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ON PROPERTIES OF SOME MODULAR SPACES OF DOUBLE SEQUENCES

ABSTRACT: We consider modular spaces of strong (A,φ) – summable and $|A,\varphi|$ - summable double sequences. The main results are two theorems in which are given the necessary conditions for inclusion between the spaces T_{φ}^{*} and T_{Ψ}^{b}. These theorems are generalization of theorems given by J. Musielak and W. Orlicz in [6].

KEY WORDS: sequence spaces, modular spaces.

1. INTRODUCTION

Modular spaces of strongly summable sequences and integrable functions are applied in various problems in mathematical analysis. In order to built up a general theory of modular spaces it is advisable to investigate special cases of modular spaces. In papers of J. Musielak [3], W. Orlicz [9], R. Taberski [10], J. Musielak and W. Orlicz [6] and myself [11], [12] there are considered and investigated some modular spaces connected with strong (A,φ) and $|A,\varphi|$ summability of sequences. Continuing the investigations of J. Musielak and W. Orlicz there are given necessary conditions of these spaces but for double sequences.

2. PRELIMINARIES

Let T denotes the space of all real double sequences. Sequences belonging to T will be denoted by $x=(t_{\mu\nu})$, $y=(s_{\mu\nu})$, $|x|=(|t_{\mu\nu}|)$, $e=(1)$, $\Theta=(0)$ and $x^{j}=(t_{\mu\nu}^{j})$, where $j=1,2,\ldots$. Moreover, we shall denote e_{pq}, $e_{pq}^{p'q'}$, $e_{p\cdot}$, $e_{\cdot q}$ the sequences having 1 at the intersection point of the p-th row and q-th column and 0 elsewhere, having 1 at the intersection of p-th, ..., $(p+p'-1)$-th rows and q-th, ..., $(q+q'-1)$-th columns and 0 elsewhere, having 1 in the p-th row and 0 elsewhere, and having 1 in q-th column and 0 elsewhere, respectively. If $x=(t_{\mu\nu})$ is a given sequence then x^{mn} will mean the sequence having $t_{\mu\nu}$ at the intersection of 1-st, ..., m-th rows and 1-st, ..., n-th columns and 0 elsewhere.

By a convergent sequence we shall mean double sequence $x=(t_{\mu\nu})$ converging in the sense of Pringsheim i.e. for every $\varepsilon>0$ there exists an integer

N such that $|t_{\mu\nu} - t_{..}| < \varepsilon$ for every $\mu, \nu > N$, where $t_{..} = \lim_{\mu,\nu\to\infty} t_{\mu\nu}$ denotes the limit of the sequence x.

If $x, y \in T$ the inequality $x \geq y$ will mean $t_{\mu\nu} \geq s_{\mu\nu}$ for all μ and ν. The relation $\geq$ is a semiorder relation in T. Let us remark that an arbitrary nonvoid set of elements from T bounded from above has a least upper bounded belonging to T, i.e. T is a linear lattice complette with respect to the relation $\geq$.

Moreover, let T_0, T_b, T_f denote spaces of all real double sequences convergent to zero, bounded real double sequences and finite double sequences (i.e. real double sequences with a finite number of elements different from zero), respectively.

By a φ-function we understood a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. φ-function will be denoted by φ, Ψ, ..., and their inverse functions by φ_{-1}, Ψ_{-1}, ..., respectively.

The symbol $\varphi(|x|)$ means the function $\varphi(|x(t)|)$. For more properties of φ-function see e.g. [1], [2], [8] and also [4], [5].

Throughout this paper we shall need the following hypotheses:

(α) there exists a constant $\delta > 0$ satisfying the inequality

(1) $$\Psi(\delta uv) \leq \varphi(u)\Psi(v)$$

for all $u, v > 0$ such that $\varphi(u)\Psi(v) \leq \delta$, $\varphi(u) > 1$.

(β) if there exists a positive constant δ satisfying the condition

(2) $$\varphi(u)\Psi(v) \leq \Psi(\tfrac{1}{\delta}uv)$$

for all $u, v > 0$ such that $u, v \leq \delta$ and $u \geq 1$.

Moreover, $A = (a_{mn\mu\nu})$ denotes a four-dimensional nonnegative matrix (i.e. $a_{mn\mu\nu} \geq 0$ for $m, n, \mu, \nu = 1,2,....$) and such that for arbitrary two positive integers m, n there exists a pair of positive integers μ_0 and ν_0 such that $a_{mn\mu_0\nu_0} \neq 0$.

