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PROPERTIES OF RECORD VALUES WITH RANDOM INDEX CHARACTERIZING EXPONENTIAL DISTRIBUTION

ABSTRACT: In this paper we give some characterizations of the exponential distribution by distributional property, expected value and failure rate of record values. The index of record values has the geometric distribution

KEY WORDS: characterization, record value, random index.

Let X be a nonnegative random variable, and let $F(x) = P(X < x)$ be its distribution function. Let $K(x) = 1 - F(x) = P(X \geq x)$ be the survival function of X, and let $f(x)$ be the density of X, and let $E(X)$ be the expected value of X.

A distribution F is NBU (NWU) if $K(x+y) \leq (\geq) K(x)K(y)$ for $x \geq 0$ $y \geq 0$ ([1]).

An absolutely continuous distribution F is an increasing failure rate (IFR) distribution (a decreasing failure rate (DFR) distribution), if the failure rate $r(x) = f(x)/K(x)$, $x \in D$, $D = \{u : K(u) > 0\}$, is nondecreasing (nonincreasing) ([1]).

We say that F has increasing failure rate average (IFRA) (decreasing failure rate average (DFRA)), if $-(1/x)\log K(x)$ is nondecreasing (nonincreasing) in $x > 0$ ([1]).

We say that X is exponentially distributed if

(1) $$F(x) = 1 - e^{-\lambda x}, \quad x > 0, \quad \text{for some } \lambda > 0.$$

We say that ν is geometrically distributed if

(2) $$P(\nu = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots, \quad \text{for some } 0 < p < 1.$$

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed random variables. Define the sequence of record times $(L(n), n \geq 1)$ in the following way: $L(1) = 1$, $L(n) = \{j : X_j > X_{L(n-1)}\}$, $n \geq 2$. Then the sequence $(R_n, n \geq 1)$, where $R_n = X_{L(n)}$, is called the sequence of record values of $(X_n, n \geq 1)$.

The following lemma is valid:

LEMMA 1 ([2]). *Let* $X \geq 0$ *be a random variable with a distribution function* F. *Assume that* $E(X)$ *is finite. Then*

$$E(X) = \int_0^\infty [1 - F(x)]dx.$$

The following theorem is given in [5] (Theorem 4.5.2, p.129):

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed positive random variables with a continuous distribution function F. Assume that the limit

$$\lim_{x \to 0^+} \frac{F(x)}{x} = \lambda, \quad 0 < \lambda < \infty.$$

Moreover, assume that ν is a geometric random variable independent of the sequence $(X_n, n \geq 1)$, and condition (2) holds. The random variables X_1 and pR_ν are identically distributed if and only if F is a distribution function of the exponential law.

With the additional assumptions: $F \in$ IFRA or $F \in$DFRA we can characterize the exponential distribution (1) by $E(X_1) = E(pR_\nu)$ ([4]). From [3] we get, that property

The random variables X_1 and $R_{\nu+1} - R_\nu$ are identically distributed, characterizes the exponential distribution in the classes NBU and NWU. The above property can be replaced by

$$E(X_1) = E(R_{\nu+1} - R_\nu).$$

In the present paper we will characterize the exponential distribution (1) by statistics pR_ν and $R_{\nu+1} - R_\nu$.

THEOREM 1. *Let* $(X_n, n \geq 1)$ *be a sequence of independent and identically distributed nonnegative random variables with an absolutely continuous distribution function F such that* $\inf\{x : F(x) > 0\} = 0$. *Moreover, let the density function* f *of* X_n's *satisfy the condition* $f(x) > 0$ *for* $x > 0$. *Assume that* ν *is a geometric random variable independent of the sequence* $(X_n, n \geq 1)$, *and the condition* (2) *holds. Moreover, let the following condition hold*

$$K(u+z) \leq (\geq) K(u)\left[K\left(\frac{z}{p}\right)\right]^p \quad \text{for } u, z \geq 0. \tag{3}$$

Then the random variables

(4) $$pR_\nu \quad and \quad R_{\nu+1} - R_\nu$$

are identically distributed if and only if F *is the exponential distribution function* (1).

Proof. The density function of pR_ν is of the form

(5) $$f_{pR_\nu}(z) = \left[K\left(\frac{z}{p}\right)\right]^{p-1} f\left(\frac{z}{p}\right) \quad \text{for } z > 0,$$

and characteristic function

(6) $$\varphi_{pR_\nu}(t) = \int_0^\infty e^{itz}\left[K\left(\frac{z}{p}\right)\right]^{p-1} f\left(\frac{z}{p}\right)dz = 1 + it\int_0^\infty e^{itz}\left[K\left(\frac{z}{p}\right)\right]^{p} dz, \quad t \in R.$$

The density function of $R_{\nu+1} - R_\nu$ is of the form

(7) $$f_{R_{\nu+1}-R_\nu}(z) = p\int_0^\infty f(u)[K(u)]^{p-2} f(u+z)du, \qquad z > 0,$$

and characteristic function

(8) $$\varphi_{R_{\nu+1}-R_\nu}(t) = p\int_0^\infty\int_0^\infty e^{itz} f(u)[K(u)]^{p-2} f(u+z)dudz =$$

$$= 1 + pit\int_0^\infty\int_0^\infty e^{itz} f(u)[K(u)]^{p-2} K(u+z)dudz, \qquad t \in R.$$

First suppose that the statistics (4) are identically distributed. Then $\varphi_{pR_\nu}(t) = \varphi_{R_{\nu+1}-R_\nu}(t)$ for $t \in R$. Comparing (6) and (8), and using the following formula:

