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HYPERBOLIC TRANSFORMATION AND HYPERBOLIC DIFFERENCE SYSTEMS

ABSTRACT: Transformations of symplectic difference systems

$$z_{k+1} = S_k z_k, \quad S_k^T J S_k = J, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

are investigated. It is shown that symplectic systems satisfying certain additional condition can be transformed (using a transformation that preserves oscillation properties of transformed systems) into the so-called hyperbolic difference system. Basic properties of solutions of hyperbolic systems are established.

KEY WORDS: symplectic difference system, hyperbolic system, recessive solution, hyperbolic transformation.

1. INTRODUCTION

In this paper we deal with transformations and qualitative properties of solutions of a certain class of symplectic difference systems (further SdS)

$$z_{k+1} = S_k z_k, \tag{1}$$

where $z \in R^{2n}$ and S is a real, symplectic $2n \times 2n$-matrix, i.e.,

$$S_k^T J S_k = J \quad \text{with} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2}$$

where I denotes the identity matrix and ^{T} stands for the transpose of the indicated matrix. Sometimes, it will be convenient to write (1) in the from

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} \tag{3}$$

with $x, u \in R^n$ and real $n \times n$-matrices A, B, C, D. Substituting into (2), it is easy to see that the matrix S in (1) is symplectic if and only if

$$A^T D - C^T B = I = D^T A - B^T C, \quad A^T C - C^T A = 0 = B^T D - D^T B. \tag{4}$$

Here and throughout we use the convention that no index at a matrix or a vector actually means the index $k \in Z$. Consequently,

$$\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D^TA-B^TC & D^TB-B^TD \\ -C^TA+A^TC & -C^TB+A^TD \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

That is, $S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$. Now, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ implies that (4) is equivalent to

(5) $$AD^T - BC^T = I = DA^T - CB^T, \quad AB^T - BA^T = 0 = CD^T - DC^T$$

and (2) is equivalent to $S_k J S_k^T = J$. Furthermore, (3) can be written as

(6) $$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} D_k^T & -B_k^T \\ -C_k^T & A_k^T \end{pmatrix}\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix}.$$

In this paper we show that every symplectic difference system (1) satisfying certain additional condition can be transformed, using the transformation preserving oscillatory behaviour, into the so-called hyperbolic symplectic system. We also study basic properties of solutions of these hyperbolic systems.

In the continuous case, it is shown in [7] that any nonoscillatory self-adjoint equation of the form

(7) $$(R(t)x')' + P(t)x = 0$$

with symmetric $n \times n$ matrices R, P and R nonsingular, can be transformed using the transformation $x = H(t)y$, H being a nonsingular $n \times n$ matrix, into the hyperbolic system

(8) $$(Q^{-1}(t)y')' - Q(t)y = 0, \qquad Q = (H^T R H)^{-1}.$$

Here the terminology "hyperbolic system" is justified by the fact that in the scalar case $n = 1$ linearly independent solutions of (8) are

$$y_1(t) = \sinh\left(\int^t Q(s)ds\right), \quad y_2(t) = \cosh\left(\int^t Q(s)ds\right).$$

Basic properties of solutions of hyperbolic system (8) are established in the recent paper [11]. Here it is also shown that even a more general system than (8), namely the linear Hamiltonian system

$$x' = A(t)x + B(t)u, \qquad u' = C(t)x - A^T(t)u,$$

with $n \times n$ matrices A, B, C and B, C symmetric, can be transformed into the hyperbolic Hamiltonian system. More details along this linea are given in the next section.

Oscillation properties and transformations of SdS have been investigated in [4] (see also [1, 3, 5, 6, 8]), where the so-called Reid Roundabout Theorem for (1) is established. This theorem relates oscillation properties of (1) to positivity of the corresponding quadratic functional and to the solvability of the associated Riccati-type matrix difference equation.

A hyperbolic SdS is the system

$$z_{k+1} = S_k z_k, \qquad S = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \tag{9}$$

with $n \times n$ matrices P, Q satisfying

$$P^T P - Q^T Q = I = PP^T - QQ^T, \quad P^T Q - Q^T P = 0 = PQ^T - QP^T. \tag{10}$$

The terminology will be justified in the third section. The first equality in (10) imply that matrix P is nonsingular and, since

$$(P^T + Q^T)(P - Q) = P^T P + Q^T P - P^T Q - Q^T Q = I,$$

matrices $P + Q,\ P - Q$ are nonsingular, too. Further

$$(P - Q)^{-1} = P^T + Q^T, \quad (P + Q)^{-1} = P^T - Q^T. \tag{11}$$

Moreover, left multiplication by P^{-1} and right multiplication by $P^{T-1} (= (P^T)^{-1} = (P^{-1})^T)$ of the equality $PQ^T = QP^T$ gives $Q^T P^{T-1} = P^{-1} Q$, that is

$$(P^{-1}Q)^T = P^{-1}Q, \tag{12}$$

$P^{-1}Q$ is symmetric.

In this paper, we investigate properties of hyperbolic SdS and we also show that a certain class of general symplectic systems (1) can be transformed into a hyperbolic system (9). We follow essentially the recent papers [2] and [4] where the so-called trigonometric systems and trigonometric transformations are investigated. Recall that the trigonometric system is a symplectic difference system (1) whose matrix is of the from $S = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$. The terminology trigonometric system comes from the fact that in the scalar case $n = 1$ solutions of this system can be expressed via the classical sine and cosine functions. Here we establish “hyperbolic analogies” of the results given in these papers.

The paper is organized as follows. In the next section we recall basic properties of symplectic difference systems and we also give, for the sake of comparision, some statements concerning transformations of differential Hamiltonian systems. Section 3 is devoted to the investigation of hyperbolic symplectic systems and basic properties of their solutions. Oscilation properties of these solutions are studied in the fourth section and a necessary and sufficient condition for nonoscillation of hyperbolic systems is given there. In the last section, it is shown that any nonoscillatory symplectic system (1) can be transformed into hyperbolic symplectic system without changing oscillatory properties of transformed systems.

2. AUXILIARY RESULTS

We start with some basic properties of solutions of SdS (1). Let $\begin{pmatrix} X \\ U \end{pmatrix}$, $\begin{pmatrix} \tilde{X} \\ \tilde{U} \end{pmatrix}$ be matrix solutions of (1), i.e., X, U, $\tilde{X}$, $\tilde{U}$ are $n \times n$-matrices satisfying

$$\begin{pmatrix} X_{k+1} & \tilde{X}_{k+1} \\ U_{k+1} & \tilde{U}_{k+1} \end{pmatrix} = \mathcal{S}_k \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}.$$

Then symplecticity of the matrix $\mathcal{S}$ implies that

$$\begin{pmatrix} X_{k+1} & \tilde{X}_{k+1} \\ U_{k+1} & \tilde{U}_{k+1} \end{pmatrix}^T \mathcal{J} \begin{pmatrix} X_{k+1} & \tilde{X}_{k+1} \\ U_{k+1} & \tilde{U}_{k+1} \end{pmatrix} = \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}^T \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix} =$$

$$= \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}^T \mathcal{J} \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix},$$

i.e.