In the following we will denote

$$A_{\mu\nu} = \sup_{m,n} a_{mn\mu\nu},$$

$$A_{\mu\cdot} = \sup_{m,n} \sum_{\nu=1}^{\infty} a_{mn\mu\nu},$$

$$A_{\cdot\nu} = \sup_{m,n} \sum_{\mu=1}^{\infty} a_{mn\mu\nu},$$

$$A_{p,q}^{p',q'} = \sup_{m,n} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} a_{mn\mu\nu}.$$

Besides the above assumptions, the following properties of the matrix A play an essential role in theory of summability of double sequences:

a. there exists $\lim_{m,n\to\infty} a_{mn\mu\nu} = 0$ for $\mu,\nu = 1,2,\ldots$

b. $\sup_{m,n} \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu} = K < \infty$,

c. there exists $\lim_{m,n\to\infty} \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu} = a$,

d. there exists a constant C such that

$$\sum_{\mu,\nu=1}^{\infty} \Psi_{-1}(A_{\mu\nu}) < C$$

for an arbitrary φ -function Ψ,

e. there exist constants K_1, K_2 and K_3 such that:

$$\sum_{\mu=1}^{\infty} a_{mn\mu\nu} \le K_1 \quad \text{for fixed } m, n \text{ and } \nu,$$

$$\sum_{\nu=1}^{\infty} a_{mn\mu\nu} \le K_2 \quad \text{for fixed } m, n \text{ and } \mu,$$

$$\sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu} \le K_3 \quad \text{for all } m \text{ and } n.$$

3. STRONG φ -SUMMABILITY OF DOUBLE SEQUENCES

Let the matrix $A = (a_{mn\mu\nu})$ and the φ -functions φ, Ψ be given.

3.1. THE MODULAR SPACE OF SEQUENCES OF STRONGLY (A,φ) -SUMMABLE TO ZERO

For $x = (t_{\mu\nu}) \in T$ we define a map $(t_{\mu\nu}) \to (\sigma_{mn}^{\varphi})$ by the formula

$$\sigma_{mn}^{\varphi}(x) = \sum_{\mu,\nu=1}^{\infty} a_{mn\mu\nu}\varphi(|t_{\mu\nu}|) \quad \text{for} \quad m,n = 1,2,\ldots.$$

In the following we introduce subspaces of the space T

$$T_{\varphi_0} = \{x \in T : \sigma^{\varphi}_{mn}(x) < \infty \text{ for } m,n = 1,2,\ldots \text{ and } \sigma^{\varphi}_{mn}(x) \to 0 \text{ as } m,n \to \infty\},$$

$$T^*_{\varphi} = \{x \in T : \lambda x \in T_{\varphi_0} \text{ for a certain } \lambda > 0\}.$$

It is well known that in the space T^*_{φ} first we may introduce the following modular

$$\rho_{\varphi}(x) = \begin{cases} \sup\limits_{m,n} \sigma^{\varphi}_{mn}(x) & \text{for } x \in T_{\varphi_0}, \\ \infty & \text{for } x \in T^*_{\varphi} \setminus T_{\varphi_0} \end{cases}$$

and next a norm which is an F-norm

$$\| x \|_{\varphi} = \inf\{\varepsilon > 0 : \rho_{\varphi}(\tfrac{x}{\varepsilon}) \le \varepsilon\}.$$

Moreover, if φ is a convex φ-function then homogeneous F-norm can be introducend in the space T_{φ} by means of the formula

$$\| x \|^1_{\varphi} = \inf\{\varepsilon > 0 : \rho_{\varphi}(\tfrac{x}{\varepsilon}) \le 1\}.$$

In these notions we did not mention the dependence on the matrix A, since in our considerations we shall deal with a fixed matrix A only.

If the series $\sigma^{\varphi}_{mn}(x)$ are defined for every m and n, we say that the method (A,φ) transforms sequence $x = (t_{\mu\nu})$ into the sequence $(\sigma^{\varphi}_{mn}(x))$. Double sequences x belonging to T^*_{φ} are called strongly (A,φ)-summable to zero. Let us remark that this definition of strongly (A,φ)-summability to zero of double sequences is generalization of the definition introduced in [11] for singular sequences (compare also [3], [6]).

