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SOME REMARKS ON ALMOST PERIODIC FUNCTIONS

ABSTRACT: In this paper we present the definition and some properties of (*IC*)-a.p. functions, i.e. uniformly almost periodic (*B*-a.p.) functions with their indefinite integrals. Next, we give the definition and some properties of $(IC)^{(n)}$-a.p. functions, i.e. uniformly almost periodic functions with their n derivatives and indefinite indegrals, and $(IC)_a^{(\infty)}$-a.p. functions, i.e. uniformly almost periodic functions with their every derivatives, with respect to a positive sequence $a=(a_i)$, and indefinite integrals.

KEY WORDS: uniformly almost periodic function, derivative of order n, indefinite integral.

1. (IC) – ALMOST PERIODIC FUNCTIONS

1.1. DEFINITIONS

We first give basic notations related to uniformly almost periodic functions with their indefinite integrals.

By $C(R)$ we denote the set of functions from R into itself which are continuous. Denote for $f,g\in C(R)$

$$T(f_a,g_b)(u) = |F_a(u)-G_b(u)| \qquad \text{for} \qquad u\in R,$$

where $a,b\in R$, $f_a(x)\equiv f(x+a)$, $g_b(x)\equiv g(x+b)$, $F_a(x)\equiv F(x+a)=\int_0^{x+a} f(s)\,ds$, $G_b(x)\equiv G(x+b)=\int_0^{x+b} g(s)\,ds$, and in the follow we define the (ID) – distans, putting

$$(ID)(f,g) = \sup_{t\in R}(|f(t)-g(t)| + T(f,g)(t)).$$

We say that an $f\in C(R)$ is (IC) – *bounded* iff $(ID)(f)<\infty$, where $(ID)(f)=(ID)(f,0)$. Let $f_h(x)\equiv f(x+h)$. We say that $f\in C(R)$ is an (*IC*)-*continuous function* iff $\lim_{h\to 0}(ID)(f,f_h)=0$. A sequence (f_k) in $C(R)$ will be called (*ID*)-*convergent to an* $f\in C(R)$ iff $\lim_{k\to\infty}(ID)(f,f_k)=0$.

THEOREM 1.1. *If a sequence* (f_k) *of (IC)-continuous functions is (ID)-convergent to a function* $f\in C(R)$, *then an f is (IC)-continuous.*

PROOF. Since a sequence (f_k) is (*ID*)-convergent to an $f \in C(R)$, it follows that for an arbitrary $\varepsilon > 0$ there exists a $k_0 > 0$ such that

$$(ID)(f, f_{k_0}) \leq \frac{\varepsilon}{3}.$$

By (*IC*)-continuity of an f_{k_0}, there exists a $\delta = \delta(\varepsilon, k_0) > 0$ such that

$$(ID)((f_{k_0})_h, f_{k_0}) \leq \frac{\varepsilon}{3} \quad \text{for} \quad h \in R, \ |h| < \delta,$$

where $(f_{k_0})_h(x) \equiv f_{k_0}(x+h)$. Hence for $|h| < \delta$ we obtain

$$(ID)(f, f_h) \leq (ID)(f, f_{k_0}) + (ID)(f_{k_0}, (f_{k_0})_h) + (ID)((f_{k_0})_h, f_h) \leq \varepsilon,$$

because $T((f_{k_0})_h, f_h) = |(F_{k_0})_h - F_h|$, and so $(ID)((f_{k_0})_h, f_h) = (ID)(f, f_{k_0})$. Thus an f is (*IC*)-continuous.

A set $E \subset R$ is called *relatively dense* iff there exists a positive number l such that in every open interval $(\alpha, \alpha + l)$, $\alpha \in R$, there is at least one element of the set *E*. A number $\tau \in R$ is called *an* $((ID), \varepsilon)-$*almost period* $(((ID), \varepsilon)-$a.p.) of a function $f \in C(R)$ iff $(ID)(f, f_\tau) \leq \varepsilon$, for $\varepsilon > 0$. Let $(IE)\{\varepsilon; f\}$ denote the set of $((ID), \varepsilon)$ -a.periods of an f.

A function $f \in C(R)$ is called (*IC*)-*almost periodic* ((*IC*)-a.p.) iff for each $\varepsilon > 0$ the set $(IE)\{\varepsilon; f\}$ is relatively dense. By $\overline{(IC)}$ we denote the set of (*IC*)-a.p. functions.

It is obvious that every (*IC*)-a.p. function is uniformly almost periodic (*B*-a.p.)

