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ON A PARABOLIC PROBLEM TOGETHER WITH A SEMILINEAR EQUATION AND A SEMILINEAR INITIAL CONDITION

ABSTRACT: The existence and uniqueness of integral and classical solutions of a parabolic problem together with a semilinear equation and a semilinear initial condition, for the operator $\sum_{i=1}^{n} a_i D_{x_i}^2 - D_t$, in the domain $(X_{i=1}^{n}(-c_i, c_i)) \times (0,T]$, where $T < \infty$, are studied.

KEY WORDS: parabolic problem, semilinear equation, semilinear initial condition, integral and classical solutions, existence and uniqueness of the solutions.

1. INTRODUCTION

Let

$$D_0 = \{ x = (x_1, \ldots, x_n) \in R^n : |x_i| < c_i \ \ (i = 1, \ldots, n)\},$$

where c_i $(i = 1, \ldots, n)$ are positive constants and let $D = D_0 \times (0,T]$, $T < \infty$.

We study the existence and uniqueness of integral and classical solutions of the following parabolic problem together with a semilinear equation and a semilinear initial condition:

$$\left(\sum_{i=1}^{n} a_i D_{x_i}^2 - D_t \right) u(x,t) = f(x,t,u(x,t)) \quad \text{for } (x,t) \in D, \tag{1.1}$$

$$u(x,t) + g(x, u(x,T_1), \ldots, u(x,T_p)) = f_0(x) \quad \text{for } (x,t) \in \overline{S}_0, \tag{1.2}$$

$$u(x,t) = f_i^j(x^i, t) \quad \text{for } (x,t) \in \overline{S}_i^j, \ i \in I_n, \ j \in I_2, \tag{1.3}$$

where a_i $(i = 1, \ldots, n)$ are positive constants, f, g, f_0, f_i^j $(i \in I_n, j \in I_2)$ are given real functions satisfying some assumptions,

$$S_0 = \{(x,0) : x \in D_0\},$$

$$S_i^j = \{(x,t) : |x_k| < c_k, \ k \in I_n, k \neq i, x_i = (-1)^j c_i, \ t \in (0,T]\},$$

$$0 < T_1 < \ldots < T_p \leq T, \quad p \in N$$

and

$$I_r = \{1, \ldots, r\}, \quad r \in N.$$

In the sequel problem (1.1) – (1.3) is said to be the problem of type (FNH).

To study the solutions of the problem considered, we use results from [1] – [7] and [9] – [10]. Some results of the paper base on those from monographs: by Friedman [8], by Krzyżański [11] and by Marcinkowska [12].

As in [3], the problem, considered in the paper, cannot be transformed to the operator differential problem, studied for example by Pazy in [13]. It is the reason that the authors study, in this paper, the differential problem without the theory of semigroups.

2. PRELIMINARIES

Throughout the paper we use the following notations:

$$R_- := (-\infty, 0), \quad R_+ := (0, \infty),$$

$$N := \{1, 2, \ldots\}, \quad N_0 := N \cup \{0\},$$

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n),$$

$$x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \quad (i \in I_n),$$

$$D_i := \mathop{\mathsf{X}}_{\substack{k=1 \\ k \neq i}}^{n} (-c_k, c_k) \quad (i \in I_n),$$

$$D_i^j := (-c_1, c_1) \times \ldots \times (-c_{i-1}, c_{i-1}) \times \{(-1)^j c_i\} \times (-c_{i+1}, c_{i+1}) \times \ldots$$
$$\ldots \times (-c_n, c_n) \quad (i \in I_n,\ j \in I_2),$$

$$\widetilde{S}_i^j := \overline{D}_i^j \times (0, T] \quad (i \in I_n,\ j \in I_2),$$

$$Z_i := \partial(\overline{D}_i \times [0, T]) \setminus \{(x^i, 0)\} \quad (i \in I_n),$$

$$P := \sum_{i=1}^{n} a_i D_{x_i}^2 - D_t.$$

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in N_0^n$ we put $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Moreover, $D_{x,t}^{\alpha} := D_x^{\widetilde{\alpha}} D_t^{\alpha_*}$, where $\alpha = (\widetilde{\alpha}, \alpha_*)$, $\widetilde{\alpha} \in N_0^n$, $\alpha_* \in N_0$, $x \in R^n$ and $t \in [0, T]$.

