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APPROXIMATION OF FUNCTIONS OF ONE AND TWO VARIABLES BY SOME OPERATORS

ABSTRACT: We consider some operatos of Szasz-Mirakyan type in exponential weighted spaces of functions of one and two variables.

We give theorems on the degree of approximation of functions by these operators and the Voronovskaya type theorems.

KEY WORDS: Szasz-Mirakyan operator, exponential weighted space, degree of approximation, Voronovskaya theorem.

I. APPROXIMATION OF FUNCTIONS OF ONE VARIABLE

1. INTRODUCTION

1.1. Similarly as in [1] let $q>0$ be a fixed number,

$$v_q(x) := e^{-qx}, \qquad x \in R_0 := [0,+\infty), \tag{1.1}$$

and let C_q be the space of all real-valued functions f continuous on R_0 for which $v_q f$ is uniformly continuous and bounded on R_0 and the norm is defined by the formula

$$\| f \|_q \equiv \| f(\cdot) \|_q := \sup_{x \in R_0} v_q(x) \, | f(x) | . \tag{1.2}$$

As in [1] for $f \in C_q$, $q>0$, we define the modulus of continuity $\omega_1(f;C_q;\cdot)$ and the modulus of smoothness $\omega_2(f;C_q;\cdot)$:

$$\omega_k(f;C_q;t) := \sup_{0 \le h \le t} \| \Delta_h^k f(\cdot) \|_q, \qquad t \ge 0, \ k=1,2, \tag{1.3}$$

where

$$\Delta_h^1 f(x) := f(x+h) - f(x), \quad \Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h),$$

for $x, h \in R_0$. It is known ([1], [6]) that if $f \in C_q$, $q>0$, and $k=1,2$, then

(a) $0 \le \omega_k(f;C_q;t_1) \le \omega_k(f;C_q;t_2)$ if $0 \le t_1 < t_2$,

(b) $\lim_{t\to 0^+} \omega_k(f;C_q;t) = 0$

Moreover let $C_q^m := \{f \in C_q : f^{(k)} \in C_q$ for $k = 1,2,...,m\}$, for fixed $m \in N := \{1,2,...\}$ and $q > 0$.

In the paper [1] were given approximation theorems for the Szasz-Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad x \in R_0, \quad n \in N,$$

for functions $f \in C_q$, $q > 0$.

In [2], for $f \in C_q$, were considered operators

$$A_n(f;x) := \sum_{k=0}^{\infty} a_k(nx) f\left(\frac{2k}{n}\right), \qquad x \in R_0, \quad n \in N, \tag{1.4}$$

$$a_k(t) := \frac{1}{\cosh t} \frac{t^{2k}}{(2k)!}, \qquad t \in R_0, \quad k \in N_0 := N \cup \{0\}, \tag{1.5}$$

and in [3] were examined operators

$$B_n(f;x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} b_k(nx) f\left(\frac{2k+1}{n}\right), \tag{1.6}$$

$x \in R_0$, $n \in N$, $f \in C_q$, where

$$b_k(t) := \frac{1}{1+\sinh t} \frac{t^{2k+1}}{(2k+1)!}, \qquad t \in R_0, \quad k \in N_0, \tag{1.7}$$

and $\sinh x$, $\cosh x$, $\operatorname{tgh} x$ are elementary hyperbolic functions.

In [1] was proved that S_n is a positive linear operator form the space C_q into C_r provided that $r > q > 0$ and $n > n_0$, where n_0 is a fixed natural number such that $n_0 > q/\ln(r/p)$. In [2] and [3] were proved that these properties have also operators A_n and B_n defined by (1.4) and (1.6).

In [2] and [3] was proved the following theorem

THEOREM I. *Let* $f \in C_q$, $q > 0$. *Then*

$$\lim_{n\to\infty} A_n(f;x) = f(x) = \lim_{n\to\infty} B_n(f;x)$$

at every $x \in R_0$. *This convergence is uniform on every interval* $[x_1, x_2]$, $x_1 \ge 0$.

In [4] was proved the following

THEOREM II. *Let* $f \in C_q^2$, $q > 0$. *Then*

$$\lim_{n\to\infty} n\{A_n(f;x) - f(x)\} = \frac{x}{2} f''(x),$$

$$\lim_{n\to\infty} n\{B_n(f;x) - f(x)\} = \frac{x}{2} f''(x),$$

at every $x \in R_0$.

1.2. In this paper we modify formulas (1.4) and (1.6) such that new operators are positive linear operators from the space C_q into C_q, $q > 0$.

Let $q > 0$ be a fixed number. For functions $f \in C_q$ and $n \in N$, $x \in R_0$ we introduce the following operators:

$$(1.8) \qquad L_n^{\{1\}}(f;q;x) := \sum_{k=0}^{\infty} a_k(nx) f\left(\frac{2k}{n+q}\right),$$

$$(1.9) \qquad L_n^{\{2\}}(f;q;x) := \frac{n+q}{2} \sum_{k=0}^{\infty} a_k(nx) \int_{2k/(n+q)}^{(2k+2)/(n+q)} f(t)dt,$$

$$(1.10) \qquad L_n^{\{3\}}(f;q;x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} b_k(nx) f\left(\frac{2k+1}{n+q}\right),$$

$$(1.11) \qquad L_n^{\{4\}}(f;q;x) := \frac{f(0)}{1+\sinh nx} + \frac{n+q}{2} \sum_{k=0}^{\infty} b_k(nx) \int_{(2k+1)/(n+q)}^{(2k+3)/(n+q)} f(t)dt,$$

where $a_k(\cdot)$ and $b_k(\cdot)$ are defined by (1.5) and (1.7).

We observe that for every $q > 0$ and $i = 1,2,3,4$, $L_n^{\{i\}}$ is positive linear operator and

$$(1.12) \qquad L_n^{\{i\}}(1;q;x) = 1, \quad x \in R_0, \quad n \in N.$$

We shall prove that $L_n^{\{i\}}(f;q)$, $n \in N$, $q > 0$, is an operator from C_q into C_q.