As regards the relation between (*IC*)-boundedness and (*IC*)-continuity, there holds the following:

REMARK 1.2. For (*IC*)-a.p. functions classes of (*IC*)-bounded functions and of (*IC*)-continuous functions are identical. Generally, the class of (*IC*)-bounded functions and the class of (*IC*)-continuous functions are different. More, there exist functions belonging to one of these classes and not belonging to the other one (see Examples:1.16,1.17).

Denote

$$F(u) = \int_0^u f(s)ds \quad \text{for} \quad u \in R.$$

There are known the following:

LEMMA 1.3. *Let f be a B-a.p. function. Then for an arbitrary* $\varepsilon > 0$ *there exists an* $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon) > 0$ *such that* $\bar{\varepsilon} < \varepsilon/3$ *and every* $\bar{\varepsilon} - a.p.$ *of an f is* $(\varepsilon/3) - a.p.$ *of the bounded indefinite integral F (*see [5], p.29).

LEMMA 1.4. *For any B-a.p. functions f, g and for an arbitrary* $\varepsilon > 0$ *there exists a relatively dense set of their commonly* ε*-a.periods* (see [5], p. 203 or [4], p. 432).

REMARK 1.5. A function *f* is (*IC*)-a.p. if and only if an *f* and its indefinite integral *F* are uniformly a.p. functions.

SUBSTANTIATION (NECESSITY) It is easily seen that if *f* is a (*IC*)-a.p. function, then functions *f* and *F* are *B*-a.p.

(SUFFICIENCY) Let $\varepsilon > 0$. Since *f*, *F* are *B*-a.p. functions, so using Lemma 1.4 we obtain that there exists the relatively dense set $E\{\varepsilon/2; f, F\}$ of their commonly ε-a.periods. Thus for $\tau \in E\{\varepsilon/2; f, F\}$ we have the estimation $(ID)(f, f_\tau) \le \varepsilon$. We conclude, $E\{\varepsilon/2; f, F\} \subset (IE)\{\varepsilon; f\}$, so $f \in \overline{(IC)}$.

Moreover, using the Bohl-Bohr Theorem on the bounded indefinite integral of a uniformly a.p. function (see [5], we have:

REMARK 1.6. A function *f* is (*IC*)-a.p. if and only if an *f* is a *B*-a.p. function and its indefinite integral *F* is bounded.

REMARK 1.7. The set of values of a *B*-a.p. function is connected.

1.2. BASIC PROPERTIES

THEOREM 1.8. *If f is an* (*IC*)*-a.p. function, then:*

(i) *an f is* (*IC*)*-bounded,*

(ii) *an f is* (*IC*)*-continuous.*

PROOF. Let $f \in \overline{(IC)}$.

(i) According to Remark 1.5, functions *f* and *F* are bounded, as uniformly a.p. functions. Thus we obtain $(ID)(f) \le M$, where $M > 0$ is a constant.

(ii) Similarly, by Remark 1.5, functions *f* and *F* are uniformly continuous (see [5], p. 22). Consequently, an *f* is (*IC*)-continuous.

Now, we shall be occupied with (*IC*)-a.periodicity of a linear combination of (*IC*)-a.p. functions, and next with (*IC*)-a.periodicity of a product of above functions.

THEOREM 1.9. *The following statements hold:*
(i) *A linear combination of (IC)-a.p. functions is an (IC)-a.p. function.*
(ii) *A product of two B-a.p. functions is (IC)-a.p. if and only if the indefinite integral of a product of these functions is bounded.*

PROOF. (i) Let $f, g \in \overline{(IC)}$. By Remark 1.5 and theorem on a sum of uniformly a.p. functions (see [5], p. 27), immediately we obtain $f + g \in \overline{(IC)}$. Moreover, $cf \in \overline{(IC)}$, where c is a constant.

(ii) Let f, g are B-a.p. functions.

(NECESSITY) Since $f g \in \overline{(IC)}$, so its indefinite integral F_{fg} is B-a.p., where $F_{fg}(u) = \int_0^u f(s)g(s)ds$ for $u \in R$. Thus F_{fg} is bounded (see [5], p. 22).

(SUFFICIENCY) Since fg is a uniformly a.p. function and its indefinite integral F_{fg} is bounded, so, by Remark 1.6, we get that a product fg is a (*IC*)-a.p. function.