Let $a_i \in R_+$ for $i \in I_n$. For every fixed index $i \in I_n$ we define the function $U : R^2 \setminus \{0\} \to R$ by the formula

$$U(\xi,\tau;a_i) = \begin{cases} (4\pi a_i\tau)^{-1/2}\exp(-(4a_i\tau)^{-1}\xi^2) & \text{for } \xi\in R,\ \tau\in R_+, \\ 0 & \text{for } \xi\in R,\ \tau\in R_- \\ & \text{or } \xi\in R\setminus\{0\},\ \tau=0. \end{cases}$$

Now, for all $x\in R^n$, $y\in R^n$, $0\le s<t$, $i\in I_n$, $j\in I_2$ and $k\in N_0$, we define the functions $U_{i,k}^{(j)}$, U_i by the formulas

$$\text{(2.1)} \qquad \begin{aligned} U_{i,k}^{(j)}(x_i,t,y_i,s) &= U(y_i - x_{i,k}^{(j)}, t-s; a_i), \\ U_i(x_i,t,y_i,s) &= U_{i,0}^{(j)}(x_i,t,y_i,s), \end{aligned}$$

where $x_{i,k}^{(j)} = (-1)^k(x_i + (-1)^{j+1}2kc_i)$.

Next, for every $x\in R^n$, $y\in R^n$ and $0\le s<t\le T$, we define the function G by the formula

$$G(x,t,y,s) = \prod_{i=1}^{n} G_i(x_i,t,y_i,s),$$

where

$$G_i(x_i,t,y_i,s) = U_i(x_i,t,y_i,s) + \sum_{k=1}^{\infty}(-1)^k(U_{i,k}^{(1)}(x_i,t,y_i,s) + U_{i,k}^{(2)}(x_i,t,y_i,s))$$

and the functions U_i, $U_{i,k}^{(j)}$ $(i\in I_n, j\in I_2, k\in N)$ are given by formulas (2.1).

3. Formulation of the homogeneous with respect to the differential equation and the initial condition Fourier's first linear problem of type (FH)

A continuous function u in $\overline{D}$ is called a regular in D if the derivatives $D_{x,t}^{\alpha}u$ $(\alpha = (\tilde\alpha,\alpha_*),\ \tilde\alpha\in N_0^n,\ \alpha_*\in N_0,\ 0<|\tilde\alpha|+2\alpha_*\le 2)$ are continuous in D.

Given the functions f_i^j $(i\in I_n, j\in I_2)$, consider the homogeneous with respect to the differential equation and the initial condition Fourier's first linear problem of the form:

$$\text{(3.1)} \qquad Pu(x,t) = 0 \quad \text{for} \quad (x,t)\in D,$$

$$\text{(3.2)} \qquad u(x,t) = 0 \quad \text{for} \quad (x,t)\in \overline{S}_0,$$

$$\text{(3.3)} \qquad u(x,t) = f_i^j(x^i,t) \quad \text{for} \quad (x,t)\in \overline{S}_i^j,\ i\in I_n,\ j\in I_2.$$

The above problem is said to be the problem of type (FH).

A regular in D function u satisfying equation (3.1) and conditions (3.2) – (3.3) is called a classical solution in D of the problem of type (FH).

4. THEOREM ON THE EXISTENCE OF A CLASSICAL SOLUTION OF THE PROBLEM OF TYPE (FH)

Now, we shall formulate the following theorem on the existence of the regular solution of the problem of type (FH):

THEOREM 4.1. *Suppose that the functions* f_i^j $(i \in I_n, j \in I_2)$ *are continuous in the domains* $\overline{D}_i \times [0,T]$, *respectively, and satisfy the equations*

$$f_i^j(x^i,t) = 0 \ \text{for} \ (x^i,t) \in Z_i \cup (\overline{D}_i \times \{0\}) \quad (i \in I_n,\ j \in I_2).$$

Then u, given by the formula

$$u(x,t) = \sum_{i=1}^{n} (u_i^1(x,t) + u_i^2(x,t)) \ \text{for} \ (x,t) \in \overline{D}, \tag{4.1}$$

where

$$u_i^j(x,t) = \begin{cases} -2a_i \int_0^t \int_{D_0} f_i^j(y^i,s) D_{y_i} G(x,t,y,s)|_{y_i=(-1)^j c_i} \, dy^i ds & \\ & \text{for} \ (x,t) \in (\overline{D}_0 \times (0,T]) \setminus \widetilde{S}_i^j, \\ f_i^j(x^i,t) & \text{for} \ (x,t) \in \widetilde{S}_i^j, \\ 0 & \text{for} \ (x,t) \in \overline{S}_0 \end{cases} \tag{4.2}$$

is the classical solution in D of the problem of type (FH).

