitrary* $X\in B$ *there exists* $Y\in A$ *such that* $\|X-Y\|_0\le\delta$.

Proof. of Lemma can be found in [4].

By the complete graph $\bar{f}$ of the function *f*, defined and bounded in the interval Δ and taking real values, we call a set of points (x,y), for which $x\in\Delta$, $I_f(x)\le y\le S_f(x)$, where I_f, S_f is the lower and upper Baire's function respectively, both with respect to the function *f*:

$$I_f(x)=\lim_{\delta\to 0}\inf_{|x'-x|\le\delta} f(x'),\qquad S_f(x)=\lim_{\delta\to 0}\sup_{|x'-x|\le\delta} f(x').$$

The Hausdorff distance between two functions f and g, both defined and bounded on Δ, is defined in the following way: it is the Hausdorff distance between their respective complete graphs, i.e. $r_\Delta(f,g)=r_\Delta(\bar{f},\bar{g})$. Let $r_{\mathfrak{R}}=r$.

A bounded function $f:\mathfrak{R}\to\mathfrak{R}$ is called H-almost periodic (H-a.p.) iff for each $\varepsilon>0$ the set

$$E_H\{\varepsilon;f\}=\{\tau\in\mathfrak{R}:r(f,f_\tau)\le\varepsilon\}$$

is relatively dense.

Let $\widetilde{H}$ be the set of H-a.p. functions. Examples, elementary properties of H-a.p. functions and the connection between H-a.p. functions and almost periodic functions of other types can be found in [2] and [5] – [8]. In particular, if a sequence (f_n), where $f_n\in\widetilde{H}$ for $n=1,2,\ldots$, is H-convergent to a function f which is defined and bounded on $\mathfrak{R}$, i.e. $r(f_n,f)\to 0$, then f is H-a.p.

Let f be a bounded function defined on the interval Δ. The following equality

$$\mu_f(\delta)=\sup\left\{\sup_{x_1\le x\le x_2}\left[|f(x_1)-f(x)|+|f_2(x)-f(x)|\right]-\right.$$
$$\left.-|f(x_1)-f(x_2)|:|x_1-x_2|\le\delta,\quad x_1,x_2\in\Delta\right\}$$

defines the modulus of non-monotonicity of f. Let us write

$$H_\mu=\{f:\mathfrak{R}\to\mathfrak{R}:f\text{ is }H\text{ - a.p. such tha }\mu_f(\delta)\le K\delta^\gamma,K\ge 0,\gamma>0\}.$$

1.6. Let Σ be the σ-algebra of subsets of the space $\mathfrak{R}^1$ which are measurable in the sense of Lebesgue, μ the Lebesgue measure, Ψ the space of Σ-measurable and finite functions $f:\mathfrak{R}^1\to\mathfrak{R}$, where $f=g\Leftrightarrow f(u)=g(u)$ μ-almost everywhere in $\mathfrak{R}^1$.

Let us write

$$\Psi_0=\{f\in\Psi:\text{for every sequence }(\lambda_n)\text{ such that }\lambda_n>0,$$
$$\lambda_n\to 0\text{ we have }\sup_{t\in\mathfrak{R}}\mu(\{u\in[t,t+1]:\lambda_n|f(u)|\ge 1\})\to 0\}.$$

For $f\in\Psi$ we define the functional

(2) $$|f|=\sup_{t\in\mathfrak{R}}\int_t^{t+1}\frac{|f(s)|}{1+|f(s)|}ds.$$

For $\eta > 0$ let us put

$$D(\eta; f, g) = \sup_{t\in\Re} \mu(u \in [t, t+1] : | f(u) - g(u) | \geq \eta\}) \text{ for } f, g \in \Psi.$$

It is known (see [10]) that for f_n, $f \in \Psi$, $n = 1,2,\ldots$, we have

$$(| f_n - f | \to 0\} \Leftrightarrow \forall_{\eta>0} \forall_{\varepsilon>0} \exists_{N>0} \forall_{n>N} D(\eta, f_n, f) \leq \varepsilon.$$

The functional $| \, | : \Psi_0 \to [0,1]$ is an F-norm. Let us denote by S_0 the space Ψ_0 in which the F-norm $| \, |$ of the form (2) is defined. Hence $S_0 = < \Psi_0, | \, | >$ is an F^*-space.

A function $f \in \Psi$ is called almost periodic in the Lebesgue measure μ (μ-a.p.) iff for any two positive numbers ε, η the set

$$E\{\varepsilon, \eta; f\} = \{\tau \in \Re : D(\eta; f, f_\tau) \leq \varepsilon\}$$

is relatively dense.

Let $\widetilde{M}$ be the set of μ-a.p. functions. It is known (see [10]) that $\widetilde{M} \not\subset S_0$. If for every $\eta > 0$ we have $D(\eta; f_n, f) \to 0$, where $f_n \in \widetilde{M}$ for $n = 1,2,\ldots$ and $f \in \Psi$, then is μ-a.p. Example and elementary properties of μ-a.p. functions can be found in [10].