COROLLARY 1.10. *A product of two (IC)-a.p. functions is (IC)-a.p. if and only if the indefinite integral of a product of these functions is bounded.*

THEOREM 1.11. *If a sequence (f_k) of (IC)-a.p. functions is (ID)-convergent to a function $f \in C(R)$, then an f is (IC)-a.p.*

PROOF. Since a sequence (f_k) is (*ID*)-convergent to an $f \in C(R)$, it follows that for an arbitrary $\varepsilon > 0$ there exists a $k_0 > 0$ such that $(ID)(f, f_{k_0}) \le \varepsilon/3$. We have $f_{k_0} \in \overline{(IC)}$, so for $\tau \in (IE)\{\varepsilon/3; f_{k_0}\}$ there holds

$$(ID)(f, f_\tau) \le (ID)(f, f_{k_0}) + (ID)(f_{k_0}, (f_{k_0})_\tau) + (ID)((f_{k_0})_\tau, f_\tau) \le \varepsilon.$$

Thus $(IE)\{\varepsilon/3; f_{k_0}\} \subset (IE)\{\varepsilon; f\}$, i.e. $f \in \overline{(IC)}$.

In the following, we shall seek for sufficient condition under which the derivative f' of a function $f \in \overline{(IC)}$ will be an (*IC*)-a.p. function, as well.

THEOREM 1.12. *If the derivative f' of a B-a.p. function f is uniformly continuous, then f' is an (IC)-a.p. function.*

PROOF. It is known that for a *B*-a.p. function *f* such that f' is uniformly continuous we have that f' is also *B*-a.p. (see [5], p. 28). Moreover, it follows

$$F_{f'}(t) = f(t) - f(0) \quad \text{for } t \in R,$$

where $F_{f'}(u) = \int_0^u f'(s)ds$, $u \in R$. Therefore $F_{f'}$ is a uniformly a.p. function. Consequently, according to Remark 1.5, we obtain $f' \in \overline{(IC)}$.

Now, we shall investigate the indefinite integral of a uniformly a.p. function. It is known that if the indefinite integral of a *B*-a.p. function is bounded, then this integral is a *B*-a.p. function (see [5], the Bohl-Bohr Theorem). We shall give more, namely there follow:

THEOREM 1.13. *If f is a B-a.p. function and its indefinite integral F is (IC)-bounded, then F is an (IC)-a.p. function and a $C^{(1)}$ – a.p. function* (see [1]).

PROOF. By theorem on the bounded indefinite integral of a uniformly a.p. function, an *F* is *B*-a.p., as a bounded function. Moreover, similarly we have F_F is a *B*-a.p. function, where $F_F(u) = \int_0^u F(s)ds$ for $u \in R$, because F_F is bounded too. We conclude, using Remark 1.5, that $F \in \overline{(IC)}$. According to Theorem 7 in [1], we obtain $F \in \overline{C^{(1)}}$. The proof is complete.

REMARK 1.14. If *f* is an (*IC*)-a.p. function and the indefinite integral F_F of a function *F* is bounded, then the indefinite integral *F* of an *f* is (*IC*)-a.p. and $C^{(1)}$ – a.p. (see [1]).

Finally, we shall be occupied with (*IC*)-a.periodicity of a function *f*.

THEOREM 1.15. *The following statements hold:*

(i) *If f is a uniformly continuous function and its indefinite integral F is uniformly a.p., then an f is (IC)-a.p.*

(ii) *If the derivative f' is a uniformly a.p. function and a function f is (IC)-bounded, then an f is (IC)-a.p. and $C^{(1)}$ – a.p.* (see [1]).

(iii) *If the derivative f' is an (IC)-a.p. function and the indefinite integral F of a function f is bounded, then an f is (IC)-a.p. and $C^{(1)}$ – a.p.* (see [1]).

1.3. EXAMPLES

First, we shall give examples of: an (*IC*)-bounded function which is not (*IC*) -continuous (Example 1.16) and an (*IC*)-continuous function which is not (*IC*)-bounded (Example 1.17).

Let (*NQ*) denote the set of irrational numbers.

EXAMPLE 1.16. Let f be the function defined by $f(x) = \sin\varphi(|x|)$ for $x \in R$, where φ is a φ-function (see [6] or [7]) strictly increasing such that the inverse function (φ_{-1}) has the finite derivative $(\varphi_{-1})'$ on $(0,\infty)$, satisfying the following condition:

(1) $$(\varphi_{-1})'(t) \downarrow 0 \quad \text{with} \quad t \to \infty .$$

Then the function f is (*IC*)-bounded, because in paper [1], p. 4, there shows that the indefinite integral $F(x) = \int_0^x \sin\varphi(|s|)ds$, $x \in R$, is bounded. Moreover, f is a bounded function, as well. However, the function f is not (*IC*)-continuous, since, by the condition (1), f is not a uniformly continuous function.

