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GREEN FUNCTIONS AND PERIODIC SOLUTION TO THE *m*-PARABOLIC EQUATION WITH BOUNDARY-VALUE CONDITIONS OF RIQUIER TYPE

ABSTRACT: The subject of the paper is the construction of the periodic solutions to the m-caloric equation $P^m u(x,t)=0$, $P=D_x^2-D_t$, $P^2=P(P)$, $P^m=P(P^{m-1})$ in the strip $D=\{(x,t): x\in(0,1),\, t\in(-\infty,\infty)\}$, satisfying the periodic boundary-value conditions $P^i u(0,t)=h_{i+1,1}(t)$, $P^i u(1,t)=h_{i+1,2}(t)$, $i=0,1,\ldots,m-1$, where $h_{i,1}(t)$, $h_{i,2}(t)$ are the periodic functions with the period $p>0$.

KEY WORDS: Green functions, Green potentials, Riquier boundary-value conditions.

1. INTRODUCTION

The subject of the paper is the construction of the periodic solutions for the m-parabolic equation

(1) $$P^m u(x,t) = 0, \qquad (x,t)\in D,$$

in the unbounded strip D, satisfying the following conditions

(2) $$u(0,t)=h_{1,1}(t), \quad u(1,t)=h_{1,2}(t), \quad t\in(-\infty,\infty),$$

(3) $$Pu(0,t)=h_{2,1}(t), \quad Pu(1,t)=h_{2,2}(t), \quad t\in(-\infty,\infty),$$

(4) $$P^2u(0,t)=h_{3,1}(t), \quad P^2u(1,t)=h_{3,2}(t), \quad t\in(-\infty,\infty),$$

$$\ldots$$

(5) $$P^{m-1}u(0,t)=h_{m,1}(t), \quad P^{m-1}u(1,t)=h_{m,2}(t), \quad t\in(-\infty,\infty),$$

where $h_{i,1}(t)$, $h_{i,2}(t)$, $i=1,\ldots,m$ are continuous, bounded and periodic functions with the period p.

The function

(6) $$u(x,t) = \sum_{i=0}^{m-1} C_i P^i w_{i+1}(x,t), \qquad (x,t)\in D,$$

where $w_i=u_{i,1}+u_{i,2}$, $i=1,\ldots,m$ are the suitable Green potentials given by formulas (12), $(12)_1-(15)$, $(15)_1$ C_i, $i=0,\ldots,m-1$ are arbitrary constants, is a periodic solution to the problem (1) – (6).

2. Green functions

To solve the last problem we shall apply the suitable Green functions

$$G(x,t,y,s), \quad G_i(x,t,y,s), \quad i=1,...,m-1.$$

Let us consider the sequences

$$x_{i,0}=x, \quad i=1,2, \quad x_{1,2n}=x+2n, \quad x_{2,2n}=x-2n, \quad n=1,2,...$$

$$x_{1,2n+1}=-x-2n, \quad x_{2,2n+1}=-x+2n+2, \quad n=0,1,...,$$

the functions

$$U(x,t,y,s)=U(x-y,t-s)=A(t-s)^{-1/2}\exp(B(t,s)(x-y)^2),$$

$$A=(2\sqrt{\pi})^{-1}, \qquad B(t,s)=B(t-s)=(-4(t-s))^{-1}$$

and the function

(7) $$G(x,t,y,s)=G(x-y,t-s)=U(x,t,y,s)+H(x,t,y,s),$$

with

(8) $$H(x,t,y,s)=-K_1(x,t,y,s)-K_3(x,t,y,s)+K_2(x,t,y,s)+K_4(x,t,y,s),$$

with

(9) $$K_1(x,t,y,s)=A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(x-2n-2-y)^2),$$

(10) $$K_3(x,t,y,s)=A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(-x+2n-y)^2),$$