A list of basic theorems and properties concerning the space T^*_{φ} is presented below (compare [13] and see also [3], [6], [9], [11]).

1. $T_f \subset T^*_{\varphi}$ if and only if the matrix A satisfies the condition **a**.

2. If the matrix A possesses the property **b**, then $T^*_{\varphi} \cap T_b = T^*_{\Psi} \cap T_b$, for arbitrary two φ-functions φ and Ψ.

3. The space T^*_{φ} is complete with respect to the norm $\| \cdot \|_{\varphi}$.

4. If φ is a convex φ-function and the matrix A satisfy the conditions **a** and **b**, then the sequences $e_{\mu\nu}$, $e_{\mu\cdot}$, $e_{\cdot\nu}$ and $e^{p'q'}_{pq}$ belong to the space T^*_{φ}.

5. The following formulas are true:

$$\| e_{\mu\nu} \|_{\varphi}^{1} = \left[\varphi_{-1}\left(\frac{1}{A_{\mu\nu}} \right) \right]^{-1},$$

$$\| e_{\mu\cdot} \|_{\varphi}^{1} = \left[\varphi_{-1}\left(\frac{1}{A_{\mu\cdot}} \right) \right]^{-1},$$

$$\| e_{\cdot\nu} \|_{\varphi}^{1} = \left[\varphi_{-1}\left(\frac{1}{A_{\cdot\nu}} \right) \right]^{-1},$$

$$\| e_{pq}^{p'q'} \|_{\varphi}^{1} = \left[\varphi_{-1}\left(\frac{1}{A_{pq}^{p'q'}} \right) \right]^{-1}.$$

3.2. THE MODULAR SPACE OF SEQUNCES OF STRONGLY $|A,\varphi|$-SUMMABLE TO ZERO

In this part first in the space T_f we may define the following functional

$$\rho_{\Psi}^{b} = \sum_{\mu,\nu=1}^{\infty} \Psi(|t_{\mu\nu}|)$$

and next a norm by means of the formula

$$\| x \|_{\Psi}^{R} = \inf \{ \varepsilon > 0 : \rho_{\Psi}^{b}(\tfrac{x}{\varepsilon}) \le 1 \}.$$

It is easily verfied that the functional ρ_{Ψ}^{b} is a modular in T_f in the sense of e.g. [4] and [5] (compare also [1], [2], [8], [16]) and moreover it is well known that the norm $\|\cdot\|_{\psi}^{R}$ is monotonic and homogenous.

Since in our considerations we shall deal with a fixed matrix A only then in these notations we did not mention the dependence on the matrix A.

If $A = (a_{mn\mu\nu})$ is a given matrix, $x = (t_{\mu\nu}) \in T$ and Ψ is a given convex φ-function, then we define the class T_{Ψ}^{b} of sequences satisfying the condition

$$\lim_{m,n\to\infty} \| \overline{x}^{mn} \|_{\Psi}^{R} = 0,$$

where $\overline{x}^{mn}$ is a new sequence such that $\overline{x}^{mn} \in T_f$ and

$$\overline{x}^{mn} = \begin{cases} \Psi_{-1}(a_{mn\mu\nu}) t_{\mu\nu} & \text{for } \mu \le m \text{ and } \nu \le n, \\ 0 & \text{elsewhere.} \end{cases}$$

Double sequences x belonging to T_Ψ^b are called strongly $|A,\Psi|$-summable to zero. Obviously this definition of strongly $|A,\varphi|$-summability to zero of double sequences is generalization of the definition introduced in [12] for singular sequences (compare also [6], [10]).

A list of the most interesting theorems and properties concerning the space T_Ψ^b is presented below (compare [16] and see also [6], [10], [12]).

1. The space T_Ψ^b with the norm

$$\| x \|_\Psi^b = \sup_{m,n} \| \bar{x}^{mn} \|_\Psi^R$$

is a Banach space,

2. If $x \in T_\Psi^b$, then $\lim_{k,l\to\infty} \| x - x^{kl} \|_\Psi^b = 0$,

3. $x \in T_\Psi^b$ if and only if $\lim_{k,k',l,l'\to\infty} \| x^{kl} - x^{k'l'} \|_\Psi^b = 0$,