In Section 2 we shall give some auxiliary results and in Section 3 we give main approximation theorems.

In this paper we shall denote by $M_k(a,b)$, $k = 1,2,\ldots$, suitable positive constants depending only on indicated parameters a, b.

2. Auxiliary results

2.1. By elementary calculations we can prove the following

Lemma 1. *For every* $q > 0$, $x \in R_0$ *and* $n \in N$ *we have*

$$L_n^{\{1\}}(t-x;q;x) = \left(\frac{n}{n+q}\operatorname{tgh} nx - 1\right)x,$$

$$L_n^{\{1\}}((t-x)^2;q;x) = \left\{\left(\frac{q}{n+q}\right)^2 + \frac{2n}{n+q}(1-\operatorname{tgh} nx)\right\}x^2 + \frac{nx}{(n+q)^2}\operatorname{tgh} nx,$$

$$L_n^{\{1\}}(e^{qt};q;x) = \frac{\cosh(nxe^{q/(n+q)})}{\cosh nx},$$

$$L_n^{\{1\}}((t-x)e^{qt};q;x) = \frac{nxe^{q/(n+q)})}{n+q}\frac{\sinh(nxe^{q/(n+q)})}{\cosh nx} - x\frac{\cosh(nxe^{q/(n+q)})}{\cosh nx},$$

$$L_n^{\{1\}}((t-x)^2e^{qt};q;x) = \left\{\frac{\cosh(nxe^{q/(n+q)})}{\cosh nx}\left(\frac{n}{n+q}e^{q/(n+q)} - 1\right)^2 + \right.$$
$$\left. + \frac{2n}{n+q}\frac{\exp(-nxe^{q/(n+q)} + q/(n+q))}{\cosh nx}\right\}x^2 +$$
$$+ \frac{nxe^{q/(n+q)}}{(n+q)^2}\frac{\sinh(nxe^{q/(n+q)})}{\cosh nx},$$

$$L_n^{\{2\}}(t-x;q;x) = L_n^{\{1\}}(t-x);q;x) + \frac{1}{n+q},$$

$$L_n^{\{2\}}((t-x)^2;q;x) = L_n^{\{1\}}((t-x)^2;q;x) + \frac{2}{n+q}L_n^{\{1\}}(t-x;q;x) + \frac{4}{3(n+q)^2},$$

$$L_n^{\{2\}}(e^{qt};q;x) = \frac{n+q}{2q}(e^{2q/(n+q)} - 1)L_n^{\{1\}}(e^{qt};q;x),$$

$$L_n^{\{3\}}(t-x;q;x) = \left(\frac{n}{n+q}\frac{\cosh nx}{1+\sinh nx} - 1\right)x,$$

$$L_n^{\{3\}}((t-x)^2;q;x) = \left\{\left(\frac{n}{n+q}\right)^2\frac{\sinh nx}{1+\sinh nx} - \frac{2n}{n+q}\frac{\cosh nx}{1+\sinh nx} + 1\right\}x^2 +$$

$$+\frac{nx}{(n+q)^2}\frac{\cosh nx}{1+\sinh nx},$$

$$L_n^{\{3\}}(e^{qt};q;x) = \frac{1+\sinh(nxe^{q/(n+q)})}{1+\sinh nx},$$

$$L_n^{\{3\}}((t-x)e^{qt};q;x) = \frac{x}{1+\sinh nx}\left\{\frac{n}{n+q}e^{q/(n+q)}\cosh(nxe^{q/(n+q)})-1-\sinh(nxe^{q/(n+q)})\right\},$$

$$L_n^{\{3\}}((t-x)^2e^{qt};q;x) = \frac{x^2}{1+\sinh nx}\left\{\left[\left(\frac{n}{n+q}\right)^2e^{2q/(n+q)}+1\right]\sinh(nxe^{q/(n+q)}) -\right.$$

$$\left.-\frac{2n}{n+q}e^{q/(n+q)}\cosh(nxe^{q/(n+q)})+1\right\}+$$

$$+\frac{nx}{(n+q)^2}e^{q/(n+q)}\frac{\cosh(nxe^{q/(n+q)})}{1+\sinh nx},$$

$$L_n^{\{4\}}(t-x;q;x) = L_n^{\{3\}}(t-x;q;x)+\frac{1}{n+q}\frac{\sinh nx}{1+\sinh nx},$$

$$L_n^{\{4\}}((t-x)^2;q;x) = L_n^{\{3\}}((t-x)^2;q;x)+\frac{2}{n+q}L_n^{\{3\}}(t-x;q;x)+$$

$$+\frac{2x}{n+q}\frac{1}{1+\sinh nx}+\frac{4}{3(n+q)^2}\frac{\sinh nx}{1+\sinh nx},$$

$$L_n^{\{4\}}(e^{qt};q;x) = \frac{n+q}{2q}(e^{2q/(n+q)}-1)L_n^{\{3\}}(e^{qt};q;x)+$$

$$+\frac{1}{1+\sinh nx}\left[1-\frac{n+q}{2q}(e^{2q/(n+q)}-1)\right].$$

Applying Lemma 1, we shall prove the main lemma.

LEMMA 2. *Let $q>0$ be a fixed number. Then*

(1.13) $$\| L_n^{\{i\}}(e^{qt};q;\cdot)\|_q \le M_i \quad \textit{for}\ \ n\in N,\ 1\le i\le 4,$$

where $M_1=2$, $M_2=2e^2$, $M_3=3$, $M_4=5e^2$. Moreover for every $f\in C_q$ we have

(1.14) $$\| L_n^{\{i\}}(f;q;x)\|_q \le M_i\|f\|_q, \qquad n\in N,\ 1\le i\le 4,$$

where M_i *are constants given in* (1.13).

The inequality (1.14) *and definitions* (1.8) – (1.11) *and* (1.5), (1.7) *show* that $L_n^{\{i\}}(f;q)$, $n \in N$, $q > 0$, *is positive linear operator from the space* C_q *into* C_q.