$$(13) \qquad \Delta \left[\begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}^T \mathcal{J} \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix} \right] = 0,$$

where Δ is the usual forward difference operator. Consequently, if $\begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}$ is symplectic at some k, then it is symplectic everywhere. If this is the case, then, according to (4), (5) we have the identites

$$(14) \qquad X^T\tilde{U} - U^T\tilde{X} = I = \tilde{U}^T X - \tilde{X}^T U, \quad X^T U - U^T X = 0 = \tilde{X}^T\tilde{U} - \tilde{U}^T\tilde{X}$$

and

$$(15) \qquad X\tilde{U}^T - \tilde{X}U^T = I = \tilde{U}X^T - U\tilde{X}^T, \qquad X\tilde{X}^T - \tilde{X}X^T = 0 = U\tilde{U}^T = \tilde{U}U^T.$$

Let $\begin{pmatrix} X \\ U \end{pmatrix}$ be a matrix solution of (1). Since the matrix $\mathcal{S}_k$ is nonsingular,

(16) $$\operatorname{rank}\begin{pmatrix} X_{k+1} \\ U_{k+1} \end{pmatrix} = \operatorname{rank}\left[S_k \begin{pmatrix} X_k \\ U_k \end{pmatrix} \right] = \operatorname{rank}\begin{pmatrix} X_k \\ U_k \end{pmatrix},$$

hence rank $\begin{pmatrix} X_k \\ U_k \end{pmatrix}$ is the same for all $k \in Z$.

The matrix solution $\begin{pmatrix} X \\ U \end{pmatrix}$ of (1) which satisfies the first part of the second identity in (14) and $\operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = n$ is said to be a *conjoined basis* of (1). The solutions $\begin{pmatrix} X \\ U \end{pmatrix}$, $\begin{pmatrix} \widetilde{X} \\ \widetilde{U} \end{pmatrix}$, such that $\begin{pmatrix} X & \widetilde{X} \\ U & \widetilde{U} \end{pmatrix}$ is symplectic, are called *normalized conjoined bases* of (1).

A conjoined basis $\begin{pmatrix} X \\ U \end{pmatrix}$ of (1) is said to be *recessive* at ∞ if X is eventually nonsingular and there exists another conjoined basis $\begin{pmatrix} \widetilde{X} \\ \widetilde{U} \end{pmatrix}$ with $\widetilde{X}$ eventually nonsingular such that $X^T \widetilde{U}^{-1} - U^T \widetilde{X}$ is nonsingular and $\lim_{t \to \infty} \widetilde{X}_k^{-1} X_k = 0$. If this is the case, $\begin{pmatrix} \widetilde{X} \\ \widetilde{U} \end{pmatrix}$ is called *dominant solution*.

Now, we turn our attention to transformations of SdS. Let $R_k = \begin{pmatrix} H_k & M_k \\ K_k & N_k \end{pmatrix}$ be a real, symplectic $2n \times 2n$-matrix, i.e.,

$$H^T N - K^T M = I, \quad H^T K = K^T H, \quad M^T N = N^T M.$$

Then the transformation

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = R_k \begin{pmatrix} \widetilde{x}_k \\ \widetilde{u}_k \end{pmatrix}$$

transforms (1) into another symplectic system

$$\begin{pmatrix} \widetilde{x}_{k+1} \\ \widetilde{u}_{k+1} \end{pmatrix} = \widetilde{S}_k \begin{pmatrix} \widetilde{x}_k \\ \widetilde{u}_k \end{pmatrix} \quad \text{with} \quad \widetilde{S}_k = \begin{pmatrix} \widetilde{A}_k & \widetilde{B}_k \\ \widetilde{C}_k & \widetilde{D}_k \end{pmatrix} = R_{k+1}^{-1} S_k R_k,$$

where in detail:

$$\widetilde{A} = N_{k+1}^T (AH + BK) - M_{k+1}^T (CH + DK),$$

$$\widetilde{B} = N_{k+1}^T (AM + BN) - M_{k+1}^T (CM + DN),$$

$$\tilde{C} = H_{k+1}^T(CH + DK) - K_{k+1}^T(AH + BK),$$
$$\tilde{D} = H_{k+1}^T(CM + DN) - K_{k+1}^T(AM + BN).$$

(Recall again the convention that no index at a matrix actually means the index k.) Moreover, if $M \equiv 0$, then this transformation preserves oscillation behavior of transformed systems (see [4, Lemma 7]).

Finally recall concepts of hyperbolic system and hyperbolic transformation of differential Hamiltonian systems

$$x' = A(t)x + B(t)u, \qquad u' = C(t)x - A^T(t)u, \tag{17}$$

where A, B, C are $n \times n$ matrices of continuous functions and B, C are symetric. Hyperbolic systems is a special system (17) of the from

$$s' = P(t)s + Q(t)c, \qquad c' = Q(t)s + P(t)c, \tag{18}$$

wher P is antisymmetric $(P^T + P = 0)$ and Q is symmetric. If $n = 1$ (then, of course, $P = 0$) solutions of (18) are

$$\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} \sinh \int^t Q \\ \cosh \int^t Q \end{pmatrix}, \qquad \begin{pmatrix} \tilde{s} \\ \tilde{c} \end{pmatrix} = \begin{pmatrix} \cosh \int^t Q \\ \sinh \int^t Q \end{pmatrix}.$$

For $n > 1$ (18) cannot be in general solved explicity, but its solutions have many of the properties of hyperbolic sine and cosine functions. In particular, the sum formulae for hyperbolic functions extend in a natural way to (18).

If (17) is nonoscillatory (i.e. there exists $2n \times n$ matrix solution $\begin{pmatrix} X \\ U \end{pmatrix}$ such that $X^T U = U^T X$ and $X(t)$ is nonsingular for large t) then (17) can be transformed into hyperbolic system (18) by a transformation preserving oscillatory properties of transformed systems. More precisely, there exists $n \times n$ matrices H, K of differentiable functions such that H is nonsingular, $H^T K = K^T H$, and the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H(t) & 0 \\ K(t) & H^{T-1}(t) \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix}$$

transforms (17) into hyperbolic system (18) (see [11] where also further properties of (18) can be found).

3. DISCRETE HYPERBOLIC MATRIX FUNCTIONS

In this section we deal with particular matrix solutions of system (9), that is, with solutions of the following system of (matrix) difference equations.

$$\begin{aligned} X_{k+1} &= P_k X_k + Q_k U_k, \\ U_{k+1} &= Q_k X_k + P_k U_k, \end{aligned} \tag{19}$$

or, of the equivalent one

$$\begin{aligned} X_k &= P_k^T X_{k+1} - Q_k^T U_{k+1}, \\ U_k &= -Q_k^T X_{k+1} + P_k^T U_{k+1} \end{aligned} \tag{20}$$

(cf. (3) and (6)). Obviously, if matrices P_k, Q_k are defined for all $k \in Z$, than a solution of (19) is defined on Z for any initial conditions.

DEFINITION 3.1. *We define the discrete hyperbolic sine and hyperbolic cosine matrix functions starting at* $m \in Z$

$$S_{k;m} = S_{k;m}(P,Q), \qquad C_{k;m} = C_{k;m}(P,Q)$$

to be the unique solution

$$X_k = S_{k;m}, \quad U_k = C_{k;m}$$

of system (19) *with the initial conditions*

$$X_m = 0, \qquad U_m = I. \tag{21}$$

For $m = 0$, *we abbreviate* $S_{k;0} = S_k$, $C_{k;0} = C_k$.