For the proof of the above theorem see [6] and [4].

5. FORMULATION OF FOURIER'S FIRST SEMILINEAR PROBLEM OF TYPE (FNH)

A continuous function u in $\overline{D}$ is called a semiregular in D if the derivatives $D_{x,t}^{\alpha} u$ $(\alpha = (\widetilde{\alpha}, \alpha_*),\ \widetilde{\alpha} \in N_0^n,\ \alpha_* \in N_0,\ 0 < |\widetilde{\alpha}| + 2\alpha_* \le 1)$ are continuous in D.

Given the functions f, g, f_0, f_i^j $(i \in I_n, j \in I_2)$, consider Fourier's first semilinear problem of the from:

$$Pu(x,t) = f(x,t,u(x,t)) \quad \text{for} \ (x,t) \in D, \tag{5.1}$$

$$u(x,t) + g(x, u(x,T_1), \ldots, u(x,T_p)) = f_0(x) \quad \text{for} \ (x,t) \in \overline{S}_0, \tag{5.2}$$

(5.3) $$u(x,t) = f_i^j(x^i,t) \quad \text{for} \quad (x,t)\in \overline{S}_i^j,\ i\in I_n,\ j\in I_2.$$

The above problem is said to be the problem of type (FNH).

A semiregular in D function u satisfying equation (5.1) and conditions (5.2) – (5.3) is called a classical solution in D of the problem of type (FNH).

6. EXISTENCE AND UNIQUENESS OF INTEGRAL AND CLASSICAL SOLUTIONS TO FOURIER'S FIRST SEMILINEAR PROBLEM OF TYPE (FNH)

Let us consider the integral equation

(6.1) $$u(x,t) = \begin{cases} v_0(x,t) + \int_{D_0} [f_0(y) - g(y,u(y,T_1),\dots,u(y,T_p))]G(x,t,y,0)\,dy - \\ - \int_0^t \int_{D_0} f(y,s,u(y,s))G(x,t,y,s)\,dyds \quad \text{for } (x,t)\in \overline{D}_0\times(0,T], \\ f_0(x) - g(x,u(x,T_1),\dots,u(x,T_p)) \quad \text{for } (x,t)\in \overline{S}_0, \end{cases}$$

where v_0 is the classical solution in D, given by (4.1) – (4.2), of the problem of type (FH), the function $y\to f_0(y)$ is continuous for $y\in\overline{D}_0$, the function $(y,z_1,\dots,z_p)\to g(y,z_1,\dots,z_p)$ is continuous for $y\in\overline{D}_0$, $z_i\in R$ $(i=1,\dots,p)$ and the function $(y,s,z)\to f(y,s,z)$ is continuous for $(y,s)\in\overline{D}$, $z\in R$.

We shall prove that a solution u of the integral equation (6.1), under some additional assumptions, is a classical solution of problem (5.1) – (5.3).

Observe that equation (6.1) can be written in the form

(6.2) $$u(x,t) = Z(x,t) + (N(u))(x,t), \quad (x,t)\in\overline{D},$$

where

(6.3) $$Z(x,t) := \begin{cases} v_0(x,t) & \text{for } (x,t)\in\overline{D}_0\times(0,T], \\ 0 & \text{for } (x,t)\in\overline{S}_0 \end{cases}$$

and

(6.4) $$(N(u))(x,t) := \\ := \begin{cases} \int_{D_0} [f_0(y) - g(y,u(y,T_1),\dots,u(y,T_p))]G(x,t,y,0)\,dy - \\ - \int_0^t \int_{D_0} f(y,s,u(y,s))G(x,t,y,s)\,dyds \quad \text{for } (x,t)\in\overline{D}_0\times(0,T], \\ f_0(x) - g(x,u(x,T_1),\dots,u(x,T_p)) \quad \text{for } (x,t)\in\overline{S}_0. \end{cases}$$