(11) $$K_2(x,t,y,s)=A(t-s)^{-1/2}\sum_{n=1}^{\infty}\exp(B(t,s)(x+2n-y)^2),$$

$(11)_1$ $$K_4(x,t,y,s)=A(t-s)^{-1/2}\sum_{n=1}^{\infty}\exp(B(t,s)(x-2n-y)^2).$$

$$(x,t,y,s)\in D_1=\{(x,t,y,s),\ (x,y)\in J\times J,\ -\infty<s<t,\ x\neq y\}.$$

3. Green function G to the equation $Pu=0$ and Dirichlet boundary-value conditions

In the sequel by C, C_i, $i=1,2,...$ we shall denote the suitable positive constants. By [2] vol. I p. 476 the function

$$G=U-K_1-K_3+K_2+K_4$$

is the Green function to the equation

$$PG(x,t,y,s)=0,\quad (x,t,y,s)\in D_1,$$

to the domain D_1, satisfying the Dirichlet boundary-value conditions

$$G(0,t,y,s)=G(1,t,y,s)=G(x,t,0,s)=G(x,t,1,s)=0,\quad (x,t,y,s)\in D_1.$$

Next we shall calculate

$$D_yG(x,t,0,s),\qquad D_yG(x,t,1,s).$$

By [4] we obtain the formulas

$$D_yG(x,t,0,s)=Q_1(x,t,s)+Q_2(x,t,s)+Q_3(x,t,s),$$

with

$$Q_1(x,t,s)=A(t-s)^{-3/2}\,x\exp\left(-\frac{x^2}{4(t-s)}\right),$$

$$\text{(I)}\qquad Q_2(x,t,s)=-A(t-s)^{-3/2}\sum_{n=0}^{\infty}(x-2n-2)\exp(B(t-s)(x-2n-2)^2),$$

$$Q_5(x,t,s)=-A(t-s)^{-3/2}\sum_{n=0}^{\infty}(-x+2n+2)\exp(B(t-s)(-x+2n+2)^2).$$

Similarly we obtain the formula

$$D_yG(x,t,1,s)=R_1(x,t,s)+R_2(x,t,s)+R_3(x,t,s),$$

with

$$R_1(x,t,s)=-A(t-s)^{-3/2}(x-1)\exp\left(-\frac{(x-1)^2}{4(t-s)}\right),$$

$$\text{(II)}\qquad R_2(x,t,s)=A(t-s)^{-3/2}\sum_{n=0}^{\infty}(x+2n+1)\exp(B(t-s)(x+2n+1)^2),$$

$$R_3(x,t,s)=A(t-s)^{-3/2}\sum_{n=0}^{\infty}(x-2n-1)\exp(B(t-s)(x-2n-1)^2),$$

4. GREEN POTENTIALS $J_1(x,t)$, $J_2(x,t)$

Next let us consider the Green potentials

$$J_1(x,t)=\int_{-\infty}^{t}(D_yG(x,t,0,s))h_{1,1}(s)ds=J_1^1(x,t)+J_1^2(x,t)+J_1^3(x,t),$$

with

$$J_1^1(x,t) = \int_{-\infty}^{t} Q_1(x,t,s)h_{1,1}(s)ds$$

$$J_1^i(x,t) = \int_{-\infty}^{t} Q_i(x,t,s)h_{1,1}(s)ds, \quad i=2,3.$$

$$J_2(x,t) = \int_{-\infty}^{t} (D_y G(x,t,1,s))h_{1,2}(s)ds = J_2^1(x,t) + J_2^2(x,t) + J_2^3(x,t),$$

with

$$J_2^1(x,t) = \int_{-\infty}^{t} (R_i(x,t,s))h_{1,2}(s)ds,$$

$$J_2(x,t) = \int_{-\infty}^{t} (D_y G(x,t,1,s))h_{1,2}(s)ds = J_2^1(x,t) + J_2^2(x,t) + J_2^3(x,t),$$

with

$$J_2^1(x,t) = \int_{-\infty}^{t} (R_1(x,t,s))h_{1,2}(s)ds,$$

$$J_2^i(x,t) = \int_{-\infty}^{t} R_i(x,t,s)h_{1,2}(s)ds, \quad i=2,3.$$

Applying the foregoing formulas (I), (II) and the formulas for $J_1(x,t)$, $J_2(x,t)$ we shall give its properties used in the sequel.