Let F_k, $k \in Z$ be matrices and $s, r \in Z$, $s \geq r$. We define the product of matrices in the following way

$$\prod_{i=r}^{s} F_i = F_s F_{s-1} \cdots F_{r+1} F_r.$$

If matrices F_k, $k \in Z$ are nonsingular and $s < r$, we put

$$\prod_{i=r}^{s} F_i = \left(\prod_{i=s+1}^{r-1} F_i \right)^{-1} \quad \text{for} \quad s < r-1, \quad \prod_{i=r}^{r-1} F_i = I.$$

Now, the solutions of the initial value problem (19), (21) can be expressed in the following way:

$$(22)\qquad S_{k;m} = \frac{1}{2}\left(\prod_{i=m}^{k-1}(P_i+Q_i)-\prod_{i=m}^{k-1}(P_i-Q_i)\right),$$

$$(23)\qquad C_{k;m} = \frac{1}{2}\left(\prod_{i=m}^{k-1}(P_i+Q_i)+\prod_{i=m}^{k-1}(P_i-Q_i)\right).$$

Indeed, matrices $(P \pm Q)$ are nonsingular according to the results presented in the first section, so that $S_{k;m}$, $C_{k;m}$ are expressed correctly. Further,

$$S_{m;m} = \frac{1}{2}\left(\prod_{i=m}^{m-1}(P_i+Q_i)-\prod_{i=m}^{m-1}(P_i-Q_i)\right) = \frac{1}{2}(I-I) = 0,$$

$$C_{m;m} = \frac{1}{2}\left(\prod_{i=m}^{m-1}(P_i+Q_i)+\prod_{i=m}^{m-1}(P_i-Q_i)\right) = \frac{1}{2}(I+I) = I,$$

hence initial conditions (21) are satisfied. If $k > m$, then

$$\begin{aligned}
P_k S_{k;m} + Q_k C_{k;m} &= \frac{1}{2}P_k\left(\prod_{i=m}^{k-1}(P_i+Q_i)-\prod_{i=m}^{k-1}(P_i-Q_i)\right) + \\
&\quad + \frac{1}{2}Q_k\left(\prod_{i=m}^{k-1}(P_i+Q_i)+\prod_{i=m}^{k-1}(P_i-Q_i)\right) = \\
&= \frac{1}{2}\left((P_k+Q_k)\prod_{i=m}^{k-1}(P_i+Q_i)-(P_k-Q_k)\prod_{i=m}^{k-1}(P_i-Q_i)\right) = \\
&= \frac{1}{2}\left(\prod_{i=m}^{k}(P_i+Q_i)-\prod_{i=m}^{k}(P_i-Q_i)\right) = S_{k+1;m}
\end{aligned}$$

and, in a similar way,

$$Q_k S_{k;m} + P_k C_{k;m} = C_{k+1;m},$$

if $k < m-1$, then

$$\begin{aligned}
Q_k S_{k;m} + P_k C_{k;m} &= \frac{1}{2}Q_k\left(\prod_{i=m}^{k-1}(P_i+Q_i)-\prod_{i=m}^{k-1}(P_i-Q_i)\right) + \\
&\quad + \frac{1}{2}P_k\left(\prod_{i=m}^{k-1}(P_i+Q_i)+\prod_{i=m}^{k-1}(P_i-Q_i)\right) = \\
&= \frac{1}{2}(P_k+Q_k)\left(\prod_{i=k}^{m-1}(P_i+Q_i)\right)^{-1} +
\end{aligned}$$

$$+\frac{1}{2}(P_k - Q_k)\left(\prod_{i=k}^{m-1}(P_i - Q_i)\right)^{-1} =$$

$$= \frac{1}{2}\left(\left(\prod_{i=k+1}^{m-1}(P_i + Q_i)\right)^{-1} + \left(\prod_{i=k+1}^{m-1}(P_i - Q_i)\right)^{-1}\right) =$$

$$= \frac{1}{2}\left(\prod_{i=m}^{k}(P_i + Q_i) + \prod_{i=m}^{k}(P_i - Q_i)\right) = C_{k+1;m},$$

$$P_k S_{k;m} + Q_k C_{k;m} = S_{k+1;m},$$

and if $k = m-1$, then

$$P_k S_{k;m} + Q_k C_{k;m} = P_{m-1} S_{m+1;m} + Q_{m-1} C_{m-1;m} =$$

$$= \frac{1}{2} P_{m-1}((P_{m-1} + Q_{m-1})^{-1} - (P_{m-1} - Q_{m-1})^{-1}) +$$

$$+ \frac{1}{2} Q_{m-1}((P_{m-1} + Q_{m-1})^{-1} + (P_{m-1} - Q_{m-1})^{-1}) =$$

$$= \frac{1}{2}(I - I) = 0 = S_{m;m} = S_{k+1;m},$$

$$Q_k S_{k;m} + P_k C_{k;m} = C_{k+1;m}.$$

Now, let p_k, q_k be real numbers satysfying

(24) $$p_k^2 - q_k^2 = 1 \quad \text{for} \quad k \in Z,$$

and let us consider the system of (scalar) difference equations

(25) $$\begin{aligned} x_{k+1} &= p_k x_k + q_k u_k, \\ u_{k+1} &= q_k x_k + p_k u_k. \end{aligned}$$

Obviously, (25) is a special case of (19) for $n = 1$. The unique solution of (25) with the initial conditions

(26) $$x_m = 0, \quad u_m = 1$$

are $x_k = s_{k;m}$, $u_k = c_{k;m}$ defined by

$$s_{k;m} = \frac{1}{2}\left(\prod_{i=m}^{k-1}(p_i + q_i) - \prod_{i=m}^{k-1}(p_i - q_i)\right),$$

$$c_{k;m} = \frac{1}{2}\left(\prod_{i=m}^{k-1}(p_i+q_i)+\prod_{i=m}^{k-1}(p_i-q_i)\right),$$

according to (22), (23). Moreover, (24) yields $|p_k|>|q_k|$ for $k\in Z$, hence

$$\operatorname{sgn}(p_k+q_k) = \operatorname{sgn}(p_k-q_k) = \operatorname{sgn} p_k.$$

Consequently,

$$p_i+q_i = \operatorname{sgn} p_i\,|p_i+q_i| = \operatorname{sgn} p_i \exp(\ln|p_i+q_i|)$$

$$p_i-q_i = \operatorname{sgn} p_i\,|p_i-q_i| = \operatorname{sgn} p_i\left|\frac{p_i^2-q_i^2}{p_i+q_i}\right| = \operatorname{sgn} p_i\,\frac{1}{|p_i+q_i|} =$$

$$= \operatorname{sgn} p_i \exp(-\ln|p_i+q_i|)$$

for $i\in Z$. Thus

$$\prod_{i=m}^{k-1}(p_i+q_i) = \left(\prod_{i=m}^{k-1}\operatorname{sgn} p_i\right)\exp\left(\sum_{i=m}^{k-1}\ln|p_i+q_i|\right),$$

$$\prod_{i=m}^{k-1}(p_i-q_i) = \left(\prod_{i=m}^{k-1}\operatorname{sgn} p_i\right)\exp\left(-\sum_{i=m}^{k-1}\ln|p_i+q_i|\right),$$

and the solution of initial value problem (25), (26) can be rewritten:

