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N. PARHI[1)] AND A.K. TRIPATHY[2)]

ON ASYMPTOTIC BEHAVIOUR AND OSCILLATION OF FORCED FIRST ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

ABSTRACT: Oscillatory and asymptotic behaviour of solutions of forced first order nonlinear neutral delay difference equations of the from

$$\Delta(y_n \pm y_{n-m}) + q_n G(y_{n-k}) = f_n, \quad n \geq 0,$$

is studied under appropriate assumptions on sequences of real numbers $\{q_n\}$ and $\{f_n\}$ and $G \in C(R, R)$. The behaviour of solutions of

$$\Delta(y_n + p_n y_{n-m}) + q_n G(y_{n-k}) = f_n, \qquad n \geq 0,$$

is also discussed where $\{p_n\}$ is allowed to change sign.

KEY WORDS: neutral difference equations, oscillation, nonoscillation, asymptotic behaviour.

1. INTRODUCTION

Several papers concerning oscillation, nonoscillation and asymptotic behaviour of solutions of dely and neutral difference equations of first order have appeared recently (see [1-3, 5-8, 10]). In [7], the present authors have studied oscillatory and asymptotic behaviour of solutions of forced first order nonlinear neutral difference equations of the form

$$\Delta(y_n - p y_{n-m}) + q_n G(y_{n-k}) = f_n, \tag{1}$$

where Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, p, f_n, q_n $(n = 0, 1, 2, \ldots)$ are real numbers with $q_n \geq 0$, $f_n \geq 0$, $G \in C(R, R)$ such that $xG(x) > 0$ for $x \neq 0$ and G is nondecreasing and $m, k \in \{0, 1, 2, \ldots\}$. Further, p is allowed to take values in different ranges, viz., $0 \leq p < 1$, $1 < p$ and $p < 0$ with $p \neq -1$. It is shown that $\sum_{n=0}^{\infty} q_n = \infty$ is sufficient for every solution of (1) to oscillate or tend to zero as $n \to \infty$.

In the present work, an attempt is made to study oscillation and asymptotic behaviour of solutions of

$$\Delta(y_n + y_{n-m}) + q_n G(y_{n-k}) = f_n \tag{2}$$

and

$$\Delta(y_n - y_{n-m}) + q_n G(y_{n-k}) = f_n \tag{3}$$

where $\{q_n\}$ and $\{f_n\}$, $n = 0, 1, 2, \ldots$, are sequences of real numbers such that $\sum_{n=0}^{\infty} |f_n| < \infty$ and G is same as in (1). Yu and Wang [10] have provided an example to show that the conditions

$$(\mathrm{H}_1) \qquad q_n \geq 0, \qquad \sum_{n=0}^{\infty} q_n = \infty$$

are not enough for every solution of

$$\Delta(y_n + y_{n-m}) + q_n y_{n-k} = 0$$

to oscillate or tend to zero as $n \to \infty$. Thus it is natural to assume conditions stronger than (H_1) for the study of oscillatory and asymptotic behaviour of solutions of (2) and (3). The results in this paper extend the work in [6,10].

Let $\ell = \max\{k, m\}$. By a solution of (2) (or (3)) on $[N, \infty) = \{N, N+1, \ldots\}$, where $N \geq 0$ is an integer, we mean a sequence $\{y_n\}$ of real numbers which is defined for $n \geq N - \ell$ and which satisfies (2) (or (3)) for $n \geq N$. A solution $\{y_n\}$ of (2) (or (3)) on $[N, \infty)$ is said to be nonoscillatory if there exists an integer $N_1 \geq N$ such that $y_n y_{n+1} > 0$ for $n \geq N_1$; otherwise, $\{y_n\}$ is said to be oscillatory.