Proof. By Lemma 1 we get

$$L_n^{\{1\}}(e^{qt};q;x) \le 2\exp[nx(e^{q/(n+q)}-1)],$$

$$L_n^{\{3\}}(e^{qt};q;x) \le \frac{2+\exp[nxe^{q/(n+q)}]}{2+e^{nx}-e^{-nx}} \le 3\exp[nx(e^{q/(n+q)}-1)],$$

for $x \in R_0$, $n \in N$ and $q > 0$. But

$$(1.15) \qquad 0 < e^{q/(n+q)} - 1 \le \sum_{k=1}^{\infty}\left(\frac{q}{n+q}\right)^k = \frac{q}{n} \quad \text{for} \quad q>0,\ \ n \in N,$$

which by (1.1) implies that

$$v_q(x)L_n^{\{1\}}(e^{qt};q;x) \le 2, \qquad v_q(x)L_n^{\{3\}}(e^{qt};q;x) \le 3,$$

for every $x \in R_0$ and $n \in N$, $q > 0$. Thus (1.13) is proved for $i = 1,3$.

In the case $i = 2$ and $i = 4$, we shall apply the following inequalities:

$$(1.16) \qquad \begin{cases} 0 \le e^t - 1 \le te^t, \qquad 0 \le e^t - 1 - t \le t^2 e^t, \\ 0 \le te^t - (e^t - 1) \le t^2 e^t \quad \text{for} \quad t \in R_0. \end{cases}$$

By Lemma 1 and (1.16) we get

$$\| L_n^{\{2\}}(e^{qt};q;\cdot)\|_q \le e^{2q/(n+q)} \| L_n^{\{1\}}(e^{qt};q;\cdot)\|_q \le$$
$$\le e^2 \| L_n^{\{1\}}(e^{qt};q;\cdot)\|_q,$$

$$\| L_n^{\{4\}}(e^{qt};q;\cdot)\|_q \le e^{2q/(n+q)} \| L_n^{\{3\}}(e^{qt};q;\cdot)\|_q + \frac{2q}{n+q}e^{2q/(n+q)} \le$$
$$\le e^2[\| L_n^{\{3\}}(e^{qt};q;\cdot)\|_q + 2],$$

for all $n \in N$ and $q > 0$, which by (1.13) for $i = 1,3$ imply (1.13) for $i = 2,4$.

From (1.8) – (1.11) it follows that

$$\| L_n^{\{i\}}(f;q;\cdot)\|_q \le \| f\|_q \| L_n^{\{i\}}(e^{qt};q;\cdot)\|_q$$

for every $f \in C_q$, $q > 0$, $n \in N$ and $1 \le i \le 4$. Now by (1.13) we obtain (1.14). Thus the proof is completed.

LEMMA 3. *Let $q>0$ be a fixed number. Then*

$$(1.17)\qquad |L_n^{\{i\}}(t-x;q;x)| \le \begin{cases} \dfrac{qx}{n+q}+\dfrac{1}{n} & \text{if } i=1, \\ \dfrac{qx}{n+q}+\dfrac{2}{n} & \text{if } i=3, \end{cases}$$

$$(1.18)\qquad L_n^{\{i\}}((t-x)^2;q;x) \le \begin{cases} \left(\dfrac{qx}{n+q}\right)^2+\dfrac{3x}{n+q} & \text{if } i=1, \\ \left(\dfrac{qx}{n+q}\right)^2+\dfrac{5x}{n+q} & \text{if } i=3, \end{cases}$$

$$(1.19)\qquad v_q(x)\left|L_n^{\{i\}}((t-x)e^{qt};q;x)\right| \le \begin{cases} \left(\dfrac{3q}{n+q}+\dfrac{1}{n}\right)x & \text{if } i=1, \\ \dfrac{2qx}{n+q}+\dfrac{2q+3}{n} & \text{if } i=3, \end{cases}$$

$$(1.20)\qquad v_q(x)\,L_n^{\{i\}}((t-x)^2e^{qt};q;x) \le \begin{cases} \dfrac{9q^4x^2}{(n+q)^4}+\dfrac{4x}{n+q} & \text{if } i=1, \\ \dfrac{9q^4x^2}{(n+q)^4}+\dfrac{6x}{n+q}+\dfrac{12}{n^2} & \text{if } i=3, \end{cases}$$

for all $x\in R_0$, $n\in N$.

PROOF. We shall prove only (1.17) – (1.20) for $i=1$, because the proof of (1.17) – (1.20) for $i=3$ is analogous.

From Lemma 1 and by $e^t>t$ for $t\in R_0$ we get

$$|L_n^{\{1\}}(t-x;q;x)| \le x|\operatorname{tgh} nx-1| + \frac{qx}{n+q}\operatorname{tgh} nx \le$$
$$\le \frac{2x}{e^{2nx}} + \frac{qx}{n+q} \le \frac{qx}{n+q}+\frac{1}{n},$$

$$L_n^{\{1\}}((t-x)^2;q;x) \le \left(\frac{q^2}{(n+q)^2}+\frac{2n}{n+q}\frac{2}{e^{2nx}}\right)x^2+\frac{nx}{(n+q)^2} \le \frac{q^2x^2}{(n+q)^2}+\frac{3x}{n+q},$$

for $x\in R_0$ and $n\in N$. By (1.16) we have

$$(1.21)\qquad \left|\frac{n}{n+q}e^{q/(n+q)}-1\right| = \left|e^{q/(n+q)} - 1 - \frac{q}{n+q}e^{q/(n+q)}\right| \le$$

$$\le \frac{q^2}{(n+q)^2}e^{q/(n+q)}, \quad n\in N.$$

Applying (1.15) and (1.21), we get by Lemma 1

$$v_q(x)|L_n^{\{1\}}((t-x)e^{qt};q;x)| \le xv_q(x)\left\{\left|\frac{n}{n+q}e^{q/(n+q)}-1\right|\frac{\sinh(nxe^{q/(n+q)})}{\cosh nx}+\right.$$

$$\left.+\frac{|\sinh(nxe^{q/(n+q)})-\cosh(nxe^{q/(n+q)})|}{\cosh nx}\right\}\le$$