(27) $$s_{k;m} = \left(\prod_{i=m}^{k-1}\operatorname{sgn} p_i\right)\sinh\left(\sum_{i=m}^{k-1}\ln|p_i+q_i|\right),$$

(28) $$c_{k;m} = \left(\prod_{i=m}^{k-1}\operatorname{sgn} p_i\right)\cosh\left(\sum_{i=m}^{k-1}\ln|p_i+q_i|\right).$$

Note, that the convention

$$\sum_{i=r}^{s} a_i = -\sum_{i=s+1}^{r-1} a_i \quad \text{for} \quad s<r-1, \qquad \sum_{i=r}^{r-1} a_i = 0$$

is used. The above considerations establish the reason, why system (25) and its generalization (19) are called the hyperbolic ones.

The following theorem shows that tha pair of functions (S,C) can be regarded as a certain fundamental system of solutions of (19).

THEOREM 3.1. *Matrices* X_k, U_k *satisfy* (19) *if and only if*

$$X_k = C_k X_0 + S_k U_0,$$
$$U_k = S_k X_0 + C_k U_0.$$

In particular, $\begin{pmatrix} C \\ S \end{pmatrix}$ *satisfies* (19) *with the initial condition*

$$X_0 = I, \qquad U_0 = 0.$$

PROOF. Let X, Y satisfy (29). Then

$$X_{k+1} = C_{k+1}X_0 + S_{k+1}U_0 = (Q_k S_k + P_k C_k)X_0 + (P_k S_k + Q_k C_k)U_0 =$$
$$= P_k(C_k X_0 + S_k U_0) + Q_k(S_k X_0 + C_k U_0) = P_k X_k + Q_k U_k,$$
$$U_{k+1} = S_{k+1}X_0 + C_{k+1}U_0 = (P_k S_k + Q_k C_k)X_0 + (Q_k S_k + P_k C_k)U_0 =$$
$$= Q_k(C_k X_0 + S_k U_0) + P_k(S_k X_0 + C_k U_0) = Q_k X_k + P_k U_k.$$

Hence, X, Y satisfy (19). Since (19) has a unique solution for each initial condition, the proof is complete.

COROLLARY 3.1. *The discrete hyperbolic matrix functions S and C satisfy the following:*

(30) $$S^T C - C^T S = 0 = SC^T - CS^T,$$

(31) $$C^T C - S^T S = I = CC^T - SS^T,$$

$\begin{pmatrix} C \\ S \end{pmatrix}$, $\begin{pmatrix} S \\ C \end{pmatrix}$ *are normalized conjoined bases of* (19).

PROOF. $\begin{pmatrix} C_0 & S_0 \\ S_0 & C_0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Thus, according to (13), $\begin{pmatrix} C_k & S_k \\ S_k & C_k \end{pmatrix}$ is symplectic for each $k \in Z$. Hence, (30), (31) are particular cases of (14). Moreover, $\operatorname{rank}\begin{pmatrix} C_k \\ S_k \end{pmatrix} = \operatorname{rank}\begin{pmatrix} I \\ 0 \end{pmatrix} = n = \operatorname{rank}\begin{pmatrix} 0 \\ I \end{pmatrix} = \operatorname{rank}\begin{pmatrix} S_k \\ C_k \end{pmatrix}$ by (16).

Identity (31) implies that C is nonsingular. Considering (27), (28), we can say that this fact generalizes the classical formula $\cosh x > 0$. (31) can be also regarded to be a generalization of the formula $\cosh^2 x - \sinh^2 x = 1$. It can be also generalized in a different way:

COROLLARY 3.2. *The discrete hyperbolic matrix functions S and C satisfy*

$$\|C\|^2 - \|S\|^2 = n,$$

where $\|\cdot\|$ *denotes the Euclidian norm of an indicated matrix.*

PROOF. Let A be a matrix. Then $AA^T = (c_{ij})$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}$. Thus

$$\operatorname{tr}(AA^T) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2 = \|A\|^2.$$

Hence, (31) implies

$$n = \operatorname{tr} I = \operatorname{tr}(CC^T - SS^T) = \operatorname{tr}(CC^T) - \operatorname{tr}(SS^T) = \|C\|^2 - \|S\|^2.$$

The following theorem shows a connection between the discrete hyperbolic matrix functions and the coefficients of system (19).

THEOREM 3.2. *The matrices* S_k *and* C_k *satisfy*

$$C_{k+1}C_k^T - S_{k+1}S_k^T = P_k, \qquad C_k C_{k+1}^T - S_k S_{k+1}^T = P_k^T, \tag{32}$$

$$S_{k+1}C_k^T - C_{k+1}S_k^T = Q_k, \qquad C_k S_{k+1}^T - S_k C_{k+1}^T = Q_k^T \tag{33}$$

for $k \in Z$.

PROOF. Since S and C satisfy (19), we have

$$S_{k+1} = P_k S_k + Q_k C_k, \qquad C_{k+1} = Q_k S_k + P_k C_k. \tag{34}$$

Right multiplication of the first equality by $-S_k^T$ and of the second one by C_k^T and subsequent addition gives

$$C_{k+1}C_k^T - S_{k+1}S_k^T = Q_k S_k C_k^T + P_k C_k C_k^T - P_k S_k S_k^T - Q_k C_k S_k^T =$$
$$= Q_k (S_k C_k^T - C_k S_k^T) + P_k (C_k C_k^T - S_k S_k^T).$$

Identities (30), (31) now yield the first equality in (32). The second one is the transpose of it.

Right multiplication of the first equality in (34) by C_k^T and of the second one by $-S_k^T$ and subsequent addition gives

$$S_{k+1}C_k^T - C_{k+1}S_k^T = P_k S_k C_k^T + Q_k C_k C_k^T - Q_k S_k S_k^T - P_k C_k S_k^T =$$
$$= P_k (S_k C_k^T - C_k S_k^T) + Q_k (C_k C_k^T - S_k S_k^T).$$

Again, we obtain (33) with the aid of (30), (31).