In Section 2 we study Eqs. (2) and (3). Section 3 deals with asymptotic and oscillatory behaviour of solutions of equations of the form

$$(4) \qquad \Delta(y_n + p_n y_{n-m}) + q_n G(y_{n-k}) = f_n,$$

where f_n, q_n and G are same as in (2) (or (3)) and $\{p_n\}$ is a sequence of real numbers with p_n changing sign. We may note that not much is known in this case. For the study of (4) where p_n lies in different ranges but with constant sign, one is referred to [5, 6, 8]. We need the following lemma for our use in the sequel:

LEMMA A ([9], p. 38) *Let* $\{u_n\}$ *and* $\{v_n\}$ *be sequences of real numbers defined for* $n \geq n_0 \geq 0$. *Then*

$$\liminf_{n\to\infty} u_n + \liminf_{n\to\infty} v_n \leq \liminf_{n\to\infty} (u_n + v_n) \leq$$

$$\leq \limsup_{n\to\infty} u_n + \liminf_{n\to\infty} v_n \left(or \quad \liminf_{n\to\infty} u_n + \limsup_{n\to\infty} v_n \right)$$

$$\leq \limsup_{n\to\infty} (u_n + v_n) \leq \limsup_{n\to\infty} u_n + \limsup_{n\to\infty} v_n$$

provided that no sum is of the form $\infty - \infty$.

2. OSCILLATION OF EQS. (2) AND (3)

We begin with the following example which is an extension of an example in [10].

EXAMPLE 1. *Define, for* $n \geq 0$,

$$B_n = \begin{cases} 0 & \text{if } n \text{ is an even integer,} \\ 1 & \text{if } n \text{ is an odd integer.} \end{cases}$$

Hence $B_n + B_{n-1} = 1$ *for* $n \geq 1$. *Consider*

$$\Delta(y_n + y_{n-1}) + q_n y_{n-1} = e^{-n}\left[e^{-1} - e + \frac{e(e^2-1)}{e^{n+1}B_{n-1} + e^2}\right], \tag{5}$$

for $n \geq 1$, *where*

$$q_n = \frac{e^2 - 1}{e^{n+1}B_{n-1} + e^2} > 0, \quad n \geq 1.$$

Hence $\sum_{n=1}^{\infty} q_n = \infty$, *because*

$$\sum_{n=1}^{\infty} q_n > \sum_{n=0}^{\infty} q_{2n+1} = \infty.$$

Further,

$$f_n = e^{-n}\left[e^{-1} - e + \frac{e(e^2-1)}{e^{n+1}B_{n-1} + e^2}\right] \quad \text{and} \quad e^{n+1}B_{n-1} + e^2 \geq e^2$$

imply that

$$\sum_{n=1}^{\infty} |f_n| \leq 2e \sum_{n=1}^{\infty} e^{-n} < \infty.$$

It is easy to verify that $y_n = B_n + 2e^{-n}$ is a positive solution of (5) with $\limsup_{n\to\infty} y_n = 1$.

REMARK. The above example indicates that the assumptions $q_n \geq 0$ and $\sum_{n=0}^{\infty} q_n = \infty$ are not sufficient for every solution of (2) to oscillate or tend to zero as $n \to \infty$.

THEOREM 1. *Suppose that*

(6) $$G(u+v) \leq \lambda(G(u)+G(v))$$

for every $u>0$ *and* $v>0$ *and for some* $\lambda>0$, *and*

(7) $$G(u+v) \geq \mu(G(u)+G(v))$$

for every $u<0$ *and* $v<0$ *and for some* $\mu>0$. *Let* $q_n \geq 0$. *If* $\sum_{n=m}^{\infty} q_n^* = \infty$, *then every solution of* (2) *oscillates or tends to zero as* $n \to \infty$, *where* $q_n^* = \min\{q_n, q_{n-m}\}$, $n \geq m$.