Lemma 1. *If the functions* $h_{1,1}(t)$, $h_{1,2}(t)$ *are continuous and bounded for* $t \in (-\infty,\infty)$ *then:* (a) $PJ_i^j(x,t) = 0$, $i=1,2$, $j=1,2,3$

1^0 $J_1^1(x,t) \to h_{1,1}(t)$ as $(x,t) \to (0,t)$, $t \in (-\infty,\infty)$,

2^0 $J_1^2(x,t) + J_1^3(x,t) \to 0$ as $(a,t) \to (0,t)$, $t \in (-\infty,\infty)$,

3^0 $J_1(x,t) \to 0$, as $(x,t) \to (0,t)$, $t \in (-\infty,\infty)$,

4^0 $J_2^1(x,t) \to h_{1,2}(t)$ as $(x,t) \to (1,t)$, $t \in (-\infty,\infty)$,

5^0 $J_2^2(x,t) + J_2^3 \to 0$ as $(x,t) \to (1,t)$, $t \in (-\infty,\infty)$,

6^0 $J_2(x,t) \to 0$ as $(x,t) \to (0,t)$, $t \in (-\infty,\infty)$.

Proof. By the properties of the potentials of the double layer, [3] vol. I. p. 482 and by formulas (I), (II) we obtain the assertions (a) – 6^0 of the Lemma 1.

Next les us consider the functions $G_1,\dots,G_{m-1}$.

$$G_1(x,t,y,s) = (t-s)G(x,t,y,s), \quad (x,t,y,s) \in D_1,$$

$$G_2(x,t,y,s) = \frac{(t-s)^2}{2!}G(x,t,y,s), \quad (x,t,y,s) \in D_1,$$

$$G_i(x,t,y,s) = \frac{(t-s)^i}{i!}G(x,t,y,s), \quad (x,t,y,s) \in D_1,$$

$$\dots$$

$$G_{m-1}(x,t,y,s) = \frac{(t-s)^{m-1}}{(m-1)!}G(x,t,y,s), \quad (x,t,y,s) \in D_1,$$

LEMMA 2. *The functions* $G_i,\ i=1,\dots,m-1$ *satisfy the equations* $P^iG_{i-1}=0$ $i=2,\dots,m$ *respectively.*

PROOF. For the function G_1 we have

$$P^2G_1(x,t,y,s) = P(P((t-s)G(x,t,y,s))) = P(D_x^2 - D_t)((t-s)G(s,t,y,s)) =$$
$$= P((t-s)PG(x,t,y,s) - G(x,t,y,s)) = -PG(x,t,y,s) = 0,$$
$$(x,t,y,s) \in D_1.$$

For the function $G_2(x,t,y,s)$ we have

$$P^3G_2(x,t,y,s) = P^2\left(P\left(\frac{(t-s)^2}{2}G(x,t,y,s)\right)\right) = P^2((D_x^2 - D_t)G_2(s,t,y,s)) =$$

$$= P^2\left(\frac{(t-s)^2}{2}PG(x,t,y,s) - (t-s)G(x,t,y,s)\right) = -P^2((t-s)G(x,t,y,s)) = 0.$$

Let us assume the implication

(III) $$P^{m-1}G_{m-2}(x,t,y,s) = 0, \quad (x,t,y,s) \in D_1,$$

then we have

(IV) $$P^mG_{m-1}(x,t,y,s) = 0, \quad (x,t,y,s) \in D_1.$$

Indeed. We have

$$P^mG_{m-1}(x,t,y,s) = P^{m-1}(PG_{m-1}(x,t,y,s)) =$$
$$= P^{m-1}(P(t-s)G_{m-2}(x,t,y,s)) =$$
$$= P^{m-1}(-G_{m-2}(x,t,y,s)) = -P^{m-1}G_{m-2}(x,t,y,s) = 0.$$

By Lemma 2 and by properties of the function G we obtain.