4. We have the formulas:

$$\| e_{\mu\nu} \|_\Psi^b = [\Psi_{-1}(1)]^{-1} \Psi_{-1}(A_{\mu\nu}),$$

$$\frac{1}{mn}\left[\Psi_{-1}\left(\frac{1}{mn}\right)\right]^{-1} \sum_{\mu,\nu=1}^{\mu=m,\nu=n} \Psi_{-1}(a_{mn\mu\nu}) \le \| \bar{e}^{mn} \|_\Psi^R,$$

$$[\Psi_{-1}(1)]^{-1} \sum_{\mu,\nu=1}^{\mu=m,\nu=n} \Psi_{-1}(a_{mn\mu\nu}) \ge \| \bar{e}^{mn} \|_\Psi^R,$$

$$\frac{1}{p'q'}\left[\Psi_{-1}\left(\frac{1}{p'q'}\right)\right]^{-1} \sup_{m,n} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(a_{mn\mu\nu}) \le \| e_{pq}^{p'q'} \|_\Psi^b,$$

$$[\Psi_{-1}(1)]^{-1} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(A_{\mu\nu}) \ge \| e_{pq}^{p'q'} \|_\Psi^b.$$

3.3. Remark

Let us remark that the spaces T_φ^* and T_Ψ^b were introduced and investigated in [6] by J. Musielak and W. Orlicz (compare aslo [3], [9]) and in [10] by R. Taberski. In these papers the authors limited themselves to investigation of the case of strong (A,φ)-summability and $|A,\varphi|$-summability to zero of single sequences by means of the firts arithmetic means.

It is easily seen that for $x=(t_\mu)$, $A=(a_{m\mu})$ where $a_{m\mu}=\frac{1}{m}$ for $\mu\leq m$ and $a_{m\mu}=0$ for $\mu>m$ and for convex φ-functions φ and Ψ we have

$$\sigma_m^\varphi(x) = \frac{1}{m}\sum_{\mu=1}^{m}\varphi(|t_\mu|) \quad \text{for} \quad m=1,2,...;$$

$$T_\varphi^* = \{x=(t_\mu):\sigma_m^\varphi(\lambda x)<\infty \ \text{ for } m=1,2,... \ \text{ and } \lim_{m\to\infty}\sigma_m^\varphi(\lambda x) = 0 \ \text{ for a certain } \lambda>0\};$$

$$\sigma_\Psi^b(x) = \sum_{\mu=1}^{m}\Psi(|t_\mu|) \quad \text{for} \quad x\in T_f,$$

$$\|x\|_\Psi^R = \inf\{\varepsilon>0 : \rho_\Psi^b(\tfrac{x}{\varepsilon}) \leq 1\},$$

$$T_\Psi^b = \{x=(t_\mu) : \lim_{m\to\infty}\Psi_{-1}(\tfrac{1}{m})\|x^m\|_\Psi^R = 0\},$$

where $x^m=(t_1,t_2,...,t_{m-1},t_m,0,0,...)$. Moreover, let us remark that if $\varphi(u)=\Psi(u)=|u|^\alpha$ where $\alpha\geq 1$, then we have

$$\sigma_m^\varphi(x) = \frac{1}{m}\sum_{\mu=1}^{m}|t_\mu|^\alpha, \quad \Psi_{-1}(\tfrac{1}{m}) = (\tfrac{1}{m})^{1/\alpha}, \quad \|x^m\|_\Psi^R = \left(\sum_{\mu=1}^{m}|t_\mu|^\alpha\right)^{1/\alpha},$$

and by the conditions

$$\lim_{m\to\infty}\sigma_m^\varphi(\lambda x) = 0 \quad \text{for a certain } \lambda>0$$

and

$$\lim_{m\to\infty}\Psi_{-1}(\tfrac{1}{m})\|x^m\|_\Psi^R = 0$$

we obtain the following condition

$$\lim_{m\to\infty}\frac{1}{m}\sum_{\mu=1}^{m}|t_\mu|^\alpha = 0.$$

Evidently, we have

$$(A,\varphi)-\lim t_\mu = |A,\varphi|-\lim t_\mu = |C^\alpha,1|-\lim t_\mu.$$

4. RELATIONS BETWEEN THE SPACES T_φ^* AND T_Ψ^b

Theorem 1. *If* $T_\varphi^* \subset T_\Psi^b$ *then the convex* φ*-functions* φ *and* Ψ *satisfy the condition* (α).