REMARK. $\sum_{n=m}^{\infty} q_n^* = \infty$ implies that $\sum_{n=0}^{\infty} q_n = \infty$. However, the converse is not necessarily true. Defining

$$q_n = \begin{cases} \frac{1}{n^2}, & \text{for n odd,} \\ n^2, & \text{for n even,} \end{cases}$$

we notice that $\sum_{n=0}^{\infty} q_n > \sum_{i=0}^{\infty} q_{2i} = 4\sum_{i=0}^{\infty} i^2 = \infty$ and, for $m=1$,

$$\sum_{n=1}^{\infty} q_n^* = 1 + 2\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} < \infty.$$

PROOF OF THE THEOREM. Let $\{y_n\}$ be a nonoscillatory solution of (2) and assume that $\{y_n\}$ is eventually positive. Hence there exists $N_1 \geq N$ such that $y_n > 0$ for $n \geq N_1$. The proof is similar for the case $y_n < 0$, $n \geq N_1$. Setting, for $n \geq N_1 + \ell$,

(8) $$z_n = y_n + y_{n-m} > 0 \quad \text{and} \quad w_n = z_n - \sum_{i=0}^{n-1} f_i,$$

we obtain from (2) that

(9) $$\Delta w_n = -q_n G(y_{n-k}) \leq 0.$$

Hence $w_n < 0$ for $n \geq N_2 \geq N_1 + \ell$ or $w_n > 0$ for $n \geq N_2$. Let $w_n < 0$ for $n \geq N_2$. We claim that the solution $\{y_n\}$ is bounded. Otherwise, $\{y_n\}$ is unbounded. Hence there exists a sub-sequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \to \infty$ as $j \to \infty$. Thus

$$w_{n_j} \geq y_{n_j} - \sum_{i=0}^{n_j-1} f_i$$

imply that $w_{n_j} > 0$ for large j, a contradiction. Consequently, $\{w_n\}$ is bounded. This implies that $\lim_{n\to\infty} w_n$ exists and hence $\lim_{n\to\infty} z_n$ exists. Suppose that $\lim_{n\to\infty} z_n = a,\ 0 < a < \infty$. Then $z_n > b > 0$ for $n \geq N_3 > N_2$. From (2), (6) and (8) we obtain, for $n \geq N_4 \geq N_3 + \ell$,

$$\begin{aligned}
\lambda(f_n + f_{n-m}) &= \lambda\Delta(z_n + z_{n-m}) + \lambda q_n G(y_{n-k}) + \lambda q_{n-m} G(y_{n-k-m}) \geq \\
&\geq \lambda\Delta(z_n + z_{n-m}) + \lambda q_n^* (G(y_{n-k}) + G(y_{n-k-m})) \geq \\
&\geq \lambda\Delta(z_n + z_{n-m}) + q_n^* G(y_{n-k} + y_{n-k-m}) = \\
&= \lambda\Delta(z_n + z_{n-m}) + q_n^* G(z_{n-k}) > \\
&> \lambda\Delta(z_n + z_{n-m}) + q_n^* G(b),
\end{aligned}$$

that is,

$$\lambda \sum_{i=N_4}^{n-1} \Delta(z_i + z_{i-m}) < \lambda \sum_{i=N_4}^{n-1} f_i + \lambda \sum_{i=N_4}^{n-1} f_{i-m} - G(b) \sum_{i=N_4}^{n-1} q_i^*$$

that is

$$\lambda(z_n + z_{n-m}) < \lambda(z_{N_4} + z_{N_4-m}) + \lambda \sum_{i=N_4}^{n-1} f_i + \lambda \sum_{i=N_4}^{n-1} f_{i-m} - G(b) \sum_{i=N_4}^{n-1} q_i^*$$

From the given hypothesis it follows that $z_n < 0$ for large n, a contradiction. Hence $\lim_{n\to\infty} z_n = 0$. Since $z_n > y_n$ for $n \geq N_2$, then $\limsup_{n\to\infty} y_n = 0$. Thus $\lim_{n\to\infty} y_n = 0$.

Next suppose that $w_n > 0$ for $n \geq N_2$. Then $\lim_{n\to\infty} w_n$ exists and hence $\lim_{n\to\infty} z_n$ exists. Proceeding as above we may show that $\lim_{n\to\infty} y_n = 0$. Thus the theorem is proved.