LEMMA 3. *The functions* G_i, $i=1,\dots,m-1$ *satisfy the conditions*

1^0 $P^i G_1(0,t,y,s) = P^i G_1(1,t,y,s) = 0, \quad i=0,1,$

2^0 $P^i G_2(0,t,y,s) = P^i G_2(1,t,y,s) = 0, \quad i=0,1,2,\dots$

3^0 $P^i G_{m-1}(0,t,y,s) = P^i G_{m-1}(1,t,y,s) = 0, \quad i=0,1,2,\dots,m-2.$

5. GREEN POTENTIALS

Let us consider the Green potentials

$$u_{1,1}(x,t) = \int_{-\infty}^{t} h_{1,1}(s) D_y G(x,t,0,s)ds, \tag{12}$$

$$u_{1,2}(x,t) = \int_{-\infty}^{t} h_{1,2}(s) D_y G(x,t,1,s)ds, \tag{12$_1$}$$

$$u_{2,1}(x,t) = \int_{-\infty}^{t}\int_{0}^{1} G_1(x,t,y,s)\left(\int_{-\infty}^{s} h_{2,1}(s_1) D_y G(y,s,0,s_1)ds_1\right)dyds \tag{13}$$

$$u_{2,2}(x,t) = \int_{-\infty}^{t}\int_{0}^{1} G_1(x,t,y,s)\left(\int_{-\infty}^{s} h_{2,2}(s_1) D_y G(y,s,1,s_1)ds_1\right)dyds, \tag{13$_1$}$$

$$u_{3,1}(x,t)) = \int_{-\infty}^{t}\int_{0}^{1} (G_2(x,t,y,s)\left(\int_{-\infty}^{s} h_{3,1}(s_1) D_y G(y,s,0,s_1)ds_1\right)dyds \tag{14}$$

$$u_{3,2}(x,t) = \int_{-\infty}^{t}\int_{0}^{1} G_2(x,t,y,s)\left(\int_{-\infty}^{s} h_{3,2}(s_1) D_y G(y,s,1,s_1)ds_1\right)dyds, \tag{14$_1$}$$

...

$$u_{m,1}(x,t) = \int_{-\infty}^{t}\int_{0}^{1} G_{m-1}(x,t,y,s)\left(\int_{-\infty}^{s} h_{m,1}(s_1) D_y G(y,s,0,s_1)ds_1\right)dyds, \tag{15}$$

$$u_{m,2}(x,t) = \int_{-\infty}^{t}\int_{0}^{1} G_{m-1}(x,t,y,s)\left(\int_{-\infty}^{s} h_{m,2}(s_1) D_y G(y,s,1,s_1)ds_1\right)dyds. \tag{15$_1$}$$

6. PROPERTIES OF THE POTENTIALS $u_{i,1}$, $u_{i,2}$, $i = 1,\dots,m$

Let (K_p) denote the class of all functions $h(t)$ continuous, bounded and periodic with the period $p>0$ for $t \in (-\infty,\infty)$ and let

$$w_i(x,t) = u_{i,1}(x,t) + u_{1,2}(x,t), \quad i = 1,\dots,m.$$

By Lemma 2 and by [3] we obtain

LEMMA 4. *If* $h_{1,1}, h_{1,2} \in (K_p)$ *then*

1^0 $Pu_{1,1}(x,t) \to 0, \quad Pu_{1,2}(x,t) = 0, \quad (x,t) \in D,$

2^0 $u_{1,1}(x,t) \to 0, \quad \text{as } (x,t) \to (1,t), \quad t \in (-\infty,\infty),$

3^0 $u_{1,1}(x,t) \to h_{1,1}(t), \quad \text{as } (x,t) \to (0,t), \quad t \in (-\infty,\infty),$

4^0 $u_{1,2}(x,t) \to 0, \quad \text{as } (x,t) \to (0,t), \quad t \in (-\infty,\infty),$

5^0 $u_{1,2}(x,t) \to h_{1,2}(t), \quad \text{as } (x,t) \to (1,t), \quad t \in (-\infty,\infty),$

6^0 $u_{1,i}$, $i = 1,2$ *are periodic functions with the period* p.

PROOF. By Lemma 2 and by [3] we obtain $1^0 - 5^0$.