(9) $$\int_0^\infty pf(u)[K(u)]^{p-1} du = 1,$$

we get

(10) $$\int_0^\infty\int_0^\infty pf(u)[K(u)]^{p-2} e^{itz}\left\{K(u)\left[K\left(\frac{z}{p}\right)\right]^p - K(u+z)\right\}dudz = 0.$$

Let the condition (3) be satisfied. Then

$$K(u)\left[K\left(\frac{z}{p}\right)\right]^p - K(u+z) \geq (\leq)\, 0.$$

By the assumptions and the condition (10) we have

(11) $$K(u+z) = K(u)\left[K\left(\frac{z}{p}\right)\right]^p, \qquad u,z \ge 0.$$

Substituting $u=0$, we get

$$K(z) = \left[K\left(\frac{z}{p}\right)\right]^p, \qquad z \ge 0.$$

Therefore the equality (11) is of the form

(12) $$K(u+z) = K(u)\,K(z), \qquad u,z \ge 0.$$

The only solution of (12) among nondegenerate distribution functions is the exponential distribution (1).

If X_1 has the distribution function (1), then the statistics (4) have the same distribution function as X_1.

THEOREM 2. *Assume that the assumptions of Theorem* 1 *are satisfied. Then* X_1 *has the exponential distribution* (1) *if and only if*

(13) $$E(pR_\nu) = E(R_{\nu+1} - R_\nu).$$

PROOF. From (5) we obtain

(14) $$K_{pR_\nu}(u) = \int_u^\infty f_{pR_\nu}(z)dz = \left[K\left(\frac{u}{p}\right)\right]^p \quad \text{for} \quad u>0.$$

By virtue of (7) we have

(15) $$K_{R_{\nu+1}-R_\nu}(u) = \int_u^\infty f_{R_{\nu+1}-R_\nu}(z)dz = p\int_0^\infty f(z)[K(z)]^{p-2}K(z+u)dz \quad \text{for} \quad u>0.$$

By Lemma 1 we can write the condition (13) as

$$p\int_0^\infty\int_0^\infty f(z)[K(z)]^{p-2}K(z+u)dzdu = \int_0^\infty \left[K\left(\frac{u}{p}\right)\right]^p du.$$

Since the formula (9) is valid, so the above condition is of the following form:

$$p\int_0^\infty\int_0^\infty f(z)[K(z)]^{p-2}K(z+u)dzdu = p\int_0^\infty\int_0^\infty f(z)[K(z)]^{p-1}\left[K\left(\frac{u}{p}\right)\right]^p dzdu, \text{ i.e.,}$$

(16) $$p\int_0^\infty\int_0^\infty f(z)[K(z)]^{p-2}\left\{K(z+u) - K(z)\left[K\left(\frac{u}{p}\right)\right]^p\right\}dzdu = 0.$$

Let F satisfy the condition (3). Then

$$K(z+u)-K(z)\left[K\left(\frac{u}{p}\right)\right]^{p}\leq(\geq)\,0.$$

By the assumptions of our theorem, and by (16) we obtain the equation (11). Next, analogously as in the proof of Theorem 1, we show that F is of the form (1).

If X_1 has the distribution function (1), then the statistics (4) are identically distributed. Therefore the equality (14) is true.

THEOREM 3. *Let* $(X_n, n\geq 1)$ *be a sequence of independent and identically distributed nonnegative random variables with an absolutely continuous distribution function* F *such that* $\inf\{x: F(x)>0\}=0$ *and* $F\in\text{IFR}$ *or* $F\in\text{DFR}$. *Moreover, let the density function* f *of* X_n'*s satisfy the condition* $f(x)>0$ *for* $x>0$. *Assume that* ν *is a geometric random variable independent of the sequence* $(X_n, n\geq 1)$, *and the condition* (2) *holds. Then* F *is the exponential distribution function* (1) *if and only if*

$$r_{pR_\nu}(0) = r_{R_{\nu+1}-R_\nu}(0). \tag{17}$$

PROOF. By formulas (5) and (14) we get

$$r_{pR_\nu} = \frac{f_{pR_\nu}(z)}{K_{pR_\nu}(z)} = \frac{f\left(\frac{z}{p}\right)}{K\left(\frac{z}{p}\right)}.$$

Hence $r_{pR_\nu}(0)=f(0)$. From (7) and (15) we obtain

$$r_{R_{\nu+1}-R_\nu}(z)=\frac{f_{R_{\nu+1}-R_\nu}(z)}{K_{R_{\nu+1}-R_\nu}(z)}=\frac{\int_0^\infty f(u)[K(u)]^{p-2}f(u+z)du}{\int_0^\infty f(u)[K(u)]^{p-2}K(u+z)du}.$$

Whence

$$r_{R_{\nu+1}-R_\nu}(0) = \frac{\int_0^\infty [f(u)]^2[K(u)]^{p-2}du}{\int_0^\infty f(u)[K(u)]^{p-1}du}.$$

The condition (17) can be written as

$$f(0) = \frac{\int_0^\infty [f(u)]^2 [K(u)]^{p-2}\, du}{\int_0^\infty f(u)[K(u)]^{p-1}\, du}, \quad \text{i.e.}$$

(18) $$\int_0^\infty f(u)[K(u)]^{p-1}\left[f(0) - \frac{f(u)}{K(u)} \right] du = 0.$$

Denoting by $r(x)$ the failure rate of random variable X_1, we can write the equality (18) in the form

(19) $$\int_0^\infty f(u)[K(u)]^{p-1}[r(0) - r(u)]du = 0.$$

Let $F \in$IFR (or $F \in$DFR). Then $r(0) \le (\ge)\, r(u)$. From the assumptions and (19) we get $r(0) = r(u)$ for $u > 0$. Therefore F is the exponential distribution function (1).

If X_1 has the distribution function (1), then the statistics (4) are identically distributed. Therefore the equality (17) is true.

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