REMARK. The prototype of G satisfying (6) and (7) (see [4, p. 292]) is

$$G(u) = |u|^\gamma \operatorname{sgn} u, \quad \gamma > 0.$$

THEOREM 2. *If* $q_n \geq 0$ *and if, for every, subsequence* $\{n_i\}$ *of* $\{n\}$, $\sum_{i=0}^{\infty} q_{n_i} = \infty$, *then every solution of* (2) *oscillates or tends to zero as* $n \to \infty$.

REMARK. If $\sum_{i=0}^{\infty} q_{n_i} = \infty$ for every subsequence $\{n_i\}$ of $\{n\}$, then $\sum_{n=0}^{\infty} q_n = \infty$.

However, the converse is not necessarily true (see Example 2 below).

PROOF OF THE THEOREM. Let $\{y_n\}$ be a nonoscillatory solution of (2) on $[N,\infty)$, $N \geq 0$, and as before let $y_n > 0$ for $n \geq N_1 \geq N$. Setting z_n and w_n as in (8) for $n \geq N_1 + \ell$, we get (9). Hence $w_n < 0$ for $n \geq N_2 \geq N_1 + \ell$ or $w_n > 0$ for $n \geq N_2$. Let $w_n < 0$ for $n \geq N_2$. Hence $\{y_n\}$ is bounded; otherwise, there exists a sub-sequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $n_j \to \infty$ and $y_{n_j} \to \infty$ as $j \to \infty$. Thus $w_{n_j} > 0$ for large *j*, a contradiction. This implies that the limit of w_n exists as $n \to \infty$. If $\limsup_{n\to\infty} y_n = \alpha$, $0 < \alpha < \infty$, then there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $n_i \to \infty$ as $i \to \infty$ and $\lim_{i\to\infty} y_{n_i - k} = \alpha$. Hence $y_{n_i - k} > \beta > 0$ for $i \geq N_3 \geq N_2$. Since $\sum_{i=0}^{\infty} q_{n_i} = \infty$, then from (9) we obtain

$$\infty = G(\beta)\sum_{i=N_3}^{\infty} q_{n_i} \leq \sum_{i=N_3}^{\infty} q_{n_i} G(y_{n_i-k}) = -\sum_{i=N_3}^{\infty} \Delta w_{n_i},$$

and

$$\sum_{i=N_3}^{r-1} \Delta w_{n_i} = \sum_{i=N_3}^{r-1} (w_{n_{i+1}} - w_{n_i}) = w_{n_r} - w_{n_{N_3}} > w_{n_r}$$

implies that

$$\sum_{i=N_3}^{\infty} \Delta w_{n_i} \geq \lim_{r\to\infty} w_{n_r} > -\infty,$$

a contradiction. Hence $\limsup_{n\to\infty} y_n = 0$. Thus $\lim_{n\to\infty} y_n = 0$. If $w_n > 0$ for $n \geq N_2$, then $\lim_{n\to\infty} w_n$ exists. Proceeding as above we get $\lim_{n\to\infty} y_n = 0$. The proof is similar for the case $y_n < 0$, $n \geq N_1$. The proof of the theorem is complete.

EXAMPLE 2. *Consider*

(10) $$\Delta(y_n + y_{n-2}) + q_n y_{n-1}^3 = f_n, \qquad n \geq 2,$$

where

$$q_n = \frac{B_{n-1}}{n^2} + n^2 B_n > 0,$$

$$f_n = \frac{1}{(n+2)^2} - \frac{1}{(n+1)^2} + \frac{1}{n^2} - \frac{1}{(n-1)^2} + \frac{B_{n-1}}{n^8} + \frac{B_n}{n^4}$$

and B_n *is same as in Example 1. Clearly,* $\sum_{n=2}^{\infty} |f_n| < \infty$.