In particular, we may also take $\varphi(u) = u^p$, with $p > 1$, or $\varphi(u) = e^u - 1$ for $u \geq 0$.

EXAMPLE 1.17. The function defined by $f(x) = 1 + \cos x$, $x \in R$, has the unbounded indefinite integral *F*, so is not (*IC*)-bounded. However, *f* is an (*IC*)-continuous function.

EXAMPLE 1.18. The function *f* in the form $f(x) = 2x\cos(x^2)$, $x \in R$, is unbounded, but its indefinite integral $F(x) = \sin(x^2)$, $x \in R$, is bounded.

It is easily given an example of an (*IC*)-a.p. function. However, let us remark that every constant function which is different from zero doesn't belong to the class $\overline{(IC)}$.

As regards the relation between (*IC*)-a.p. and *B*-a.p. functions, there holds the following contain: $\overline{(IC)} \not\subset \widetilde{B}$, where $\widetilde{B}$ denote the set of uniformly a.p. functions (Examples: 1.19, 1.20).

EXAMPLE 1.19. Let *f* be a *B*-a.p. function which has the bounded indefinite integral *F*. Then $f \in \overline{(IC)}$.

In particular, we may also take the function *f* in the form $f(x) = \sin x + \sin(\alpha x)$ for $x \in R$, where $\alpha \in NQ$, since *F* is bounded.

EXAMPLE 1.20. Let f be a B-a.p. function which has the unbounded indefinite integral F. Then $f \notin \overline{(IC)}$.

In particular, we may also take the function f defined by $f(x) = 2 + \cos x + \cos(\alpha x)$ for $x \in R$, where $\alpha \in NQ$. Then f is a uniformly a.p. function and $f \notin \overline{(IC)}$.

Let us still remark that above function f is $C^{(1)} - a.p.$ (see [1]), because the derivative f' is B-a.p.

Now, we shall give an example of an (IC)-a.p. function which is not $C^{(1)} - a.p.$ (Example 1.21 and see [1]). An example of a $C^{(1)} - a.p.$ function which is not (IC)-a.p. is the function from Example 1.20.

EXAMPLE 1.20. Let

$$f(x) = \begin{cases} \left(x - \frac{2k}{\pi}\right)^2 \sin \frac{1}{x - \frac{2k}{\pi}} & \text{for } x \in \left\langle \frac{2k-1}{\pi}, \frac{2k+1}{\pi} \right\rangle \setminus \left\{ \frac{2k}{\pi} \right\}, \\ 0 & \text{for } x = \frac{2k}{\pi}, \end{cases}$$

$$g(x) = \begin{cases} \left(\sqrt{2}x - \frac{2k}{\pi}\right)^2 \sin \frac{1}{\sqrt{2}x - \frac{2k}{\pi}} & \text{for } x \in \left\langle \frac{\sqrt{2}(2k-1)}{2\pi}, \frac{\sqrt{2}(2k+1)}{2\pi} \right\rangle \setminus \left\{ \frac{\sqrt{2}k}{\pi} \right\}, \\ 0 & \text{for } x = \frac{\sqrt{2}k}{\pi}, \end{cases}$$

where $k = 0, \pm 1, \pm 2, \dots$.

Functions f and g are periodic with periods $T_f = 2/\pi$, $T_g = \sqrt{2}/\pi$, respectively. Moreover, f and g are continuous functions on R. Derivatives f', g' exist at every point $x \in R$, but the derivative f' is not continuous at points $x = 2k/\pi$ and the derivative g' is not continuous at points $x = \sqrt{2}k/\pi$, $k = 0, \pm 1, \pm 2, \dots$. Thus the sum $h = f + g$ is a B-a.p. function and the derivative $h' \notin C(R)$. From where we obtain $h \notin \overline{C^{(1)}}$. Using Remark 1.6, we need only to see that the indefinite integral $H(x) = \int_0^x h(s)ds$, $x \in R$, is bounded. Namely, there exist positive constants M_1 and M_2 such that for each $x \in R$ there exists a $k_0 \in Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ such that

$$\left|\int_0^x f(s)ds\right| \le \left|\int_0^{1/\pi} f(s)ds\right| + \left|\int_{(2k_0+1)/\pi}^x f(s)ds\right| \le M_1$$

and

$$\left|\int_0^x g(s)ds\right| \le M_2 .$$

Consequently, taking $0 < M = 2\max\{M_1, M_2\}$, for every $x \in R$ it follows $|H(x)| \le M$. Finally, we have $h \in \overline{(IC)}$.