Now, we are going to derive analogoies of the difference and sum formulae:

$$\sinh(x \mp y) = \sinh x \cosh y \mp \cosh x \sinh y,$$

$$\cosh(x \mp y) = \cosh x \cosh y \mp \sinh x \sinh y.$$

First, let us provide a heuristic computations. Let us consider the scalar initial value problem (25), (26) and its solutions defined by (27), (28). Denote $\varphi_i = \ln|p_i + q_i|$. For $k > m > 0$ we have

$$s_{k;m} = \left(\prod_{i=m}^{k-1} \operatorname{sgn} p_i\right) \sinh\left(\sum_{i=m}^{k-1} \varphi_i\right) = \frac{\prod_{i=0}^{k-1} \operatorname{sgn} p_i}{\prod_{i=0}^{m-1} \operatorname{sgn} p_i} \sinh\left(\sum_{i=0}^{k-1} \varphi_i - \sum_{i=0}^{m-1} \varphi_i\right) =$$

$$= \prod_{i=0}^{k-1} \operatorname{sgn} p_i \prod_{i=0}^{m-1} \operatorname{sgn} p_i \left[\sinh\left(\sum_{i=0}^{k-1} \varphi_i\right) \cosh\left(\sum_{i=0}^{m-1} \varphi_i\right) - \sinh\left(\sum_{i=0}^{m-1} \varphi_i\right) \cosh\left(\sum_{i=0}^{k-1} \varphi_i\right)\right] =$$

$$= s_{k;0} c_{m;0} - s_{m;0} c_{k;0}.$$

THEOREM 3.3. (Difference formulae) *Let $m, k \in Z$. Then*

$$S_{k;m} = S_k C_m^T - C_k S_m^T, \qquad C_{k;m} = C_k C_m^T - S_k S_m^T. \tag{35}$$

PROOF. Set

$$D_k = S_k C_m^T - C_k S_m^T, \qquad E_k = C_k C_m^T - S_k S_m^T.$$

First, note that

$$D_{k+1} = S_{k+1} C_m^T - C_{k+1} S_m^T = (P_k S_k + Q_k C_k) C_m^T - (Q_k S_k + P_k C_k) S_m^T =$$
$$= P_k (S_k C_m^T - C_k S_m^T) + Q_k (C_k C_m^T - S_k S_m^T) = P_k D_k + Q_k E_k,$$

$$E_{k+1} = C_{k+1} C_m^T - S_{k+1} S_m^T = (Q_k S_k + P_k C_k) C_m^T - (P_k S_k + Q_k C_k) S_m^T =$$
$$= Q_k (S_k C_m^T - C_k S_m^T) + P_k (C_k C_m^T - S_k S_m^T) = Q_k D_k + P_k E_k.$$

Next, $D_m = 0$ and $E_m = I$ by (30). Thus D and E solve (19), (21). But the unique solution of this problem is $D_k = S_{k;m}$, $E_k = C_{k;m}$.

COROLLARY 3.3. (Sum Formulae) *Let $m, k \in Z$. Then*

$$S_k = S_{k;m} C_m + C_{k;m} S_m, \quad C_k = C_{k;m} C_m + S_{k;m} S_m.$$

PROOF. Right multiplication of the first formula in (35) by C_m and of the second one by S_m and subsequent addition gives

$$
\begin{aligned}
S_{k;m}C_m + C_{k;m}S_m &= (S_k C_m^T C_m - C_k S_m^T C_m) + (C_k C_m^T S_m - S_k S_m^T S_m) = \\
&= S_k (C_m^T C_m - S_m^T S_m) - C_k (S_m^T C_m - C_m^T S_m) = \\
&= S_k I - C_k 0 = S_k
\end{aligned}
$$

by (30). Thus, the first formula holds. The second one can be proved in a similar way.

COROLLARY 3.4. *Let $m, k \in Z$. Then*

$$S_{m;k} = -S_{k;m}^T, \qquad C_{m;k} = C_{k;m}^T.$$

PROOF. First difference formula (35) yields

$$S_{m;k} = S_m C_k^T - C_m S_k^T = (C_k S_m^T - S_k C_m^T)^T = -S_{k;m}^T$$

and the second one yields

$$C_{m;k} = C_m C_k^T - S_m S_k^T = (C_k C_m^T - S_k S_m^T)^T = C_{k;m}^T.$$

The last corollary represents an analogy between parity of the usual hyperbolic functions (sine is odd and cosine is even) and the introduced discrete hyperbolic matrix functions.

DEFINITION 3.2. *We define the discrete hyperbolic tangent matrix function to be*

$$T_k = C_k^{-1} S_k$$

for $k \in Z$. We define the discrete hyperbolic cotangent matrix function to be

$$K_k = S_k^{-1} C_k$$

for those $k \in Z$ such that S_k is nonsingular.

Let us recall that matrix C_k is nonsingular for each $k \in Z$.

THEOREM 3.4. *The discrete hyperbolic tangent matrix function T satisfies*

$$T_k^T = T_k, \tag{36}$$

$$T_k^2 + (C_k^T C_k)^{-1} = I, \tag{37}$$

(38) $$\Delta T_k = T_{k+1} - T_k = C_{k+1}^{-1} Q_k C_k^{T-1},$$

for each $k \in Z$. *The discrete hyperbolic cotangent matrix function K satisfies*

(39) $$K_k^T = K_k,$$

(40) $$K_k^2 + (S_k^T S_k)^{-1} = I,$$

for $k \in Z$ *such that* S_k *is nonsingular*

(41) $$\Delta K_k = K_{k+1} - K_k = -S_{k+1}^{-1} Q_k S_k^{T-1},$$

for $k \in Z$ *such that* S_k *and* S_{k+1} *are nonsingular.*

PROOF. If S_k is nonsingular, then K_k is well defined. Right multiplication by S^{T-1} and subsequent left multiplication by S^{-1} of the second equality in (30) give $0 = C^T S^{T-1} - S^{-1}C$. Consequently, $(S^{-1}C)^T = S^{-1}C$, which is (39).

Right multiplication by S^{T-1} and subsequent left multiplication by S^{-1} of the second equality in (31) give $S^{-1}S^{T-1} = KK^T - I$, so that (40) holds.

Since matrices S_k and S_{k+1} are nonsingular, matrices K_k and K_{k+1} exist and are symmetric by (39). Left and right multiplication of the first equation in (33) by S_{k+1}^{-1} and S_k^{T-1}, respectively, gives $C_k^T S_k^{T-1} - S_{k+1}^{-1} C_{k+1} = S_{k+1}^{-1} Q_k S_k^{T-1}$. Since $S_{k+1}^{-1} C_{k+1} = K_{k+1}$ and $C_k^T S_k^{T-1} = (S_k^{-1} C_k)^T = K_k^T = K_k$, the identity (41) holds.

The validity of remaining formulae can be verified in a similar way.

COROLLARY 3.5. *The matrix* $C_{k+1}^{-1} Q_k C_k^{T-1}$ *is symmetric for* $k \in \{0,1,2,..., a-1\}$ *and*

(42) $$T_k = \sum_{i=0}^{k-1} C_{i+1}^{-1} Q_i C_i^{T-1}$$

for $k \in \{0,1,2,...,a\}$.

If S_k *is nonsingular for* $k \in \{0,1,2,...,a\}$, *then the matrix* $S_{k+1}^{-1} Q_k S_k^{T-1}$ *is symmetric for* $k \in \{1,2,...,a-1\}$ *and*

(43) $$K_k = K_1 - \sum_{i=1}^{k-1} S_{k+1}^{-1} Q_i S_i^{T-1}$$

for $k \in \{1,2,...,a\}$.

PROOF. Since the left hand side of (38) is symmetric by (36), so is the right hand side. Now, the sum of both sides of (38) from 0 to $k-1$ gives

$$T_k - T_0 = \sum_{i=1}^{k-1} C_{k+1}^{-1} Q_i C_i^{T-1}$$

Using $T_0 = C_0^{-1} S_0 = 0$, we obtain (42). Identity (43) can be verified in a similar way.