Further,

$$q_n^* = \min\{q_n, q_{n-2}\} = \begin{cases} 1/n^2, & \text{if } n \text{ is even,} \\ (n-2)^2, & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\sum_{n=2}^{\infty} q_n^* \geq \sum_{i=0}^{\infty} (2i+1)^2 = \infty \quad \text{and hence} \quad \sum_{n=2}^{\infty} q_n = \infty.$$

However,

$$\sum_{n=1}^{\infty} q_{2n} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \tag{11}$$

Every solution of (10) oscillates or tends to zero as $n \to \infty$ by Theorem 1. In particular, $\{y_n\} = \{1/(n+1)^2\}$ is a positive solution of (10) and $y_n \to 0$ as $n \to \infty$. However, Theorem 2 fails to hold for (10) due to (11). Further, we observe that $\sum_{n=2}^{\infty} q_n = \infty$ need not imply that $\sum_{i=0}^{\infty} q_{n_i} = \infty$ for every subsequence $\{n_i\}$ of $\{n\}$.

EXAMPLE 3. *Consider*

$$\Delta(y_n + y_{n-1}) + e^{2(n-1)-e^{-3(n-1)}} y_{n-1}^3 e^{y_{n-1}^3} = e^{-(n+1)} \tag{12}$$

for $n \geq 1$. *We can choose large* $u > 0$ *and* $v > 0$ *such that for every* $\lambda > 0$,

$$G(u+v) = (u+v)^3 e^{(u+v)^3} > (u^3 + v^3) e^{(u^3+v^3)} >$$
$$> \lambda[u^3 e^{u^3} + v^3 e^{v^3}] = \lambda[G(u) + G(v)].$$

Hence Theorem 1 cannot be applied to (12). On the other hand, since

$$e^{-e^{-3(n-1)}} \to 1 \text{ as } n \to \infty$$

then, for $0<\varepsilon<1$, there exists $N>0$ such that

$$n \geq N \text{ implies that } e^{-e^{-3(n-1)}} > 1-\varepsilon.$$

Thus, for every subsequence $\{n_i\}$ of $\{n\}$, we obtain

$$\sum_{i=0}^{\infty} q_{n_i} \geq \sum_{n_i=N}^{\infty} q_{n_i} = \sum_{n_i=N}^{\infty} e^{2(n_i-1)} e^{-e^{-3(n_i-1)}} > (1-\varepsilon)\sum_{n_i=N}^{\infty} e^{2(n_i-1)} = \infty.$$

From Theorem 2 it follows that every nonoscillatory solution of (12) tends to zero as $n\to\infty$. In particular, $\{y_n\}=\{e^{-n}\}$ is a positive solution of (12) and $y_n\to 0$ as $n\to\infty$.

THEOREM 3. *Suppose that $q_n \geq 0$ and for every subseqence $\{n_i\}$ of $\{n\}$, $\sum_{i=0}^{\infty} q_{n_i}=\infty$. Then every solution of* (3) *oscillates or tends to zero as $n\to\infty$.*

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of (3) on $[N,\infty)$, $N\geq 0$, and assume that $y_n>0$ for $n\geq N_1$. Setting, for $n\geq N_1+\ell$,

$$z_n = y_n - y_{n-m} \quad \text{and} \quad w_n = z_n - \sum_{i=0}^{n-1} f_i, \tag{13}$$

we obtain

$$\Delta w_n = -q_n G(y_{n-k}) \leq 0. \tag{14}$$

Hence $w_n>0$ for $n\geq N_2>N_1+\ell$ or $w_n<0$ for $n\geq N_2$. If $w_n>0$ for $n\geq N_2$, then $\lim_{n\to\infty} w_n$ exists. If possible, let $\limsup_{n\to\infty} y_n=\alpha$, $\alpha>0$. Hence there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $n_i\to\infty$ as $i\to\infty$ and $\lim_{i\to\infty} y_{n_i-k}=\alpha$. Thus $y_{n_i-k}>\beta>0$ for $i\geq N_3\geq N_2$. Consequently from the given hypothesis and (14) it follows that

$$\infty = G(\beta)\sum_{i=N_3}^{\infty} q_{n_i} < \sum_{i=N_3}^{\infty} q_{n_i} G(y_{n_i-k}) = -\sum_{i=N_3}^{\infty} \Delta w_{n_i}$$