EXAMPLE 1.22. Let f be a B-a.p. function which have the uniformly continuous derivative f' and the bounded indefinite integral F. Then there holds $f \in \overline{(IC)} \cap \overline{C^{(1)}}$.

In particular, let f be the function in the form $f(x) = \cos(\alpha x) + \cos(\beta x)$ for $x \in R$, where $\alpha, \beta \in R \setminus \{0\}$. Then $f \in \overline{C^{(1)}}$ and $f \in \overline{(IC)}$. If, moreover, we assume that α and β are incommensurable, then the function f is not periodic.

1.4. STEKLOV FUNCTIONS

We first recall the basic notation related to Steklov functions.

For a given positive number h and for a function $f : R \to R$ which is locally integrable, put

$$S_f(h)(u) = \frac{1}{2h}\int_{u-h}^{u+h} f(s)ds \qquad \text{for} \qquad u \in R.$$

Then an $S_f(h)$ is called *the Steklov function* of an f.

It is easy to seen that it follows:

THEOREM 1.23. *The following statements hold:*

(i) *If f is an (IC)-a.p. function, then the Steklov function $S_f(h)$ is an (IC)-a.p. function and a $C^{(1)}$ – a.p. function* (*see* [1]).

(ii) *If f is an (IC)-continuous function, then* $\lim_{h\to 0}(ID)(f, S_f(h)) = 0$.

PROOF. (i) We assume that $f \in \overline{(IC)}$. Then, by Remark 1.5, f and F are B-a.p. functions. Moreover, we know that $S_f(h)$ is a uniformly a.p. function, as well (see [5]). Let us still remark that the indefinite integral $F_{S_f(h)}$ of an $S_f(h)$ is bounded, from where, according to Remark 1.6, we obtain $S_f(h) \in \overline{(IC)}$. The function $S_f(h)$ is $C^{(1)}-a.p.$ (see [2]), too.

(ii) In the same way as in [5], for an arbitrary but fixed $t \in R$ we write

$$| f(t) - S_f(h)(t)| \leq \frac{1}{2h} \int_{-h}^{h} | f(t) - f(s+t)| ds,$$

$$| F(t) - F_{S_f(h)}(t)| \leq \frac{1}{2h} \int_{-h}^{h} \left| \int_{0}^{t} (f(s) - f(s+x)) ds \right| dx$$

for $h > 0$. Since *f* is an (*IC*)-continuous function, so for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $(ID)(f, f_s) \leq \varepsilon/3$ for $s \in R$, $|s| < \delta$. Thus for all $t \in R$ and $h \in (0, \delta)$ we obtain $(ID)(f, S_f(h)) \leq \varepsilon$, and the proof is complete.

COROLLARY 1.24. *If f is an (IC)-a.p. function, then*

$$\lim_{h \to 0} (ID)(f, S_f(h)) = 0.$$

2. $(IC)^{(n)}$ - ALMOST PERIODIC FUNCTIONS

2.1. DEFINITIONS

We first present basic notations related to $(IC)^{(n)}$-almost periodic functions. Let N_0 denote the set $N \cup \{0\}$.

By $C^{(n)}(R)$ we denote the set of functions from R into itself with n-th continuous derivatives on *R*, for $n \in N_0$. For functions $f, g \in C^{(n)}(R)$ we define $(ID)^{(n)}$ – distans in the following

$$(ID)^{(n)}(f,g) = \sup_{t \in R} \left(| f(t) - g(t)| + \sum_{i=1}^{n} \left| f^{(i)}(t) - g^{(i)}(t) \right| + T(f,g)(t) \right),$$

where *T* is defined in the section 1.1.

We say that an $f \in C^{(n)}(R)$ is $(IC)^{(n)}$*-bounded* iff $(ID)^{(n)}(f) < \infty$, where $(ID)^{(n)}(f) = (ID)^{(n)}(f, 0)$. We say that $f \in C^{(n)}(R)$ is *an* $(IC)^{(n)}$*-continuous* *function* iff $\lim_{h\to 0}(ID)^{(n)}(f, f_h) = 0$. A sequence (f_k) in $C^{(n)}(R)$ will be called $(ID)^{(n)}$ *-convergent to an* $f \in C^{(n)}(R)$ iff $\lim_{k\to\infty}(ID)^{(n)}(f, f_k) = 0$.