4. A NONOSCILATION PROPERTY

We denote the fact that matrix A is symmetric and positive definite by $A > 0$. Recall some basic properties of matrices: If $A > 0$ then $A^{-1} > 0$. If matrix A is nonsingular, then $AA^T > 0$ and $A^T A > 0$. If $A > 0$ and $B > 0$, then $A + B > 0$. If matrix A is nonsingular, then $B > 0$ if only if $A^T BA > 0$.

Corollary 3.5 allows us to formulate the following definition.

DEFINITION 4.1. *A discrete hyperbolic cosine matrix function C has a generalized zero at* $k \in Z$, *if* $C_k^{-1} Q_{k-1} C_{k-1}^{T-1}$ *is not positive definite.*

A discrete hyperbolic sine matrix functions S has a generalized zero at $k \in Z$, *if* S_{k-1} *is nonsingular and either* S_k *is singular or* $S_k^{-1} Q_{k-1} S_{k-1}^{T-1}$ *is not positive definite.*

In other words, if C has not a generalized zero at k, then

$$C_k^{-1} Q_{k-1} C_{k-1}^{T-1} > 0; \tag{44}$$

if S has not a generalized zero at k and S_{k-1} is nonsingular, then S_k is nonsingular and

$$S_k^{-1} Q_{k-1} S_{k-1}^{T-1} > 0.$$

The definition of a generalized zero of C is simpler than the definition of a generalized zero of S since the matrices C_k and C_{k-1} are nonsingular everywhere.

Let $k \in Z$. The matrix C has not a generalized zero at k, if and only if

$$T_k - T_{k-1} > 0$$

by (38).

Let $k \in Z$ and let S_k and S_{k-1} be nonsingular. S has not a generalized zero at k, if only if

$$K_{k-1} - K_k > 0$$

by (41).

Relations (19) and (20) imply immediately

$$S_1 = Q_0, \quad C_1 = P_0, \quad S_{-1} = -Q_{-1}^T, \quad C_{-1} = P_{-1}^T.$$

Thus, C has not a generalized zero at 0, if and only if $Q_{-1}P_{-1}^{-1} > 0$. Since $C_0 = I$ and $C_1 = P_0$, C hast not a generalized zero at $k = 1$, if $P_0^{-1}Q_0 > 0$. S has a generalized zero at 0 if Q_{-1} is nonsingular.

THEOREM 4.1. *Let $a \in N$ and let C have not a generalized zero at $k \in \{0,1,2,...,a\}$. Then Q_k is nonsingular for $k \in \{0,1,2,...,a-1\}$ and S has not a generalized zero at $k \in \{1,2,...,a\}$.*

PROOF. Since $C_k^{-1}Q_{k-1}C_{k-1}^{T-1} > 0$ for $k \in \{1,2,...,a\}$, Q_k is nonsingular for $k \in \{0,1,2,...,a-1\}$. Further, (42) yields

$$T_k = \sum_{i=0}^{k-1} C_{i+1}^{-1} Q_i C_i^{T-1} > 0$$

for $k \in \{1,2,...,a\}$. By Definition 3.2, $S_k = C_k T_k$, thus, S_k is nonsingular for $k \in \{1,2,...,a\}$. For $k \in \{1,2,...,a-1\}$, we have using (38)

$$S_k^T Q_k^{-1} S_{k+1} = T_k^T C_k^T Q_k^{-1} C_{k+1} T_{k+1} = T_k^T C_k^T Q_k^{-1} C_{k+1} (C_{k+1}^{-1} Q_k C_k^{T-1} + T_k) =$$
$$= T_k^T + T_k^T C_k^T Q_k^{-1} C_{k+1} T_k = T_k^T + T_k^T (C_{k+1}^{-1} Q_k C_k^{T-1})^{-1} T_k > 0.$$

Hence $S_{k+1}^{-1} Q_k S_k^{T-1} > 0$ for $k \in \{1,2,...,a-1\}$. Consequently, S has not a generalized zero at $k \in \{2,3,...,a\}$.

Since $S_0 = 0$ is singular, S has not a generalized zero at $k = 1$. This observation completes the proof.

Before formulating a condition for nonexistence of generalized zeroes of C, we provide a heuristic consideration. A discrete cosine scalar function c defined by (28) does not vanish for any $k \in Z$ since $\cosh(x) > 0$ for each $x \in R$ and $|p_1| \geq 1$ according to (24). Moreover,

$$\operatorname{sgn}(c_k^{-1} q_{k-1} c_{k-1}^{-1}) =$$

$$= \operatorname{sgn}\left(\frac{q_{k-1}}{\left(\prod_{i=0}^{k-1} \operatorname{sgn} p_i \prod_{i=0}^{k-2} \operatorname{sgn} p_i \right) \cosh\left(\prod_{i=0}^{k-1} \ln|p_i + q_i| \right) \cosh\left(\prod_{i=0}^{k-2} \ln|p_i + q_i| \right)} \right) =$$

$$= \operatorname{sgn}\left(\frac{q_{k-1}}{p_{k-1}}\right).$$

Thus c has not a generalized zero at $k \in N$ if and only if $q_{k-1}/p_{k-1} > 0$.

THEOREM 4.2. *The matrix C has not a genaralized zero at* $k \in N$ *if and only if* $P_{k-1}^{-1}Q_{k-1} > 0$.

PROOF. Let $k \in N$ and let A_k, B_k be matrices defined by

$$A_k = \prod_{i=0}^{k-1}(P_i + Q_i), \qquad B_k = \prod_{i=0}^{k-1}(P_i - Q_i)$$

for $k > 0$. By (11), matrices A_k, B_k are nonsingular and

$$A_k B_k^T = A_k^T B_k = B_k A_k^T = B_k^T A_k = I.$$

Using (23), we have

$$C_k = \frac{1}{2}(A_k + B_k) = \frac{1}{2}(A_k + A_k^{T-1}) = \frac{1}{2}A_k^{T-1}(A_k^T A_k + I),$$

thus

$$C_k^{-1} = 2(A_k^T A_k + I)^{-1} A_k^T = 2(A_k^T A_k + I)^{-1} B_k^{-1},$$

$$C_k^{T-1} = 2A_k(A_k^T A_k + I)^{T-1} = 2A_k(A_k^T A_k + I)^{-1}.$$

Hence,

$$C_k^{-1}Q_{k-1}C_{k-1}^{T-1} = 4(A_k^T A_k + I)^{-1} B_k^{-1} Q_{k-1} A_{k-1}(A_{k-1}^T A_{k-1} + I)^{-1}.$$

Since $(A_k^T A_k + I)^{-1} > 0$ and $(A_{k-1}^T A_{k-1} + I)^{-1} > 0$, $C_k^{-1}Q_{k-1}C_{k-1}^{T-1} > 0$ if and only if $B_k^{-1}Q_{k-1}A_{k-1} > 0$. Further,

$$B_{k-1}B_k^{-1}Q_{k-1} = B_{k-1}B_k^{-1}Q_{k-1}B_{k-1}^{T-1}B_{k-1}^T = B_{k-1}(B_k^{-1}Q_{k-1}B_{k-1}^{T-1})B_{k-1}^T.$$

that is $B_k^{-1}Q_{k-1}A_{k-1} > 0$ if and only if $B_{k-1}B_k^{-1}Q_{k-1} > 0$.