and

$$\sum_{i=N_3}^{r-1} \Delta w_{n_i} = \sum_{i=N_3}^{r-1} (w_{n_{i+1}} - w_{n_i}) = w_{n_r} - w_{n_{N_3}} > -w_{n_{N_3}}$$

implies that

$$-\sum_{i=N_3}^{\infty} \Delta w_{n_i} \leq w_{N_3} < \infty,$$

a contradiction. Hence $\limsup_{n\to\infty} y_n = 0$. Thus $\lim_{n\to\infty} y_n = 0$. Next suppose that $w_n < 0$ for $n \geq N_2$. If $\lim_{n\to\infty} w_n = \lambda$, then $-\infty \leq \lambda < 0$. Suppose that $\lambda = -\infty$. Then $\lim_{n\to\infty} z_n = -\infty$. There exists $N_4 > N_2$ such that $z_n < 0$ for $n \geq N_4$. From (13) we obtain $y_n < y_{n-m}$ for $n \geq N_4$, that is, $\{y_n\}$ is bounded. Hence $\{z_n\}$ is bounded, a contradiction. Thus $-\infty < \lim_{n\to\infty} w_n < 0$. Then proceeding as above we obtain $\lim_{n\to\infty} y_n = 0$. The proof proceeds similarly when $y_n < 0$ for $n \geq N_1$. This completes the proof of the theorem.

EXAMPLE 4. *Every solution of*

$$\Delta(y_n - y_{n-2}) + e^{-3}(1+e)e^{2n} y_{n-1}^3 = (e^2 + e^{-1})e^{-n}, \quad n \geq 0,$$

oscillates or tends to zero as $n \to \infty$ *by Theorem 3. Clearly,* $\{y_n\} = \{e^{-n}\}$ *is a positive solution of the equation with* $y_n \to 0$ *as* $n \to \infty$. *We may note that for every subsequence* $\{n_i\}$ *of* $\{n\}$,

$$\sum_{i=0}^{\infty} q_{n_i} = e^{-3}(1+e)\sum_{i=0}^{\infty} e^{2n_i} = \infty.$$

THEOREM 4. *Let* $q_n \leq 0$ *and, for every subsequence* $\{n_i\}$ *of* $\{n\}$, $\sum_{i=0}^{\infty} q_{n_i} = -\infty$. *Then every solution of* (3) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $n \to \infty$.

PROOF. If $\{y_n\}$ is a nonoscillatory solution of (3) on $[N, \infty)$, $N \geq 0$, then we may assume (and we do) that $y_n > 0$ for $n \geq N_1$. Setting z_n and w_n as in (13), for $n \geq N_1 + \ell$, we obtain $\Delta w_n \geq 0$. Hence $w_n < 0$ or > 0 for $n \geq N_2 > N_1 + \ell$. If $w_n < 0$ for $n \geq N_2$, then $\lim_{n\to\infty} w_n$ exists. Proceeding as in the proof of Theorem 3 we may show that $\lim_{n\to\infty} y_n = 0$. Suppose that $w_n > 0$ for $n \geq N_2$. If $\lim_{n\to\infty} w_n < \infty$, then $\lim_{n\to\infty} y_n = 0$. If $\lim_{n\to\infty} w_n = \infty$, then $\lim_{n\to\infty} z_n = \infty$. Since $z_n < y_n$ for $n \geq N_1$, then $\lim_{n\to\infty} y_n = \infty$. The proof is similar in case $y_n < 0$ for $n \geq N_1$. Thus the theorem is proved.