$$\le x\left\{\frac{q^2}{(n+q)^2}e^{q/(n+q)}\exp[nx(e^{q/(n+q)}-1)-qx]+\frac{2}{e^{2nx}}\right\}\le$$

$$\le x\left\{\frac{3q}{n+q}+\frac{1}{n}\right\},$$

and analogously

$$v_q(x)|L_n^{\{1\}}((t-x)^2e^{qt};q;x)| \le x^2\left\{\frac{q^4}{(n+q)^4}e^{2q/(n+q)}\exp[nx(e^{q/(n+q)}-1)-qx]+\right.$$

$$\left.+\frac{2n}{n+q}\frac{1}{e^{2nx}}\right\}+\frac{nx}{(n+q)^2}e^{q/(n+q)}\exp[nx(e^{q/(n+q)}-1)-qx]\le$$

$$\le\frac{q^4e^2x^2}{(n+q)^4}+\frac{x}{n+q}+\frac{x}{n+q}e^{q/(n+q)}\le\frac{9q^4x^2}{(n+q)^4}+\frac{4x}{n+q}$$

for $x\in R_0$ and $n\in N$. Thus the proof is completed.

Remark. It is obvious that by formulas given in Lemma 1 and by Lemma 3 we can obtain analogies of (1.17)-(1.20) for $i=3$ and $i=4$.

3. Approximation theorems

3.1. First we shall prove four theorems on the degree of approximation.

Theorem 1. *Suppose that* $f\in C_q$ *with a fixed* $q>0$. *Then*

$$(1.22)\qquad \| L_n^{\{2\}}(f;q;\cdot) - L_n^{\{1\}}(f;q;\cdot) \|_q \le 2\omega_1\left(f;C_q;\frac{2}{n+q}\right),$$

$$\| L_n^{\{4\}}(f;q;\cdot) - L_n^{\{3\}}(f;q;\cdot) \|_q \le 3\omega_1\left(f;C_q;\frac{2}{n+q}\right),$$

for all $n \in N$.

PROOF. From (1.8) and (1.9) it follows that

$$(1.23)\qquad L_n^{\{2\}}(f;q;x) - L_n^{\{1\}}(f;q;x) =$$

$$= \frac{n+q}{2}\sum_{k=0}^{\infty} a_k(nx) \int_{2k/(n+q)}^{(2k+2)/(n+q)} \left[f(t) - f\left(\frac{2k}{n+q}\right)\right] dt$$

for $x \in R_0$ and $n \in N$. But for fixed *k*, *n* and $2k/(n+q) \le t \le (2k+2)/(n+q)$ we have by (1.1) – (1.3)

$$\left| f(t) - f\left(\frac{2k}{n+q}\right)\right| = \left| \Delta^1_{t-2k/(n+q)} f\left(\frac{2k}{n+q}\right)\right| \le$$

$$\le \omega_1(f;C_q;t-2k/(n+q))\left(v_q\left(\frac{2k}{n+q}\right)\right)^{-1} \le$$

$$\le \omega_1(f;C_q;2/(n+q))e^{2kq/(n+q)},$$

which implies that

$$| L_n^{\{2\}}(f;q;x) - L_n^{\{1\}}(f;q;x) | \le \omega_1(f;C_q;2/(n+q))\, L_n^{\{1\}}(e^{qt};q;x)$$

for $x \in R_0$ and $n \in N$. From this we immediately obtain (1.22) by (1.2) and (1.13) with $i = 1$.

The proof for $L_n^{\{4\}}$ and $L_n^{\{3\}}$ is analogous.

Theorem 1 and properties of $\omega_1(f;C_q;\cdot)$ imply the following

COROLLARY 1. *For every* $f \in C_q$, $q > 0$, *we have*

$$\lim_{n\to\infty} \left\| L_n^{\{2\}}(f;q;\cdot) - L_n^{\{1\}}(f;q;\cdot) \right\|_q = 0 = \lim_{n\to\infty} \left\| L_n^{\{4\}}(f;q;\cdot) - L_n^{\{3\}}(f;q;\cdot) \right\|_q .$$

THEOREM 2. *Suppose that* $f \in C_q^2$, $q > 0$. *Then*

(1.24) $$v_q(x)|\,L_n^{\{i\}}(f;q;x)-f(x)\,| \le \|\,f'\,\|_q\,|\,L_n^{\{i\}}(t-x;q;x)\,| \;+$$
$$+\;\|\,f''\,\|_q\,\{L_n^{\{i\}}((t-x)^2;q;x) \;+\; v_q(x)\,L_n^{\{i\}}((t-x)^2 e^{qt};q;x)\},$$

for all $x \in R_0$, $n \in N$ *and* $1 \le i \le 4$.

In particular

(1.25) $$v_q(x)\,|\,L_n^{\{1\}}(f;q;x)-f(x)| \le \|\,f'\,\|_q\left(\frac{qx}{n+q}+\frac{1}{n}\right) +$$
$$+\;\|\,f''\,\|_q\left(\frac{10q^2x^2}{(n+q)^2}+\frac{7x}{n+q}\right),$$

(1.26) $$v_q(x)\,|\,L_n^{\{3\}}(f;q;x)-f(x)| \le \|\,f'\,\|_q\left(\frac{qx}{n+q}+\frac{2}{n}\right) +$$
$$+\;\|\,f''\,\|_q\left(\frac{10q^2x^2}{(n+q)^2}+\frac{11x}{n+q}+\frac{12}{n^2}\right),$$

for $x \in R_0$ *and* $n \in N$.