1.7. A function $f \in L^p_{loc}$, where $p \geq 1$, is called Weyl almost periodic (W^p-a.p.) iff for each $\varepsilon > 0$ there exists a number $d = d(\varepsilon) > 0$ such that the set

$$E_{pd}\{\varepsilon; f\} = \left\{ \tau \in \Re : D_S p_d(f, f_\tau) = \right.$$

$$\left. = \sup \left\{ \left(\frac{1}{d} \int_t^{t+d} | f(s) - f_\tau(s) |^p \, ds \right)^{1/p} : t \in \Re \right\} \leq \varepsilon \right\}$$

is relatively dense.

Let $\widetilde{W}^p$ be the set of W^p-a.p. functions. Elementary properties of W^p-a.p. functions can be found in [3]. In particular, if $\overline{\lim}_{\varepsilon\to 0} d(\varepsilon) < \infty$, then a W^p-a.p. function is S^p-a.p. Every W^p-a.p. function f is W^p-bounded, i.e. $D_W p(f) = D_W p(f, 0) < \infty$, where $D_W p(f, g) = \lim_{d\to\infty} D_S p_d(f, g)$. A sequence (f_n), where $f_n \in L^p_{loc}$ for $n = 1,2,\ldots$, is called W^p-convergent to $f_n \in L^p_{loc}$ if for an arbitrary $\varepsilon > 0$ there exists $N > 0$ such that for $n > N$ we have $D_W p(f_n, f) \leq \varepsilon$.

1.8. Let $C^{(n)}(\mathfrak{R})$ be the set of functions $f:\mathfrak{R}\to\mathfrak{R}$ which have first n derivatives continuous on $\mathfrak{R}$. Let us put for $f\in C^{(n)}(\mathfrak{R})$

$$D^{(n)}(f) = \sup_{t\in\mathfrak{R}}\left(|f(t)|+\sum_{k=1}^{n}|f^{(k)}(t)|\right).$$

A function $f\in C^{(n)}(\mathfrak{R})$ is called $C^{(n)}$-almost periodic ($C^{(n)}$-a.p.) iff for each $\varepsilon>0$ the set

$$E^{(n)}\{\varepsilon;f\}=\{\tau\in\mathfrak{R}:D^{(n)}(f-f_\tau)\leq\varepsilon\}$$

is relatively dense (see [1]).

2. Completeness of almost periodic functions spaces

Every nonempty and closed subset A of the complete metric space $<X,\rho>$ is the complete space.

Because $\widetilde{B}\not\subset BC(\mathfrak{R})$ is a closed subspace of the space of continuous and bounded functions on $\mathfrak{R}$ and the metric space $<BC(\mathfrak{R}),|\,|>$ is complete, so the space $<\widetilde{B},|\,|>$ of uniformly a.p. functions is complete, where $|f-g|=\sup_{t\in\mathfrak{R}}|f(t)-g(t)|$ for $f,g\in BC(\mathfrak{R})$.

It is known (see [3]) that the metric space $S^p=<L^p_{loc},D_Sp>$ is complete. Therefore the space $<\widetilde{S}^p,D_Sp>$ of S^p-a.p. functions is complete, as the subset $\widetilde{S}^p$ of the S^p is closed.

Let us put $X_{BV}=\{f\in X_0:V(f)<\infty\}$. In [14] it has been shown that the metric space $<X_{BV},\rho_v>$, where $\rho_v(f,g)=V(f-g)$, is complete. Because the subset $\widetilde{V}\not\subset X_{BV}$ is closed, so the space $<\widetilde{V},\rho_v>$ of V-a.p. functions is complete.

Let us write $BX_0^\alpha=\{f:\mathfrak{R}\to\mathfrak{R}:L^0_\alpha(f)<\infty\}$. The metric space $<BX_0^\alpha,\rho^0_\alpha>$, where $\rho^0_\alpha(f,g)=L^0_\alpha(f-g)$, is complete. The proof of this theorem can be found in [16]. Therefore the closed space $<\widetilde{L}^0_\alpha,\rho^0_\alpha>$ of L^0_α-a.p. functions is complete.

Because the metric space $<F_\mathfrak{R},r>$ is complete (see [6]), so the space $<\widetilde{H},r>$ of H-a.p. functions is complete, as the subset $\widetilde{H}\not\subset F_\mathfrak{R}$ is closed.

The space S_0 is complete (see [10]), and so the closed metric space $<\widetilde{M},|\,|>$ of μ-a.p. functions is complete.

The example of the sequence of uniformly a.p. functions, which satisfies the W^p-Cauchy condition and is not W^p-convergent, is contained in [3], p. 244. Hence the space $<\widetilde{W}^p, D_{W^p}>$ is not complete.

3. A connection between almost periodicity of the derivative and the indefinite integral of the function and almost periodicity of this function.

In [3] are contained sufficient conditions for almost periodicity of the derivative and the indefinite integral of *B*-a.p. and *S*-a.p. functions. In the following we prove the inverse to the above theorem in cases: Bohr a.p. functions, Stepanov a.p. functions and functions a.p. in the sense of the Hausdorff metric.