EXAMPLE 5. *Every nonoscilatory solution of*

$$\Delta(y_n - y_{n-2}) - (e(e-1-e^{-1}+e^{-2}) + e^{-2n})y_{n-1} = -e^{-(n+1)},$$

$n \geq 0$, *tends to zero or* $\pm\infty$ *as* $n \to \infty$ *by Theorem 4. Clearly,* $\{y_n\} = \{e^n\}$ *is such a solution of the equation. We may note that, for every sequence* $\{n_i\}$ *of* $\{n\}$

$$\sum_{i=0}^{\infty} q_{n_i} < -\sum_{i=0}^{\infty} e(e-1-e^{-1}+e^{-2}) = -\infty.$$

COROLLARY 5. *Suppose that the conditions of Theorem 4 are satisfied. Then every bounded solution of* (3) *oscillates or tends to zero as* $n \to \infty$.

This follows from Theorem 4.

THEOREM 6. *If the conditions of Theorem 4 are satisfied, then every solution* $\{y_n\}$ *of* (2) *oscillates or tends to zero as* $n \to \infty$ *or* $\limsup_{n\to\infty} |y_n| = \infty$.

The proof is similar to that of Theorem 4 and hence is omitted.

3. OSCILLATION OF EQ. (4)

In this section we study oscillatory and asymptotic behaviour of solutions of Eq. (4).

THEOREM 7. *Let* $-1 < -p_1 \leq p_n \leq p_2 < 1$ *with* $0 < p_1 + p_2 < 1$, *where* p_1 *and* p_2 *are positive reals. If* $q_n \geq 0$ *and* $\sum_{n=0}^{\infty} q_n = \infty$, *then every solution of* (4) *oscillates or tends to zero as* $n \to \infty$.

PROOF. Let $\{y_n\}$ be a solution of (4) on $[N, \infty)$, $N \geq 0$. If $\{y_n\}$ oscillates, then there is nothing to prove. Suppose that $\{y_n\}$ is a nonoscillatory solution of (4) and assume that $\{y_n\}$ is eventually positive. Hence there exists $N_1 \geq N$ such that $y_n > 0$ for $n \geq N_1$. Setting

(15) $$z_n = y_n + p_n y_{n-m} \quad \text{and} \quad w_n = z_n - \sum_{i=0}^{n-1} f_i$$

for $n \geq N_1 + \ell$, we obtain

(16) $$\Delta w_n = -q_n G(y_{n-k}) \leq 0.$$

Hence $w_n > 0$ for $n \geq N_2 \geq N_1 + \ell$ or $w_n < 0$ for $n \geq N_2$. Let $w_n > 0$ for $n \geq N_2$. Then $\lim_{n\to\infty} w_n$ and $\lim_{n\to\infty} z_n$ exist. If $\{y_n\}$ is unbounded, then there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $n_j \to \infty$ as $j \to \infty$, $y_{n_j} \to \infty$ as $j \to \infty$ and

$$y_{n_j} = \max\{y_n : N_2 \leq n \leq n_j\}.$$

If $\lim_{j\to\infty} y_{n_j - m} = \infty$, then choosing *j*, large enough such that $n_j - m > N_2$, we have from (15) that

$$w_{n_j} = y_{n_j} + p_{n_j} y_{n_j - m} - \sum_{i=0}^{n_j - 1} f_i \geq (1 - p_1) y_{n_j - m} - \sum_{i=0}^{n_j - 1} f_i.$$

Thus $\lim_{j\to\infty} w_{n_j} = \infty$, a contradiction. If $\lim_{j\to\infty} y_{n_j - m} < \infty$, then

$$w_{n_j} \geq y_{n_j} - p_1 y_{n_j - m} - \sum_{i=0}^{n_j - 1} f_i$$

implies that $\lim_{j\to\infty} w_{n_j} = \infty$, a contradiction again. Hence $\{y_n\}$ is bounded. We claim that $\liminf_{n\to\infty} y_n = 0$. Otherwise, $\liminf_{n\to\infty} y_n = \alpha$, $0 < \alpha < \infty$. Then $y_n > \beta > 0$ for $n \geq N_3 > N_2$. Hence, for $n \geq N_4 > N_3 + \ell$, we get

$$\sum_{n=N_4}^{\infty} q_n G(y_{n-k}) > G(\beta) \sum_{n=N_4}^{\infty} q_n = \infty.$$