Proof. Let $x \in R_0$ be a fixed point. Then for $f \in C_q^2$ and $t \in R_0$ we have

$$f(t) \;=\; f(x) \;+\; f'(x)(t-x) \;+\; \int_x^t \int_x^s f''(u)\,du\,ds\,,$$

which yields

$$f(t) \;=\; f(x) \;+\; f'(x)(t-x) \;+\; \int_x^t du \int_u^t f''(u)\,ds \;=$$
$$=\; f(x) \;+\; f'(x)(t-x) \;+\; \int_x^t (t-u)\,f''(u)\,du.$$

From this and by (1.12) we get

$$L_n^{\{i\}}(f(t);q;x) \;=\; f(x) \;+\; f'(x)\,L_n^{\{i\}}(t-x;q;x) \;+$$
$$+\;L_n^{\{i\}}\left(\int_x^t (t-u)\,f''(u)\,du;q;x\right), \quad n \in N, \quad 1 \le i \le 4.$$

Consequently,

$$v_q(x)|\,L_n^{\{i\}}(f;q;x)-f(x)\,| \le \|\,f'\,\|_q\,|\,L_n^{\{i\}}(t-x;q;x)\,| \;+$$

$$+\ v_q(x)\,L_n^{\{i\}}\left(\left|\int_x^t (t-x)\,f''(u)du\right|;q;x\right)$$

for $n\in N$ and $1\le i\le 4$. But

$$\left|\int_x^t (t-u)f''(u)du\right| \le \|f''\|_q\,(e^{qt}+e^{qx})(t-x)^2, \qquad t\in R_0,$$

which implies that

$$v_q(x)\,L_n^{\{i\}}\left(\left|\int_x^t (t-u)\,f''(u)du\right|;q;x\right)\le$$

$$\le \|f''\|_q\,\{v_q(x)\,L_n^{\{i\}}((t-x)^2e^{qt};q;x)\ +\ L_n^{\{i\}}((t-x)^2;q;x)\}$$

for $n\in N$ and $1\le i\le 4$. Combining the above, we obtain (1.24).

Estimations (1.25) and (1.26) follow from (1.24) and (1.17), (1.18) and (1.20).

THEOREM 3. *Let* $f\in C_q$, $q>0$, *and let*

$$\Phi_n(x;q):=\left(\frac{10q^2x^2}{(n+q)^2}+\frac{7x}{n+q}\right)^{1/2}, \qquad x\in R_0,\quad n\in N. \tag{1.27}$$

Then

$$v_q(x)|\,L_n^{\{1\}}(f;q;x)-f(x)\,|\le 12\omega_2(f;C_q;\Phi_n(x;q))\ + \tag{1.28}$$
$$+\ 16\left(\frac{qx}{n+q}+\frac{1}{n}\right)(\Phi_n(x;q))^{-1}\omega_1(f;C_q;2\Phi_n(x;q)),$$

for all $x>0$ *and* $n\in N$.

PROOF. From (1.8)-(1.11) we get

$$L_n^{\{i\}}(f;q;0)\ =\ f(0) \qquad \text{for}\ \ n\in N,\quad i=1,3. \tag{1.29}$$

Similarly as in [1] we shall apply the Stieklov function

$$f_h(x):=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}[2f(x+s+t)-f(x+2(s+t))]dsdt, \qquad x\in R_0,\quad h>0,$$

of function $f\in C_q$. It is known ([1]) that for $h>0$:

(1.30) $$\|f_h - f\|_q \le \omega_2(f;C_q;h),$$

(1.31) $$\| f_h' \|_q \le 16h^{-1}\omega_1(f;C_q;2h),$$

(1.32) $$\| f_h'' \|_q \le 9h^{-2}\omega_2(f;C_q;h),$$

which show that $f_h \in C_q^2$ if $f \in C_q$, $q > 0$. Hence for $x \in R_0$ and $n \in N$ we can write

$$v_q(x)| L_n^{\{1\}}(f;q;x) - f(x) | \le v_q(x)\{| L_n^{\{1\}}(f - f_h;q;x) | +$$
$$+ | L_n^{\{1\}}(f_h;q;x) - f_h(x) | + | f_h(x) - f(x) |\} := W_1 + W_2 + W_3.$$

By (1.14) and (1.30) we have

$$W_1 \le 2\| f - f_h \|_q \le 2\omega_2(f;C_q;h),$$
$$W_3 \le \omega_2(f;C_q;h).$$

Applying (1.25), (1.27) and (1.31), we get

$$W_2 \le \| f_h' \|_q \left(\frac{qx}{n+q} + \frac{1}{n}\right) + \| f_h'' \|_q \, \Phi_n^2(x;q) \le$$
$$\le 16\left(\frac{qx}{n+q} + \frac{1}{n}\right) h^{-1}\omega_1(f;C_q;2h) + 9h^{-2}\Phi_n^2(x;q)\omega_2(f;C_q;h).$$

Collecting the above results, we obtain

$$v_q(x)| L_n^{\{1\}}(f;q;x) - f(x) | \le 16\left(\frac{qx}{n+q} + \frac{1}{n}\right) h^{-1}\omega_1(f;C_q;2h) +$$
$$+ (3 + 9h^{-2}\Phi_n^2(x;q))\omega_2(f;C_q;h),$$

for $h > 0$, $x > 0$ and $n \in N$. Now, for given $q, x > 0$ and $n \in N$, choosing $h = \Phi_n(x;q)$, we obtain the estimation (1.28).

Analogously we can prove the following

THEOREM 4. *Let* $f \in C_q$, $q > 0$, *and let*

$$\Psi_n(x;q) := \left\{ \frac{10q^2x^2}{(n+q)^2} + \frac{11x}{n+q} + \frac{12}{n^2} \right\}^{1/2}, \qquad x \in R_0, \quad n \in N.$$

Then

$$v_q(x)|\,L_n^{\{3\}}(f;q;x)-f(x)\,| \le 12\omega_2(f;C_q;\Psi_n(x;q)) +$$

$$+\ 16\left(\frac{qx}{n+q}+\frac{2}{n}\right)(\Psi_n(x;q))^{-1}\omega_1(f;C_q;2\Psi_n(x;q)),$$

for all $x>0$ *and* $n\in N$. *Moreover for* $x=0$ *we have* (1.29).