Let

(6.5) $$M := \max\left\{ \sup_{x\in D_0} |f_0(x)| + \sup_{\substack{x\in D_0 \\ w\in C(\overline{D}_0\times(0,T],R)}} |g(x,w(x,T_1),\dots,w(x,T_p))|, \right.$$

$$\sup_{\substack{(x,t)\in\overline{D}_0\times(0,T] \\ w\in C(\overline{D}_0\times(0,T],R)}} \left| \int_{D_0} [f_0(y) - g(y,w(y,T_1),\dots,w(y,T_p))] G(x,t,y,0)\,dy \right| +$$

$$+ \sup_{\substack{(x,t)\in\overline{D}_0\times(0,T] \\ w\in C(\overline{D}_0\times(0,T],R)}} \left| \int_0^t \int_{D_0} f(y,s,w(y,s)) G(x,t,y,s)\,dyds \right|,$$

$$\sup_{(x,t)\in\overline{D}_0\times(0,T]} \int_{D_0} |G(x,t,y,0)|\,dy +$$

$$\left. + \sup_{(x,t)\in\overline{D}_0\times(0,T]} \int_0^t \int_{D_0} |G(x,t,y,s)|\,dyds \right\}.$$

Consider the transformation

(6.6) $$u \xrightarrow[T]{} \mathcal{T}(u) := Z + N(u).$$

Denote by B_r the complete metric space, with the metric ρ, of the functions w such that

(i) $w \in C(\overline{D}, R)$,

(ii) $w(x,t) + g(x,w(x,T_1),\dots,w(x,T_p)) = f_0(x)$ for $(x,t)\in\overline{S}_0$, where the regularity of the function g is given in Lemma 6.1,

(iii) $\rho(w,0) = \sup_{(x,t)\in\overline{D}} |w(x,t)| \le r$, where r is a positive number.

Moreover, denote by $\mathcal{K}$ the class of all functions $w\in C(\overline{D},R)$ such that

$$\rho(w,0) = \sup_{(x,t)\in\overline{D}} |w(x,t)| \le (1-\kappa)r,$$

where κ is a number belonging to the interval $(0,1)$ and r is the constant from the definition of space B_r.

LEMMA 6.1. *Assume that:*

(i) v_0 *is the classical solution in D of the problem of thpe (FH) and* $v_0 \in \mathcal{K}$,

(ii) the function $y \to f_0(y)$ *is continuous for* $y \in \overline{D}_0$,

(*iii*) *the function* $(y, z_1, ..., z_p) \to g(y, z_1, ..., z_p)$, *where* $y \in \overline{D}_0$, $z_i \in R$ $(i = 1, ..., p)$, *is continuous with respect to the first variable,*

(*iv*) *the function g satisfies the Lipschitz condition*

$$| g(y, w(y,T_1),..., w(y,T_p)) - g(y, \widetilde{w}(y,T_1),..., \widetilde{w}(y,T_p))| \le$$
$$\le L_1 \sum_{i=1}^{p} | w(y,T_i) - \widetilde{w}(y,T_i)| \quad for \quad y \in \overline{D}_0, \quad w, \widetilde{w} \in C(\overline{D}_0 \times (0,T], R),$$

where L_1 *is a positive constant,*

(*v*) *the function* $(y, s, z) \to f(y, s, z)$, *where* $(y, s) \in \overline{D}$, $z \in R$, *is continuous with respect to the first and second variables,*

(*vi*) *the function f satisfies the Lipschitz condition*

$$| f(y,s,z) - f(y,s,\widetilde{z})| \le L_2 | z - \widetilde{z} | \quad for \quad (y,s) \in D, \quad z, \widetilde{z} \in R,$$

where L_2 *is a positive constant,*

(*vii*) $(pL_1 + L_2)M \le \kappa$, $pL_1 \le \kappa$ and $M \le \kappa r$.

Then:

(T_1) *transformation* $\mathcal{T}$ *is invariant in space* B_r,

(T_2) *transformation* $\mathcal{T}$ *is a contraction on space* B_r.

Consquently, there exists a unique semiregular in D function satisfying the integral equation (6.1).