On the other hand, (16) yields

$$\sum_{n=N_4}^{r-1} q_n G(y_{n-k}) = -\sum_{n=N_4}^{r-1} \Delta w_n = w_{N_4} - w_r < w_{N_4}.$$

Hence

$$\sum_{n=N_4}^{\infty} q_n G(y_{n-k}) \leq w_{N_4} < \infty,$$

a contradiction. Thus our claim holds. Since $\{y_n\}$ is bounded, then, using Lemma A, we obtain from (15) that

$$\lim_{n\to\infty} z_n = \limsup_{n\to\infty} z_n \geq \limsup_{n\to\infty} [y_n - p_1 y_{n-m}] \geq$$

$$\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} (-p_1 y_{n-m}) =$$

$$= \limsup_{n\to\infty} y_n - p_1 \limsup_{n\to\infty} y_{n-m} =$$

$$= (1-p_1) \limsup_{n\to\infty} y_n$$

and, since $\liminf_{n\to\infty} y_n = 0$,

$$\lim_{n\to\infty} z_n = \liminf_{n\to\infty} z_n \le \liminf_{n\to\infty} [y_n + p_2 y_{n-m}] \le$$

$$\le \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} (p_2 y_{n-m}) =$$

$$= p_2 \limsup_{n\to\infty} y_{n-m} = p_2 \limsup_{n\to\infty} y_n .$$

Hence

$$(1-p_1) \limsup_{n\to\infty} y_n \le p_2 \limsup_{n\to\infty} y_n ,$$

that is,

$$0 \le (p_1 + p_2 - 1) \limsup_{n\to\infty} y_n \le 0 .$$

Consequently, $\limsup_{n\to\infty} y_n = 0$. Thus $\lim_{n\to\infty} y_n = 0$.

Next suppose that $w_n < 0$ for $n \ge N_2$. If $\{y_n\}$ is unbounded, then proceeding as above we obtain $w_{n_j} > 0$ for large j, a contradiction. Hence $\{y_n\}$ is bounded. From this it follows that $\{w_n\}$ is bounded. Thus $\lim_{n\to\infty} w_n$ and $\lim_{n\to\infty} z_n$ exist. Proceeding as above we show that $\lim_{n\to\infty} y_n = 0$. The proof is similar when $y_n < 0$ for $n \ge N_1$. Thus the theorem is proved.

EXAMPLE 6. *Consider*

$$\Delta\left(y_n + \frac{1}{3}(-1)^n y_{n-2} \right) + (e-1)\, e^{2(n-2)} y_{n-1}^3 = -\frac{1}{3}(-1)^n (e+1)\, e^{-(n-1)},$$

for $n \ge 0$. *Clearly,* $-1 < -\frac{1}{3} \le \frac{1}{3}(-1)^n \le \frac{1}{3} < 1$. *Every nonoscillatory solution of the equation tends to zero as* $n \to \infty$. *In particular,* $\{y_n\} = \{e^{-n}\}$ *is a positive solution of the equation with* $y_n \to 0$ *as* $n \to \infty$.

REMARK. In [2], Graef and Spikes have studied boundedness and asymptotic behaviour of solutions of Eq. (4). However, the technique employed in our work is different from theirs.

THEOREM 8. *Suppose that* $-1 < -p_1 \le p_n \le p_2 < 1$ *with* $0 < p_1 + p_2 < 1$. *If* $q_n \le 0$ *and* $\sum_{n=0}^{\infty} q_n = -\infty$, *then every solution* $\{y_n\}$ *of* (4) *oscillates or tends to zero as* $n \to \infty$ *or* $\limsup_{n\to\infty} |y_n| = \infty$.

The proof is similar to that of Theorem 7 and hence is omitted.

COROLLARY 9. *If the conditions of Theorem 8 are satisfied, then every bounded solution of* (4) *oscillates or tends to zero as* $n \to \infty$.

This follows from Theorem 8.

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([1]) Department of Mathematics, Berhampur University, Berhampur – 760007, INDIA;
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