Similarly as in Theorem 1.1, we obtain:

THEOREM 2.1. *If a sequence* (f_k) *of* $(IC)^{(n)}$*-continuous functions is* $(ID)^{(n)}$ *-convergent to a function* $f \in C^{(n)}(R)$*, then an* f *is* $(IC)^{(n)}$*-continuous.*

A function $f \in C^{(n)}(R)$ is called $(IC)^{(n)}$ *–almost periodic* ($(IC)^{(n)}$ –a.p.) iff an f is (IC)-a.p. and $C^{(n)}$ – a.p. (see [1]), $n \in N_0$. By $\overline{(IC)^{(n)}}$ we denote the set of $(IC)^{(n)}$ –a.p. functions, i.e. we have

$$\overline{(IC)^{(n)}} = \overline{(IC)} \cap \overline{C^{(n)}}.$$

For an arbitrary $n \in N_0$ every $(IC)^{(n+1)}$ –a.p. function is $(IC)^{(n)}$ –a.p. Moreover, every $(IC)^{(1)}$ –a.p. function is an L-a.p. function (see [1], [8]).

2.2. BASIC PROPERTIES

Properties of $(IC)^{(n)}$ –a.p. functions we obtain using known theorems related to (IC)-a.p. and $C^{(n)}$ – a.p. functions (see [1], [2]).

THEOREM 2.2. *If f is an* $(IC)^{(n)}$ *– a.p. function, then*:

(i) *an f is* $(IC)^{(n)}$ *–bounded,*

(ii) *an f is* $(IC)^{(n)}$ *–continuous.*

PROPOSITION 2.3. *If a sequence* (f_k) *of* $(IC)^{(n)}$ *– a.p. functions* $(ID)^{(n)}$ *– convergent to a function* $f \in C^{(n)}(R)$*, then an* f *is* $(IC)^{(n)}$ *– a.p.*

THEOREM 2.4. *The following statements hold:*

(i) *A linear combination of* $(IC)^{(n)}$ *–a.p. functions is an* $(IC)^{(n)}$ *–a.p. function.*

(ii) *A product of two* $C^{(n)}$ *– a.p. functions is* $(IC)^{(n)}$ *– a.p. if and only if the indefinite integral of a product of these functions is bounded.*

(iii) *If the derivative* $f^{(n+1)}$ *of a* $C^{(n)}$ *– a.p. function f is uniformly continuous, then the derivative* f' *is an* $(IC)^{(n)}$ *– a.p. function.*

(iv) *If f is an* $(IC)^{(n)}$ *– a.p. function and the indefinite integral* F_F *of an F, where F is the indefinite integral of an f, is bounded, then F is* $(IC)^{(n+1)}$ *– a.p. function.*

Now, we shall be occupied with $(IC)^{(n)}$ –a.periodicity of a function *f*.

THEOREM 2.5. *The following statements hold:*

(i) *If the derivative* $f^{(n)}$ *of a function f is uniformly continuous and the indefinite integral F of an f is* $C^{(n)}$ *– a.p., then f is an* $(IC)^{(n)}$ *– a.p. function.*

(ii) *If the derivative* f' *is an* $(IC)^{(n)}$ *– a.p. function and the indefinite integral F of a function f is bounded, then f is an* $(IC)^{(n+1)}$ *– a.p. function.*

Finally, we shall give an example of an $(IC)^{(n)}$ – a.p. function.

EXAMPLE 2.6. Let *f* be the function defined by $f(x) = \cos(\alpha x) + \sin(\beta x)$ for $x \in R$, where $\alpha, \beta \in R \setminus \{0\}$ are incommensurable. Then $f \in \overline{C^{(n)}}$ and $f \in \overline{(IC)}$, because *F* is a bounded function. Thus $f \in C^{(n)}(R)$ is a $(IC)^{(n)}$ – a.p. function, but not periodic.

2.3. STEKLOV FUNCTIONS

In this section we shall give the theorem on $(IC)^{(n)}$ –periodicity of Steklov functions. Using Theorem 1.23 and the theorem on approximation of $C^{(n)}$ – a.p. functions by their Steklov functions (see [2]), similarly as in the part 1.4, we obtain:

THEOREM 2.7. *The following statements hold:*

(i) *If f is an* $(IC)^{(n)}$ *– a.p. function, then the Steklov function* $S_f(h)$ *is an* $(IC)^{(n+1)}$ *– a.p. function.*

(ii) *If f is an* $(IC)^{(n)}$ *–continuous function, then* $\lim_{h\to 0}(ID)^{(n)}(f, S_f(h)) = 0$.