Let $\lambda_1, \lambda_1, \dots, \lambda_n$ be eigenvalues of matrix $P_{k-1}^{-1}Q_{k-1}$. Since $P^{-1}Q$ is symmetric by (12), $\lambda_1, \lambda_1, \dots, \lambda_n$ are real. There exists orthogonal matrix G (i.e. $G^{-1} = G^T$) such that

$$P_{k-1}^{-1}Q_{k-1} = G^T \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\} G.$$

By (10), $I = P_{k-1}P_{k-1}^T - Q_{k-1}Q_{k-1}^T$, thus

$$0 < P_{k-1}^{-1}P_{k-1}^{T-1} = I - P_{k-1}^{-1}Q_{k-1}Q_{k-1}^{T}P_{k-1}^{T-1} = I - (P_{k-1}^{-1}Q_{k-1})(P_{k-1}^{-1}Q_{k-1})^{T} =$$
$$= G^{T}G - G^{T}\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}G[G^{T}\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}G]^{T} =$$
$$= G^{T}G - G^{T}(\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\})^{2}G = G^{T}[I - \operatorname{diag}\{\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2\}]G =$$
$$= G^{T}\operatorname{diag}\{1-\lambda_1^2, 1-\lambda_2^2, \ldots, 1-\lambda_n^2\}]G.$$

Hence, $1-\lambda_i^2 > 0$, that is $-1 < \lambda_i < 1$ and, consequently,

$$1-\lambda_i > 0, \qquad i = 1, 2, \ldots, n.$$

Now, we have

$$B_{k-1}B_k^{-1}Q_{k-1} = (P_{k-1} - Q_{k-1})^{-1}B_kB_k^{-1}Q_{k-1} =$$
$$= [P_{k-1}(I - P_{k-1}^{-1}Q_{k-1})]^{-1}Q_{k-1} = (I - P_{k-1}^{-1}Q_{k-1})^{-1}P_{k-1}^{-1}Q_{k-1} =$$
$$= [G^{T}G - G^{T}\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}G]^{-1}G^{T}\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}G =$$
$$= [G^{T}(\operatorname{diag}\{1-\lambda_1, 1-\lambda_2, \ldots, 1-\lambda_n\}G]^{-1}G^{T}\operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}G =$$
$$= G^{T}\operatorname{diag}\left\{\frac{\lambda_1}{1-\lambda_1}, \frac{\lambda_2}{1-\lambda_2}, \ldots, \frac{\lambda_n}{1-\lambda_n}\right\}G.$$

Thus, $B_{k-1}B_k^{-1}Q_{k-1} > 0$ if and only if $\lambda_1 > 0$, $\lambda_2 > 0$, ..., $\lambda_n > 0$. This observation completes the proof.

Following [4], we say that the conjoined basis $\begin{pmatrix} X \\ U \end{pmatrix}$ of (19) has a focal point in an interval $(k-1, k]$ if

(45) $$\operatorname{Ker} X_k \subset \operatorname{Ker} X_{k-1} \quad \text{and} \quad X_{k-1}X_k^{\Diamond}Q_{k-1} \geq 0$$

does not hold. Here, ◊ denotes the Moore-Penrose generalized inverse, the inequality ≥ means nonnegative definiteness, and Ker stands for the kernel of a matrix. Note that if the kernel condition in (45) holds, then the matrix $X_{k-1}X_k^{\Diamond}Q_{k-1}$ is really symmetric, see [4, Section 3].

System (19) is said to be nonoscillatory if there exists $n_0 \in N$ and a conjoined basis $\begin{pmatrix} X \\ U \end{pmatrix}$ which has not a focal point in an interval $(k-1, k]$ for any $k \geq n_0$, in the opposite case (19) is said to be oscillatory.

The $2n \times n$ matrix $\binom{C}{S}$ is a conjoined basis of (19) by Corollary 3.1. Since C_k is nonsingular for each $k \in Z$, condition (45) ensuring that the conjoined basis $\binom{C}{S}$ has not a focal point in an interval $(k-1, k]$ can be reduced to

$$C_{k-1} C_k^{-1} Q_{k-1} \geq 0.$$

Since $C_k^{-1} Q_{k-1} C_{k-1}^{T-1} = C_{k-1}^{-1} (C_{k-1} C_k^{-1} Q_{k-1}) C_{k-1}^{T-1}$, the conjoined basis $\binom{C}{S}$ has not a focal point in $(k-1, k]$ if and only if C has not a generalized zero at k or Q_{k-1} is singular (cf. (44)).

Observe that in contrast to continuous hyperbolic system (18) which is always nonoscillatory (see e.g. [10]) discrete hyperbolic system can be oscillatory and the next corollary presents a necessary and sufficient condition for nonoscillation of (19).

COROLLARY 4.1. *Hyperbolic system* (19) *is nonoscillatory if and only if* $P_k^{-1} Q_k \geq 0$ *eventually.*

5. HYPERBOLIC TRANSFORMATION

In this section we prove that any symplectic difference systems (1) satisfying certain additional condition can be transformed into a hyperbolic system.

THEOREM 5.1. *Suppose that symplectic system* (1) *possesses normalized conjoined bases* $Z = \binom{X}{U}$, $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ *such that* $X\tilde{X}^T$ *is positive definite in a given discrete interval. Then, in this interval, there exists* $n \times n$*-matrices* H *and* K *such that* H *is nonsingular,* $H^T K = K^T H$, *and the trnasformation*

$$\begin{pmatrix} S \\ C \end{pmatrix} = \begin{pmatrix} H^{-1} & 0 \\ -K^T & H^T \end{pmatrix} \begin{pmatrix} X \\ U \end{pmatrix} \tag{46}$$

transforms symplectic system (1) *into the hyperbolic system* (9) *without changing the oscillatory behavior, i.e., a conjoined basis* $\binom{X}{U}$ *of* (1) *has a focal* *point in* $(k-1,k]$ *if and only if* $\begin{pmatrix} H^{-1}X \\ -K^T X + H^T U \end{pmatrix}$ *has there a focal point. Moreover, the matrices* $\mathcal{P}$ *and* $\mathcal{Q}$ *are given by formulae*

$$\mathcal{P}_k = H_{k+1}^{-1}\mathcal{A}_k H_k + H_{k+1}^{-1}\mathcal{B}_k K_k, \qquad \mathcal{Q}_k = H_{k+1}^{-1}\mathcal{B}_k H_k^{T-1}. \tag{47}$$