THEOREM 1. *Let us assume that f is a bounded and differentiable function on* $\Re$. *Then:*

(a) *If the derivative* f' *is B-a.p. function, then f is* $C^{(1)}$*-a.p.*

(b) *If the derivative* f' *is continuous and S-a.p., then f is V-a.p.*

(c) *If the derivative* f' *is continuous and* $f' \in H_\mu$, *then f is V-a.p.*

PROOF. It is known (see [1]) that the bounded indefinite integral of a *B*-a.p. function is a $C^{(1)}$-a.p. function. Since

$$g(x) = \int_0^x f'(u)\,du \quad \text{for every} \quad x \in \Re,$$

where $g = f - f(0)$, so *g* is $C^{(1)}$-a.p. Hence it follows (a).

Because the indefinite integral of a continuous and *S*-a.p. derivative f' is bounded, so, by [12], we obtain that *f* is *V*-a.p. Therefore the part (b) is true.

If the indefinite integral of a function $f' \in H_\mu$, which is locally integrable, is bounded, then this integral is *V*-a.p. (see [15]). Hence it follows (c).

Let us put

$$F(x) = \int_0^x f(u)\,du \quad \text{for} \quad x \in \Re,$$

where the function $f : \Re \to \Re$ is measurable in the sense of Lebesgue and locally integrable, and for an arbitrary sequence (h_n), where $h_n \neq 0$, $h_n \to 0$, let us write

$$G_n(x) = \frac{F(x+h_n)-f(x)}{h_n} \quad \text{for } x \in \Re.$$

THEOREM 2. *If f is a uniformly continuous function on* $\Re$ *and the indefinite integral F is B-a.p., then f is a B-a.p. function.*

PROOF. Because f is uniformly continuous, so we have

$$\sup_{x\in\Re} |G_n(x)-f(x)| \le \sup_{x\in R}\frac{1}{h_n}\int_0^{h_n} |f(u+x)-f(x)|du \to 0 \quad \text{as} \quad n\to\infty.$$

Hence the sequence $(G_n(x))$, $x \in R$, of B-a.p. functions tends to f uniformly with respect to $x \in R$, and so f is B-a.p.

The inverse theorem to the above is not true. It is known (see [3]) that the indefinite integral F of a B-a.p. function f is B-a.p. iff F is bounded.

THEOREM 3. *If a function f is S-continuous and the indefinite integral F is S-a.p., then f is an S-a.p. function.*

PROOF. For an arbitrary $t \in \Re$ we obtain

$$\int_t^{t+1} |G_n(x)-f(x)|\,dx \le \frac{1}{h_n}\int_0^{h_n}\left(\int_t^{t+1} |f(u+x)-f(x)|\,dx\right)du.$$

Because f is S-continuous, so for an arbitrary $\varepsilon > 0$ there exists $N > 0$ such that $|h_n| \le \delta = \delta(\varepsilon)$ for $n > N$ and for this n we have

$$\sup_{t\in\Re}\int_t^{t+1} |G_n(x)-f(x)|\,dx \le \varepsilon.$$

Hence the sequence (G_n) of S-a.p. functions is S-convergent to f, i.e. f is S-a.p.

THEOREM 4. *If a function f is continuous and bounded on* $\Re$, *the indefinite integral F is H-a.p., then f is an H-a.p. function.*

PROOF. Since for $n = 1,2,\ldots$ we have $r(G_n, f) = r(f_{\vartheta h_n}, f)$, where $f_{\vartheta h_n}(x) = f(x+\vartheta h_n)$, $|\vartheta| \le 1$, so, by Lemma, we obtain $r(G_n, f) \le |h_n| \to 0$ as $n \to \infty$. It is known (see [13]), that G_n, which is a linear combination of continuous H-a.p. functions, is H-a.p. Therefore the sequence (G_n) is H-convergent to f, and so f is H-a.p.

THEOREM 5. *If $f \in X_0$ is V-continuous and the indefinite integral F is V-a.p., then f is a V-a.p. function.*

PROOF. Since F is absolutely continuous, so F is V-continuous (see [9]). Hence G_n, $n = 1,2,\ldots$, are V-a.p. functions. Mereover, because f is V-continuous, we obtain

$$V(G_n - f) \le \sup_{t\in\Re} |G_n(t) - f(t)| + \sup_{t\in\Re} V(t; G_n - f),$$

$$\sup_{t\in\Re} |G_n(t) - f(t)| \le \sup_{t\in\Re} \frac{1}{h_n} \int_0^{h_n} |f(u+t) - f(t)|\, du \to 0,$$

$$\sup_{t\in\Re} V(t; G_n - f) \le \sup_{t\in\Re} \frac{1}{h_n} \int_0^{h_n} V(t; f_n - f)\, du \to 0$$

as $n \to \infty$, and so the sequence (G_n) of V-a.p. functions is V-convergent to f, i.e. f is V-a.p.

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