From Lemma 2 and Theorem 1 we derive

COROLLARY 2. *Let* $f\in C_q$, $q>0$. *Then*

$$v_q(x)|\,L_n^{\{2\}}(f;q;x)-f(x)\,| \le$$

$$\le 2\omega_1(f;C_q;2/(n+q)) + v_q(x)|\,L_n^{\{1\}}(f;q;x)-f(x)\,|,$$

$$v_q(x)|\,L_n^{\{4\}}(f;q;x)-f(x)\,| \le 3\omega_1(f;C_q;2/(n+q)) +$$

$$+\ v_q(x)|\,L_n^{\{3\}}(f;q;x)-f(x),$$

for all $x>0$ *and* $n\in N$.

Theorem 3, Theorem 4, Corollary 2 and properties of ω_1 and ω_2 imply

COROLLARY 3. *Let* $q>0$ *be a fixed number. Then for every* $f\in C_q$ *and* $x\in R_0$ *we have*

$$\lim_{n\to\infty} L_n^{\{i\}}(f;q;x) = f(x), \qquad 1\le i\le 4.$$

This convergence is uniform on every interval $[x_1,x_2]$, $x_1\ge 0$.

3.2. In this section we shall give the Voronovskaya type theorems.

THEOREM 5. *Suppose that* $f\in C_q^2$, $q>0$. *Then*

$$\lim_{n\to\infty} n\,\{L_n^{\{2\}}(f;q;x)-L_n^{\{1\}}(f;q;x)\} = f'(x), \tag{1.33}$$

$$\lim_{n\to\infty} n\,\{L_n^{\{4\}}(f;q;x)-L_n^{\{3\}}(f;q;x)\} = f'(x), \tag{1.34}$$

at every $x\in R_0$.

Proof. We apply the formula (1.23). But for fixed k, n, q we get by the Taylor formula for $f \in C_q^2$ and $t \in [2k/(n+q),(2k+2)/(n+q)]$:

$$f(t) = f\left(\frac{2k}{n+q}\right) + f'\left(\frac{2k}{n+q}\right)\left(t-\frac{2k}{n+q}\right) + \frac{1}{2}f''(\xi_{k,t})\left(t-\frac{2k}{n+q}\right)^2,$$

where $\xi_{k,t}$ is a point such that $2k/(n+q) < \xi_{k,t} < t$. From this and by (1.23) it follows that

$$L_n^{\{2\}}(f;q;x) - L_n^{\{1\}}(f;q;x) =$$

$$= \frac{n+q}{2}\sum_{k=0}^{\infty} a_k(nx)\, f'\left(\frac{2k}{n+q}\right)\int_{2k/(n+q)}^{(2k+2)/(n+q)}\left(t-\frac{2k}{n+q}\right)dt +$$

$$+ \frac{n+q}{4}\sum_{k=0}^{\infty} a_k(nx)\int_{2k/(n+q)}^{(2k+2)/(n+q)} f''(\xi_{k,t})\left(t-\frac{2k}{n+q}\right)^2 dt := Y_1 + Y_2$$

for $x \in R_0$ and $n \in N$. It is obvious that

$$Y_1 = \frac{1}{n+q} L_n^{\{1\}}(f'(t);q;x), \qquad x \in R_0, \quad n \in N,$$

which by Corollary 3 implies that

$$\lim_{n\to\infty} nY_1 = \lim_{n\to\infty} L_n^{\{1\}}(f';q;x) = f'(x), \qquad x \in R_0.$$

Moreover we observe that

$$|Y_2| \le \|f''\|_q \frac{n+q}{2}\sum_{k=0}^{\infty} a_k(nx)\int_{2k/(n+q)}^{(2k+2)/(n+q)} e^{qt}\left(t-\frac{2k}{n+q}\right)^2 dt \le$$

$$\le \|f''\|_q \frac{n+q}{2}\,\frac{1}{3}\left(\frac{2}{n+q}\right)^3 \sum_{k=0}^{\infty} a_k(nx)\, e^{q(2k+2)/(n+q)} \le$$

$$\le \|f''\|_q \frac{2e^{2q/(n+q)}}{3(n+q)^2} L_n^{\{1\}}(e^{qt};q;x) \le$$

$$\le \frac{6}{(n+q)^2}\|f''\|_q \|L_n^{\{1\}}(e^{qt};q;\cdot)\|_q\, e^{qx},$$

which by Lemma 2 yields

$$|Y_2| \le \frac{12e^{qx}}{(n+q)^2} \| f'' \|, \qquad x \in R_0, \quad n \in N.$$

From this we get

$$\lim_{n\to\infty} nY_2 = 0 \quad \text{for every} \qquad x \in R_0.$$

Combining these, we obtain (1.33).

The proof of (1.34) is analogous by definitions (1.8)-(1.11).

THEOREM 6. *Suppose that* $f \in C_q^2$, $q > 0$. *Then*

$$\lim_{n\to\infty} n\{L_n^{\{1\}}(f;q;x) - A_n(f;x)\} = -qxf'(x), \tag{1.35}$$

$$\lim_{n\to\infty} n\{L_n^{\{3\}}(f;q;x) - B_n(f;x)\} = -qxf'(x), \tag{1.36}$$

at every $x \in R_0$.

PROOF. From (1.4) and (1.8) we get

$$L_n^{\{1\}}(f;q;x) - A_n(f;x) = \sum_{k=0}^{\infty} a_k(nx)\left\{f\left(\frac{2k}{n+q}\right) - f\left(\frac{2k}{n}\right)\right\},$$

for $x \in R_0$ and $n \in N$. Moreover we have

$$L_n^{\{1\}}(f;q;0) - A_n(f;0) = 0, \qquad n \in N.$$

Arguing as in the proof of Theorem 5, we get by the Taylor formula for $f \in C_q^2$ and fixed *k*, *n*, *q*:

$$f\left(\frac{2k}{n+q}\right) = f\left(\frac{2k}{n}\right) + f'\left(\frac{2k}{n}\right)\left(\frac{2k}{n+q} - \frac{2k}{n}\right) + \frac{1}{2}f''(\gamma_{k,n})\left(\frac{2k}{n+q} - \frac{2k}{n}\right)^2,$$