COROLLARY 2.8. *If f is an $(IC)^{(n)}$ – a.p. function, then*

$$\lim_{h\to 0}(ID)^{(n)}(f, S_f(h)) = 0.$$

3. $(IC)_a^{(\infty)}$ – ALMOST PERIODIC FUNCTIONS

3.1. DEFINITIONS

We first recall basic notations related to $(IC)_a^{(\infty)}$ – almost periodic functions.

By $C^{(\infty)}(R)$ we denote the set of functions from R into itself which have every derivatives. For functions $f, g \in C^{(\infty)}(R)$ and a sequence $a = (a_i)$ such that $a_i > 0$, $i = 1, 2, \ldots$, we define $(ID)_a^{(\infty)}$ – distans in the following

$$(ID)_a^{(\infty)}(f,g) = \sup_{t\in R}\left(|f(t)-g(t)| + \sum_{i=1}^{\infty} a_i\left|f^{(i)}(t)-g^{(i)}(t)\right| + T(f,g)(t)\right),$$

where T is defined in the section 1.1.

We say that an $f \in C^{(\infty)}(R)$ is $(IC)_a^{(\infty)}$ – *bounded* iff $(ID)_a^{(\infty)}(f) < \infty$, where $(ID)_a^{(\infty)}(f) = (ID)_a^{(\infty)}(f,0)$. We say that $f \in C^{(\infty)}(R)$ is *an* $(IC)_a^{(\infty)}$ – *continuous function iff* $\lim_{h\to 0}(ID)_a^{(\infty)}(f, f_h) = 0$. A sequence (f_k) in $C^{(\infty)}(R)$ will be called $(ID)_a^{(\infty)}$ – *convergent to an* $f \in C^{(\infty)}(R)$, iff $\lim_{h\to 0}(ID)_a^{(\infty)}(f, f_k) = 0$.

Similarly as in Theorem 1.1, we get:

THEOREM 3.1. *If a sequence (f_k) of $(IC)_a^{(\infty)}$ – continuous functions is $(ID)_a^{(\infty)}$ – convergent to a function $f \in C^{(\infty)}(R)$, then an f is $(IC)_a^{(\infty)}$ – continuous.*

We say that an $f \in C^{(\infty)}(R)$ is *conditionally locally bounded with respect to a sequence* $a = (a_i)$: $a_i > 0$, $a_{i+1} \le a_i$, $a = 1, 2, \ldots$, (i.e. $f \in (CBC_{a,loc}^{(\infty)})$) iff for an arbitrary closed interval $<x, y>$ and for each $i = 0, 1, 2, \ldots$ there exists a positive number $M_i = M_{i,f}^{<x,y>}$ such that

$$\max_{t\in<x,y>}\left|f^{(i)}(t)\right| = M_i \quad\text{and}\quad \sum_{i=1}^{\infty} a_i M_{i+1} < \infty .$$

A function $f \in (CBC_{a,loc}^{(\infty)})$ is called $(IC)_a^{(\infty)}$ – *almost periodic* ($(IC)_a^{(\infty)}$ – a.p.) iff an *f* is (IC) – a.p. and $(C)_a^{(\infty)}$ – a.p. (see [3]). By $\overline{(IC)_a^{(\infty)}}$ we denote the set of $(IC)_a^{(\infty)}$ – a.p. functions, i.e. we have

$$\overline{(IC)_a^{(\infty)}} = \overline{(IC)} \cap \overline{C_a^{(\infty)}}.$$

It is seen that every $(IC)_a^{(\infty)}$ – a.p. function is $(IC)^{(n)}$ – a.p., for an arbitrary fixed $n \in N_0$. Every $(IC)_a^{(\infty)}$ – a.p. function is an *L*-a.p. function (see [3], [8]), as well.

3.2. Basic properties

Properties of $(IC)_a^{(\infty)}$ – a.p. functions we obtain in the same way as properties of $(IC)^{(n)}$ – a.p. functions, using known theorems related to (IC) – a.p. and $C_a^{(\infty)}$ – a.p. functions (see [3]).