PROOF. Let $Z = \begin{pmatrix} X \\ U \end{pmatrix}$, $\widetilde{Z} = \begin{pmatrix} \widetilde{X} \\ \widetilde{U} \end{pmatrix}$ be normalized conjoined bases of (1) such that $X\widetilde{X}^T$ is positive definite, H be any $n \times n$ matrix satisfying $HH^T = 2X\widetilde{X}^T$ and let $K = (U\widetilde{X}^T + \widetilde{U}X^T)H^{T-1}$. Then H is nonsingular and using (14), (15)

$$\begin{aligned} H^T K - K^T H &= H^T(U\widetilde{X}^T + \widetilde{U}X^T)H^{T-1} - H^{-1}(\widetilde{X}U^T + X\widetilde{U}^T)H = \\ &= H^{-1}[HH^T(U\widetilde{X}^T + \widetilde{U}X^T) - (\widetilde{X}U^T + X\widetilde{U}^T)HH^T]H^{T-1} = \\ &= 2H^{-1}[X\widetilde{X}^T(U\widetilde{X}^T + \widetilde{U}X^T) - (\widetilde{X}U^T + X\widetilde{U}^T)X\widetilde{X}^T]H^{T-1} = 0. \end{aligned}$$

Denote $Z_k = H_k^{-1}X_k$, then

$$\begin{aligned} Z_{k+1} &= H_{k+1}^{-1}X_{k+1}X_k^{-1}H_k H_k^{-1} = H_{k+1}^{-1}X_{k+1}X_k^{-1}H_k H_k^{T-1}Z_k = \\ &= 2H_{k+1}^{-1}(\mathcal{A}_k X_k + \mathcal{B}_k U_k)X_k^{-1}X_k\widetilde{X}_k^T H_k^{T-1}Z_k = \\ &= H_{k+1}^{-1}[\mathcal{A}_k H_k H_k^T + \mathcal{B}_k U_k\widetilde{X}_k^T) + \mathcal{B}_k(I + \widetilde{U}_k X^T)]H_k^{T-1}Z_k = \\ &= \{H_{k+1}^{-1}[\mathcal{A}_k H_k + \mathcal{B}_k(U_k\widetilde{X}_k^T + \widetilde{U}_k X^T)H_k^{T-1}] + H_{k+1}^{-1}\mathcal{B}_k H_k^{T-1}\}Z_k = \\ &= (\mathcal{P}_k + \mathcal{Q}_k)Z_k \end{aligned}$$

with $\mathcal{P}$, $\mathcal{Q}$ given by (47). Similarly, for $\widetilde{Z} = H^{-1}\widetilde{X}$ we have (by the same computation as above)

$$\widetilde{Z}_{k+1} = (\mathcal{P}_k - \mathcal{Q}_k)\widetilde{Z}_k.$$

Consequently, if we denote

$$C_k = \frac{Z_k + \widetilde{Z}_k}{2}, \qquad S_k = \frac{Z_k - \widetilde{Z}_k}{2},$$

then S, C satisfy the hyperbolic system (9) with $\mathcal{P}$, $\mathcal{Q}$ given by (47). To finish the proof, i.e. to show that (46) really transforms (1) into (9), we need to verify that

$$\frac{U + \widetilde{U}}{2} = KC + H^{T-1}S, \qquad \frac{U - \widetilde{U}}{2} = KS + H^{T-1}C.$$

For example, concerning the second identity (computations in proving the first identity are the same), we have

$$KS + H^{T-1}C = 2(\widetilde{U}X^T + U\widetilde{X}^T)H^{T-1}H^{-1}\left(\frac{X-\widetilde{X}}{2}\right) + 2H^{T-1}H^{-1}\left(\frac{X+\widetilde{X}}{2}\right) =$$

$$= \frac{1}{2}\{(\widetilde{U}X^T + U\widetilde{X}^T)\widetilde{X}^{T-1}X^{-1}(X-\widetilde{X}) + \widetilde{X}^{T-1}X^{-1}(X+\widetilde{X})\} =$$

$$= \frac{1}{2}\{(\widetilde{U}X^T + U\widetilde{X}^T)(\widetilde{X}^{T-1} - X^{T-1}) + \widetilde{X}^{T-1} + X^{T-1}\} =$$

$$= \frac{1}{2}\{\widetilde{U}X^T\widetilde{X}^{T-1} + U - \widetilde{U} - U\widetilde{X}^TX^{T-1} + X^{T-1} + \widetilde{X}^{T-1}\} =$$

$$= \frac{1}{2}\{U - \widetilde{U} + (\widetilde{U}X^T + I)\widetilde{X}^{T-1} - (U\widetilde{X}_I^T)X^{T-1}\} = \frac{1}{2}(U-\widetilde{U}),$$

here we have used identities (15). The proof is complete.

REMARK 1. Observe that any hyperbolic system (9) can be transformed into another hyperbolic system with symmetric and positive semidefinite matrices $\widetilde{Q}_k$ at the position of the matrices Q_k, using the transformation

$$\begin{pmatrix}\widetilde{x}\\ \widetilde{u}\end{pmatrix} = \begin{pmatrix}H_k^{-1} & 0\\ 0 & H_k^T\end{pmatrix}\begin{pmatrix}x\\ u\end{pmatrix},$$

where the matrices H_k are recursively defined by $H_0 = I$ and $H_{k+1} = G_k^{-1}H_k$ with orthogonal matrices G_k, i.e., $G_k^TG_k = I$, such that G_kQ_k are symmetric and positive semidefinite. Such matrices G_k exist according to the well-known principle of polar decomposition, see e.g. [9, Theorem 3.1.9 (c)]. This setting implies that all matrices H_k are orthogonal and hence that the transformation matrices $\begin{pmatrix}H_k^{-1} & 0\\ 0 & H_k^T\end{pmatrix}$ are symplectic. The transformed system then reads

$$\widetilde{x}_{k+1} = \widetilde{P}_k\widetilde{x}_k + \widetilde{Q}_k\widetilde{u}_k, \qquad \widetilde{u}_{k+1} = \widetilde{Q}_k\widetilde{x}_k + \widetilde{P}_k\widetilde{u}_k,$$

where

$$\widetilde{P}_k = H_{k+1}^{-1}P_kH_k \quad \text{and} \quad \widetilde{Q}_k = H_{k+1}^{-1}Q_kH_k = H_k^{-1}G_kQ_kH_k^{T-1}$$

so that indeed all matrices $\widetilde{Q}_k$ are symmetric and positive semidefinite.

REMARK 2. This remark concerns the assumption of the existence of a pair of normalized conjoined bases $\begin{pmatrix} X \\ U \end{pmatrix}$, $\begin{pmatrix} \widetilde{X} \\ \widetilde{U} \end{pmatrix}$ such that $X\widetilde{X}^T$ is positive definite. If (1) is nonoscillatory, this system possesses recessive and dominant solutions $\begin{pmatrix} \overline{X} \\ \overline{U} \end{pmatrix}$, $\begin{pmatrix} \hat{X} \\ \hat{U} \end{pmatrix}$, respectively, such that $\overline{X}^T\hat{U} - \overline{U}\hat{X} = I$. Then

$$\begin{pmatrix} X \\ U \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \hat{X} + \overline{X} \\ \hat{U} + \overline{U} \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{X} \\ \widetilde{U} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \hat{X} - \overline{X} \\ \hat{U} - \overline{U} \end{pmatrix}.$$

is the normalized pair of conjoined bases for which $X\widetilde{X}^T = \hat{X}\hat{X}^T - \overline{XX}^T$ is positive definite eventually, since

$$\lim_{k\to\infty} \hat{X}_k^{-1}\overline{X}_k = 0.$$

Consequently, any nonoscillatory symplectic difference system (1) can be transformed into hyperbolic system.

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