where $\gamma_{k,n}$ is a point such that $2k/(n+q) < \gamma_{k,n} < 2k/n$. Consequently,

$$L_n^{\{1\}}(f;q;x) - A_n(f;x) = \frac{-q}{n+q}\sum_{k=0}^{\infty} a_k(nx) f'\left(\frac{2k}{n}\right)\frac{2k}{n} + \frac{q^2}{(n+q)^2}\sum_{k=0}^{\infty} a_k(nx) f''(\gamma_{k,n})\left(\frac{2k}{n}\right)^2 := Z_1 + Z_2, \tag{1.37}$$

for $x \in R_0$ and $n \in N$. By (1.4) we have

$$Z_1 = \frac{-q}{n+q} A_n(tf'(t);x),$$

$$|Z_2| \le \|f''\|_q \frac{q^2}{(n+q)^2} \sum_{k=0}^{\infty} a_k(nx) e^{q2k/n} \left(\frac{2k}{n}\right)^2 \le$$

$$\le \|f''\|_q \frac{q^2}{(n+q)^2} A_n(t^2 e^{qt};x),$$

for $x \in R_0$ and $n \in N$. Since the functions $g_1(x) = xf'(x)$ and $g_2(x) = x^2 e^{qx}$ belong to the space C_{q+1} if $f \in C_q^2$, we have by Theorem I given in §1:

$$\lim_{n\to\infty} A_n(tf'(t);x) = xf'(x), \qquad \lim_{n\to\infty} A_n(t^2 e^{qt};x) = x^2 e^{qx},$$

for $x \in R_0$. Hence for every $x \in R_0$ we get

$$\lim_{n\to\infty} n Z_1 = -qxf'(x), \qquad \lim_{n\to\infty} n Z_2 = 0,$$

and by (1.37) we obtain (1.35).

The proof of (1.36) is identical.

Now we shall prove the main Voronovskaya type theorem for operators $L_n^{\{i\}}$.

THEOREM 7. *Let $f \in C_q^2$, $q > 0$. Then for every $x \in R_0$ we have:*

$$\lim_{n\to\infty} n\{L_n^{\{i\}}(f;q;x) - f(x)\} = -qxf'(x) + \frac{x}{2} f''(x), \tag{1.38}$$

if $i = 1,3$, and

$$\lim_{n\to\infty} n\{L_n^{\{i\}}(f;q;x) - f(x)\} = (1-qx)f'(x) + \frac{x}{2} f''(x), \tag{1.39}$$

if $i = 2,4$.

PROOF. If $i = 1,3$, then we write

$$L_n^{\{1\}}(f;q;x) - f(x) = [L_n^{\{1\}}(f;q;x) - A_n(f;x)] + [A_n(f;x) - f(x)],$$

$$L_n^{\{3\}}(f;q;x) - f(x) = [L_n^{\{3\}}(f;q;x) - B_n(f;x)] + [B_n(f;x) - f(x)],$$

$x \in R_0$, $n \in N$. Applying Theorem 6 and Theorem II given in §1, we obtain (1.38).

For $i=2$ and $i=4$ we have

$$L_n^{\{2\}}(f;q;x)-f(x) = [L_n^{\{2\}}(f;q;x)-L_n^{\{1\}}(f;q;x)] + [L_n^{\{1\}}(f;q;x)-f(x)],$$

$$L_n^{\{4\}}(f;q;x)-f(x) = [L_n^{\{4\}}(f;q;x)-L_n^{\{3\}}(f;q;x)] + [L_n^{\{3\}}(f;q;x)-f(x)],$$

$x \in R_0$, $n \in N$. Now, by Theorem 5 and (1.38), we immediately obtain (1.39).

Theorem 7 implies the following

COROLLARY 4. *For every function* $f \in C_q^2$, $q>0$, *and for* $i=1,2,3,4$ *we have*

$$L_n^{\{i\}}(f;q;x)-f(x) = O(1/n)$$

in every fixed $x \in R_0$.

II. APROXIMATION OF FUNCTIONS OF TWO VARIABLES

1. INTRODUCTION

1.1. In the paper [2] were examined approxinmation properties of operators of the Szasz-Mirakjan type

$$(2.1) \qquad A_{m,n}(f;x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j(mx)a_k(ny) f\left(\frac{2j}{m},\frac{2k}{n}\right),$$

$$(2.2) \qquad \overline{A}_{m,n}(f;x,y) := \frac{mn}{4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j(mx)a_k(ny) \int_{2j/m}^{(2j+2)/m}\int_{2k/n}^{(2k+2)/n} f(t,z)dtdz\,,$$

$(x,y) \in R_0^2 := R_0 \times R_0$ and $m,n \in N$ where R_0, N, N_0 and $a_i(t)$ are defined in §1, Section I.

These operators were considered in exponential weighted space $C_{p,q}$, $p,q>0$, connected with the weighted function

$$(2.3) \qquad v_{p,q}(x,y) := e^{-px-qy}, \qquad (x,y) \in R_0^2.$$

$C_{p,q}$ is the set of all real-valued functions f continuous on R_0^2 for which $v_{p,q}f$ is uniformly continuous and bounded function on R_0^2. The norm in $C_{p,q}$ is defined by

(2.4) $$\| f \|_{p,q} \equiv \| f(\cdot,\cdot) \|_{p,q} := \sup_{(x,y)\in R_0^2} v_{p,q}(x,y) | f(x,y) |.$$

It is obvious that

$$C_{p,q} \subset C_{r,q} \subset C_{r,s}, \quad C_{p,q} \subset C_{p,s} \subset C_{r,s}, \quad \text{if } 0<p<r \text{ and } 0<q<s.$$

As in [2] we shall denote by $C^m_{p,q}$, with fixed $m \in N$ and $p,q>0$, the set of all functions $f \in C_{p,q}$ which have partial derivatives of the order $\le m$ on R_0^2 and these belong also to $C_{p,q}$.