PROOF. Let φ and Ψ be two convex φ-functions. It is well known that if $T_\varphi^* \subset T_\Psi^b$ then there is an arbitrary sufficiently positive number η such that if $\| x \|_\varphi^1 \le \eta$ than $\| x \|_\Psi^1 \le 1$ for every $x \in T_f$ (compare e.g. [6] and also [13] or [16]).

Now, we choose $u, v > 0$ such that

(3) $$\varphi(u)\Psi(v) \le \eta, \qquad \varphi(u) \ge 1.$$

Thus, $\Psi(v) \le \eta$ and there exist natural numbers $p,\ p',\ q,\ q'$ such that

(4) $$\eta A_{pq} < \Psi(v) \le \eta p' q' A_{pq}$$

and

(5) $$\tfrac{1}{2} \le A_{pq}^{p'q'} \varphi(u) \le 1.$$

It is easily seen that by (3) and (4) we obtain $A_{pq}\varphi(u) < 1$ and $u < \varphi_{-1}(\frac{1}{A_{pq}})$.

In the following applying the definition of $\| \cdot \|_\varphi^1$, the formulas 3.1.5 and the inequality (5) we obtain

$$\| u\eta e_{pq}^{p'q'} \|_\varphi^1 = u\eta \| e_{pq}^{p'q'} \|_\varphi^1 \ u\eta[\varphi_{-1}(\tfrac{1}{A_{pq}^{p'q'}})]^{-1} \le \eta.$$

Moreover, by the formulas 3.2.4 we have

$$\begin{aligned} & u\eta[p'q'\varphi_{-1}(1)]^{-1}\Psi_{-1}(A_{pq}) \le \\ & \le u\eta[p'q'\Psi_{-1}(1)]^{-1} \sup_{m,n} \sum_{\mu=m,\nu=n}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(a_{mn\mu\nu}) \le \\ & \le \| u\eta e_{pq}^{p'q'} \|_\Psi^b \le 1. \end{aligned}$$

In consequence we obtain

$$u\eta\Psi_{-1}(A_{pq})) \le u\eta \sup_{m,n} \sum_{\mu=p,\nu=q}^{p+p'-1,q+q'-1} \Psi_{-1}(a_{mn\mu\nu}) \le p'q'\Psi_{-1}(1),$$

$$\Psi(\tfrac{1}{p'q'} u\eta\Psi_{-1}(A_{pq})) \le \Psi(\tfrac{1}{p'q'} u\eta \sup_{m,n} \sum_{\mu=p,\nu=q}^{p+p'-1,q+q'-1} \Psi_{-1}(a_{mn\mu\nu})) \le 1,$$

(6) $$\Psi(\tfrac{1}{p'q'}u\eta\Psi_{-1}(A_{pq})) \leq 1.$$

But Ψ is a convex φ-function and Ψ_{-1} is the inverse function to Ψ then it is easily seen that

(7) $$\Psi_{-1}(A_{pq}) \geq p'q'\Psi_{-1}(\eta p'q'A_{pq})$$

where $n \leq (p'q')^{-2}$.

In the following by (6) and (7) we have

$$\Psi(u\eta\Psi_{-1}(\eta p'q'A_{pq})) \leq 1.$$

and next by (4) we obtain

(8) $$\Psi(\eta uv) \leq 1.$$

Applying the properties of the matrix A and the inequalities (4) and (5) we get

$$\eta < \frac{1}{A_{pq}}\Psi(v) \leq p'q'\frac{1}{A_{pq}^{p'q'}}\Psi(v) < 2p'q'\varphi(u)\Psi(v),$$

$$\eta \leq 2p'q'\varphi(u)\Psi(v).$$

Hence applying (8) we have

$$\eta\frac{1}{2p'q'}\Psi(\eta uv) \leq \varphi(u)\Psi(v).$$

But φ-function Ψ is convex then we get

$$\Psi\left(\frac{1}{2}\eta^2\frac{1}{p'q'}uv\right) \leq \varphi(u)\Psi(v).$$

Choosing $\eta^2\frac{1}{p'q'} = 2\delta$, we conclude the inequality (1)

THEOREM 2. *If $T_\psi^b \subset T_\varphi^*$ then the convex functions φ and Ψ satisfy the condition (β).*