Ad. 6^0 We shall give the proof anly for $u_{1,1}$ because for $u_{1,2}$ the proof is similar. We have

$$u_{1,1}(x,t+p) = \int_{-\infty}^{t+p} h_{1,1}(s) D_y G(x-0, t+p-s) ds.$$

Applying in the last integral the change of the integral variable

$$s = p + z, \qquad ds = dz, \qquad z \in (-\infty,\infty),$$

we obtain

$$u_1(x, z+p) = \int_{-\infty}^{t} h_{1,1}(p+z) D_y G(x-0, t-z) dz =$$

$$= \int_{-\infty}^{t} h_{1,1}(z) D_y G(x-0, t-z) dz = u_{1,1}(x,t).$$

By the last Lemma we obtain

LEMMA 5. *The function* $w_1(x,t)$ *is a periodic function with the period* p.

Next let us consider the potentials $u_{2,1}(x,t)$, $u_{2,2}(x,t)$ and $w_2(x,t) = u_{2,1}(x,t) + u_{2,2}(x,t)$.

LEMMA 6. *If the functions* $h_{2,1}, h_{2,2} \in (K_p)$ *then*

1^0 $P^2 u_{2,i}(x,t) = 0,\quad i = 1,2,\quad P^2 w_2(x,t) = 0,\quad (x,t) \in D,$

2^0 $Pw_2(x,t) = Pu_{2,1}(x,t) + Pu_{2,2}(x,t),\quad (x,t) \in D,$

3^0 $Pw_2(0,t) = h_{2,1}(t),\quad Pw_2(1,t) = h_{2,2}(t),\quad t \in (-\infty,\infty),$

4^0 $Pw_2(x,t)$ is a periodic function with a period p.

PROOF. Since

$$Pw_2(x,t) = \int_{-\infty}^{t} h_{2,1}(s) D_y G(x,t,0,s) ds + \int_{-\infty}^{t} h_{2,2}(s) D_y G(x,t,1,s) ds$$

thus

$$P^2 w_2(x,t) = 0,\ (x,t) \in D.$$

The proof of the conditions 3^0, 4^0 is similar to those in Lemma 5.

LEMMA 7. *If functions* $h_{m,1}, h_{m,2} \in (K_P)$ *then the function* w_3 *satisfies the conditions*

1^0 $P^m w_m(x,t) = 0,\quad (x,t) \in D,$

2^0 $P^{m-1} w_m(0,t) = h_{m,1}(t),\quad t \in (-\infty,\infty),$

3^0 $P^{m-1} w_m(1,t) = h_{m,2}(t),\quad t \in (-\infty,\infty),$

4^0 $P^{m-1} w_m(x,t)$ is a periodic function with a period p.

PROOF. Since by formulas (15), $(15)_1$ we obtain

$$P^{m-1} w_m(x,t) = \int_{-\infty}^{t} h_{m,1}(s) D_y G(x,t,0,s) ds + \int_{-\infty}^{t} h_{m,2}(s) D_y G(x,t,1,s) ds$$

thus by the last formula we obtain the assertion of the Lemma 7.

7. FUNDAMENTAL THEOREM

THEOREM 1. *If* $h_{i,1}$, $h_{1,2} \in (K_p)$ *then the functions* w_i, $i = 1,\dots,m$ *satisfy the conditions*

1^0 $P^m \sum_{n=1}^{m} w_n(x,t) = 0, \quad (x,t) \in D,$

2^0 $P^i w_i(x,t) = 0, \quad (x,t) \in D, \quad i = 1,..,m,$

3^0 $P^i w_{i+1}(0,t) = h_{i+1,1}(t), \quad t \in (-\infty,\infty), \quad i = 0,1,\ldots,m-1,$

4^0 $P^i w_{i+1}(1,t) = h_{i+1,2}(t), \quad t \in (-\infty,\infty), \quad i = 0,1,\ldots,m-1,$

5^0 *the functions* $P^i w_{i+1}(x,t)$, $i = 0,1,\ldots,m-1$, *are periodic with the period* p,

6^0 *the function* $u(x,t) = \sum_{i=0}^{3} C_i P^i w_{i+1}(x,t)$ *is a periodic solution with the period* p *to the problem* (1) – (6).

COLLORARY. *If at last two functions* $h_{i,1}$, $h_{1,2}$, $i = 1,\ldots,m$ *has the incommensurable periods* $p_1 > 0$, $p_2 > 0$ *then the function* $u(x,t)$ *is almost periodic function.*

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