Theorem 3.2. *The following statements hold*:

(i) *Every* $(IC)_a^{(\infty)}$ – *a.p. function is* $(IC)_a^{(\infty)}$ – *bounded and* $(IC)_a^{(\infty)}$ – *continuous.*

(ii) *A linear combination of* $(IC)_a^{(\infty)}$ – *a.p. functions is an* $(IC)_a^{(\infty)}$ – *a.p. function.*

(iii) *Let be given the sequence* $b=(b_i): b_i = a_{i+1}^2 / c_i$ *with* $c_i > 0$, $c_i \le c_{i+1}$, $c_i \ge \binom{i+1}{[(i+1)/2]}$, $i = 1, 2, \ldots$, *and* $\sum_{i=1}^{\infty} 2^i / c_i < \infty$. *Moreover, let the indefinite integral of a product of two* $(IC)_a^{(\infty)}$ – *a.p. functions f, g is bounded. Then a product fg is an* $(IC)_b^{(\infty)}$ – *a.p. function.*

(iv) *If a sequence* (f_k) *of* $(IC)_a^{(\infty)}$ – *a.p. functions is* $(ID)_a^{(\infty)}$ – *convergent to a function* $f \in (CBC_{a,loc}^{(\infty)})$, *then an f is* $(IC)_a^{(\infty)}$ – *a.p.*

(v) *Let* $\sup\{a_i / a_{i+1} : i = 1, 2, \ldots\} < \infty$ *and let f be a* $C_a^{(\infty)}$ – *a.p. function. Then the derivative* f' *is an* $(IC)_a^{(\infty)}$ – *a.p. function.*

(v̇i) *If f is an $(IC)_a^{(\infty)}$ – a.p. function and the indefinite integral F_F of an F, where F is the indefinite integral of an f, is bounded, then F is also an $(IC)_a^{(\infty)}$ – a.p. function.*

(v̇ii) *If the indefinite integral F of a function f is a $C_a^{(\infty)}$ – a.p. function and* $\sup\{a_i / a_{i+1} : i = 1, 2, \ldots\} < \infty$ *then an f is $(IC)_a^{(\infty)}$ – a.p.*

(v̇iii) *If the derivative f' is an $(IC)_a^{(\infty)}$ – a.p. function and the indefinite integral F of a function f is bounded, then f is an $(IC)_a^{(\infty)}$ – a.p. function.*

PROOF. For example, we shall show (v̇i). Let $f \in \overline{(IC)_a^{(\infty)}}$. Since an F is uniformly a.p. and the indefinite integral F_F is bounded, so, according to Remark 1.6, we get $F \in \overline{(IC)}$. Moreover, $F \in \overline{C_a^{(\infty)}}$ as the bounded indefinite integral F of a $C_a^{(\infty)}$ – a.p. function f (see [3]). Consequently, an F is $(IC)_a^{(\infty)}$ – a.p.

Now, we shall give an example of an $(IC)_a^{(\infty)}$ – a.p. function.

EXAMPLE 3.3. Let f be the function defined by $f(x) = \sin(\alpha x) + \sin(\beta x)$ for $x \in R$, with $\alpha, \beta \in (-1,1)$ which are incommensurable. Then we have $f \in (CBC_{a,loc}^{(\infty)})$ and $f \in \overline{C_a^{(\infty)}}$, for a positive sequence $a = (a_i)$ such that $\sum_{i=1}^{\infty} a_i < \infty$. Moreover, $f \in \overline{(IC)}$, since F is bounded. Thus we obtain that f is an $(IC)_a^{(\infty)}$ – a.p. function, but not periodic.

3.3. STEKLOV FUNCTIONS

Finally, we shall be occupied with $(IC)_a^{(\infty)}$ – a.periodicity of Steklov functions. Using Theorem 1.23 and the theorem on approximation of $C_a^{(\infty)}$ – a.p. functions by their Steklov functions (see [3]), we obtain:

THEOREM 3.4. *The following statements hold:*

(i) *If f is an $(IC)_a^{(\infty)}$ – a.p. function, then the Steklov function $S_f(h)$ is also $(IC)_a^{(\infty)}$ – a.p.*

(ii) *If f is an $(IC)_a^{(\infty)}$ – continuous function, then* $\lim_{h\to 0}(ID)_a^{(\infty)}(f, S_f(h)) = 0$.

COROLLARY 3.5. *If f is an* $(IC)_a^{(\infty)}$ *– a.p. function, then*

$$\lim_{h\to 0}(ID)_a^{(\infty)}(f, S_f(h)) = 0.$$

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