Proof. (T_1) : Let $u \in B_r$. Observe that, by (6.6), (6.3), (6.4), by assumption (*i*), by (6.5) and by assumption (*vii*),

$$|(\mathcal{T}(u))(x,t)| \le | v_0(x,t) +$$
$$+ \left| \int_{D_0} [f_0(y) - g(y, u(y,T_1),..., u(y,T_p))] G(x,t,y,0) \, dy \right| +$$
$$+ \left| \int_0^t \int_{D_0} f(y,s,u(y,s)) G(x,t,y,s) \, dyds \right| \le$$
$$\le (1-\kappa) r + M \le (1-\kappa) r + \kappa r = r \quad \text{for} \quad (x,t) \in \overline{D}_0 \times (0,T].$$

Moreover, from (6.6), (6.3), (6.4), (6.5) and from assumption (*vii*),

$$|(\mathcal{T}(u))(x,t)| \le |f_0(x)| + |g(x,u(x,T_1),...,u(x,T_p))| \le$$
$$\le M \le \kappa r < r \quad \text{for} \quad (x,t) \in \overline{S}_0 .$$

(T_2): Suppose that $u_i \in B_r$ $(i = 1,2)$. Then, by (6.6), (6.4), assumptions (*vi*) and (*iv*), formula (6.5) and assumption (*vii*),

$$|(\mathcal{T}(u_1))(x,t) - (\mathcal{T}(u_2))(x,t)| =$$
$$= \left| \int_0^t \int_{D_0} [f(y,s,u_1(y,s)) - f(y,s,u_2(y,s))] G(x,t,y,s)\, dyds - \right.$$
$$- \int_{D_0} [g(y,u_1(y,T_1),...,u_1(y,T_p)) - g(y,u_2(y,T_1),...,u_2(y,T_p))] \times$$
$$\left. \times\, G(x,t,y,0)\, dy \right| \le$$
$$\le L_2 \int_0^t \int_{D_0} |u_1(y,s) - u_2(y,s)|\, |G(x,t,y,s)|\, dyds +$$
$$+ L_1 \int_{D_0} \sum_{i=1}^{p} |u_1(y,T_i) - u_2(y,T_i)|\, |G(x,t,y,0)|\, dy \le$$
$$\le (pL_1 + L_2) M \rho(u_1,u_2) \le \kappa \rho(u_1,u_2) \quad \text{for} \quad (x,t) \in \overline{D}_0 \times (0,T].$$

Moreover, from (6.6), (6.4), (6.3), assumptions (*iv*) and (*vii*),

$$|(\mathcal{T}(u_1))(x,t) - (\mathcal{T}(u_2))(x,t)| =$$
$$= |g(x,u_1(x,T_1),...,u_1(x,T_p)) - g(x,u_2(x,T_1),...,u_2(x,T_p))| \le$$
$$\le L_1 \sum_{i=1}^{p} |u_1(x,T_i) - u_2(x,T_i)| \le L_1 p \rho(u_1,u_2) \le$$
$$\le \kappa \rho(u_1,u_2) \quad \text{for} \quad (x,t) \in \overline{S}_0 .$$

The proof of Lemma 6.1 is complete.

From Theorem 4.1, from Lemma 6.1 and on the base of the potential theory (see: Krzyżański [11] or Marcinkowska [12]), we get the following theorem on the existence and uniqueness of a semiregular solution in D of the problem of type (FNH):

THEOREM 6.1. *Suppose that the assumptions from Theorem* 4.1 *are satisfied. Then* v_0, *given by formula*

$$v_0(x,t) = u(x,t) \quad \textit{for} \quad (x,t) \in \overline{D}, \tag{6.7}$$

where u is defined by (4.1) *and* (4.2), *is the classical solution in D of the problem of type* (*FH*).

Suppose, additionally, that $v_0 \in K$ *and suppose that assumptions* (*ii*) – (*vii*) *of Lemma* 6.1 *are satisfied and that:*

(*viii*) $f_0(y) - g(y, w(y,T_1),..., w(y,T_p)) = 0$ *for all* $w \in C(\partial D_0 \times (0,T], R)$ *and* $y \in \partial D_0$.

Then there exists a unique semiregular in D function v satisfying the integral equation (6.1).

Finally, if additionally the functions

$$(y,s,z) \to \frac{\partial f(y,s,z)}{\partial y_i} \quad (i \in I_n), \qquad (y,s,z) \to \frac{\partial f(y,s,z)}{\partial z}$$

are continuous for $(y,s) \in D$, $z \in R$, *then v is a semiregular solution in D of the problem of type* (*FNH*).

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