PROOF. We suppose that $T_\psi^b \subset T_\varphi^*$ where φ-functions φ and Ψ are convex. Then there is an $\eta > 0$ such that $\| x\|_\Psi^b \leq \eta$ implies $\| x\|_\varphi^1 \leq 1$ for every $x \in T_f$ (compare e.g. [6], [12], [15] or [16]). In the following we choose $\eta \in (0,1>$. Now take $u, v > 0$ satisfying the condition

(9) $$uv \le \eta\Psi_{-1}(1), \qquad u \ge \eta.$$

Then $v \le \Psi_{-1}(1)$ and there exist the integers p, p', q and q' such that

(10) $$\sup_{m,n} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(a_{mn\mu\nu}) < v \le \Psi_{-1}(p'q'A_{pq}^{p'q'}).$$

In consequence by (9) and (10) we have $u[\Psi_{-1}(1)]^{-1}v \le \eta$ and

(11) $$u[\Psi_{-1}(1)]^{-1} \sup_{m,n} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(a_{mn\mu\nu}) < \eta.$$

Hence by the definition of $\|\cdot\|_{\Psi}^{b}$ and the formulas 3.2.4 we get

$$\| e_{pq}^{p'q'} \|_{\Psi}^{b} \le u[\Psi_{-1}(1)]^{-1} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(a_{mn\mu\nu}) < \eta.$$

In the following by the condition 3.1.5 we obtain

$$\| u e_{pq}^{p'q'} \|_{\varphi}^{1} = u\left\{\varphi_{-1}\left(\frac{1}{A_{pq}^{p'q'}}\right)\right\}^{-1} \le 1$$

and

(12) $$\varphi(u) \le \frac{1}{A_{pq}^{p'q'}}.$$

Finally, inequalities (10) and (12), give

(13) $$\varphi(u)\Psi(v) \le \frac{1}{A_{pq}^{p'q'}} p'q'A_{pq}^{p'q'} = p'q'.$$

Moreover, the inequality (11) give

$$\frac{1}{\eta} \sup_{m,n} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(a_{mn\mu\nu}) \le \Psi_{-1}(1),$$

$$\Psi\left(\frac{1}{\eta} u \sup_{m,n} \sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1} \Psi_{-1}(a_{mn\mu\nu})\right) \le 1,$$

and for a certain pair of natural numbers (p'', q'') such that $1 \le p'' \le p'$, $1 \le q'' \le q'$ we have

(14)
$$\frac{1}{2} \le p''q''\Psi\left(\frac{1}{\eta}u\Psi_{-1}(A_{p''q''})\right) < 1.$$

However, by the property of φ-function Ψ and by the conditions (10) and (14) we obtain

$$\begin{aligned} p'q' &= 2p'q'p''q''\Psi\left(\frac{1}{\eta}u\Psi_{-1}(A_{p''q''})\right) \le \\ &\le 2p'q'p''q''\Psi\left(\frac{1}{\eta}u\sup_{m,n}\sum_{\mu=p,\nu=q}^{\mu=p+p'-1,\nu=q+q'-1}\Psi_{-1}(a_{mn\mu\nu})\right) \le \\ &\le 2(p'q')^2\Psi\left(\frac{1}{\eta}uv\right) \le \Psi\left(2\frac{1}{\eta}(p'q')^2uv\right). \end{aligned}$$

Thus

(15)
$$p'q' \le \Psi\left(2\frac{1}{\eta}(p'q')^2uv\right)$$

where u and v satisfy the condition (9).

Now, choosing

$$\delta = \eta\min\left\{\Psi_{-1}(1), \frac{1}{2(p'q')^2}\right\} \le \eta\min\left\{\Psi_{-1}(1), \frac{1}{2}\right\}$$

and applying the inequalities (13) and (15) we obtain the condition (2).

Let us remark that the above theorems give the necessary conditions for inclusion between the spaces T_φ^* and T_Ψ^b. These theorems are generalization (on the double sequences) of the theorems due to Julian Musielak and Władysław Orlicz (see [6], pp. 135-139). The paper [15] contains the theorems in which there are given the sufficient conditions for the inclusions of the spaces T_φ^* and T_Ψ^b of double sequences (compare also [6]).

THEOREM 3. *Let us suppose that the φ-functions φ and Ψ are convex function and let the matrix A has the properties given in part* **2.**

(a) $T_\varphi^* \subset T_\Psi^b$ *if and only if the condition* (α) *holds,*

(b) $T_\varphi^* \subset T_\Psi^b$ *if and only if the condition* (β) *holds.*

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