In [3]-[5] were examined operators

(2.5) $$B_{m,n}(f;x,y) := \frac{f(0,0)}{(1+\sinh mx)(1+\sinh ny)} + \frac{1}{1+\sinh mx}\sum_{k=0}^{\infty} b_k(ny) f\left(0, \frac{2k+1}{n}\right) + \frac{1}{1+\sinh ny}\sum_{j=0}^{\infty} b_j(mx) f\left(\frac{2j+1}{m}, 0\right) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_j(mx) b_k(ny) f\left(\frac{2j+1}{m}, \frac{2k+1}{n}\right),$$

(2.6) $$\overline{B}_{m,n}(f;x,y) := \frac{f(0,0)}{(1+\sinh mx)(1+\sinh ny)} + \frac{1}{1+\sinh mx}\sum_{k=0}^{\infty} b_k(ny)\frac{n}{2}\int_{(2k+1)/n}^{(2k+3)/n} f(0,z)dz + \frac{1}{1+\sinh ny}\sum_{j=0}^{\infty} b_j(mx)\frac{m}{2}\int_{(2j+1)/m}^{(2k+3)/m} f(t,0)dt + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_j(mx) b_k(ny)\frac{mn}{4}\int_{(2j+1)/m}^{(2j+3)/m}\int_{(2k+1)/n}^{(2k+3)/n} f(t,z)dtdz,$$

$(x,y) \in R_0^2$, $m,n \in N$, $f \in C_{p,q}$ with $p,q>0$, where $b_k(t)$ is defined by (1.7).

In [2] and [4] was proved that operators $A_{m,n}$, $\overline{A}_{m,n}$, $B_{m,n}$ and $\overline{B}_{m,n}$ are positive linear operators from the space $C_{p,q}$, $p,q>0$, into $C_{r,s}$ provided that

$r > p$, $s > q$ and $m > m_0 > p/\ln(r/p)$, $n > n_0 > q/\ln(s/q)$. In [2] and [4] was proved the following

THEOREM I. *Let* $f \in C_{p,q}$, $p,q > 0$. *Then*

$$\lim_{m,n\to\infty} A_{m,n}(f;x,y)=f(x,y) = \lim_{m,n\to\infty} \overline{A}_{m,n}(f;x,y),$$

$$\lim_{m,n\to\infty} B_{m,n}(f;x,y)=f(x,y) = \lim_{m,n\to\infty} \overline{B}_{m,n}(f;x,y),$$

at every point $(x,y) \in R_0^2$. *The above convergence is uniform on every rectangle* $D = \{(x,y): 0 \le x_1 \le x \le x_2,\ 0 \le y_1 \le y \le y_2\}$.

In [5] was proved the Voronovskaya type theorem.

THEOREM II. *Suppose that* $f \in C^2_{p,q}$, $p,q > 0$. *Then for operators* $A_{n,n}$ *defined by* (2.1) *we have*

$$\lim_{n\to\infty} n\{A_{n,n}(f;x,y)-f(x,y)\} = \frac{x}{2} f''_{xx}(x,y) + \frac{y}{2} f''_{yy}(x,y)$$

at every $(x,y) \in R_0^2$. *The identical property have operators* $B_{n,n}$ *defined by* (2.5).

Theorems on the degree of approximation of $f \in C_{p,q}$ by the above operators were formulated in [2] and [4] by the metric of the space $C_{r,s}$, $r > p$, $s > q$.

1.2. In this paper we modify formulas (2.1), (2.2), (2.5) and (2.6). We introduce the following operators in the space $C_{p,q}$ with fixed $p,q > 0$:

(2.7) $$L^{\{1\}}_{m,n}(f;p,q;x,y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_j(mx) a_k(ny) f\left(\frac{2j}{m+p}, \frac{2k}{n+q}\right),$$

(2.8) $$L^{\{2\}}_{m,n}(f;p,q;x,y) :=$$

$$:= \frac{(m+p)(n+q)}{4} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_j(mx) a_k(ny) \iint_{D_{jk}} f(t,z)\,dt\,dz,$$

(2.9) $$L^{\{3\}}_{m,n}(f;p,q;x,y) := \frac{f(0,0)}{(1+\sinh mx)(1+\sinh ny)} +$$

$$+\frac{1}{1+\sinh mx}\sum_{k=0}^{\infty} b_k(ny) f\left(0, \frac{2k+1}{n+q}\right)+$$

$$+\frac{1}{1+\sinh ny}\sum_{j=0}^{\infty} b_j(mx) f\left(\frac{2j+1}{m+p}, 0\right)+$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_j(mx) b_k(ny) f\left(\frac{2j+1}{m+p}, \frac{2k+1}{n+q}\right),$$

(2.10) $$L_{m,n}^{\{4\}}(f;p,q;x,y) := \frac{f(0,0)}{(1+\sinh mx)(1+\sinh ny)}+$$

$$+\frac{1}{1+\sinh mx}\sum_{k=0}^{\infty} b_k(ny)\frac{n+q}{2}\int_{(2k+1)/(n+q)}^{(2k+3)/(n+q)} f(0,z)dz+$$

$$+\frac{1}{1+\sinh ny}\sum_{j=0}^{\infty} b_j(mx)\frac{m+p}{2}\int_{(2j+1)/(m+p)}^{(2j+3)/(m+p)} f(t,0)dt+$$

$$+\frac{(m+p)(n+q)}{4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_j(mx)b_k(ny)\iint_{E_{jk}} f(t,z)dt,$$

$(x,y)\in R_0^2$, $m,n\in N$, $f\in C_{p,q}$, where

$$D_{jk} := \{(x,y)\in R_0^2 : 2j/(m+p)\le x\le (2j+2)/(m+p),$$
$$2k/(n+q)\le y\le (2k+2)/(n+q)\},$$

$$E_{jk} := \{(x,y)\in R_0^2 : (2j+1)/(m+p)\le x\le (2j+3)/(m+p),$$
$$(2k+1)/(n+q)\le y\le (2k+3)/(n+q)\}.$$

It is easily verified that operators $L_{m,n}^{\{i\}}$, $1\le i\le 4$, and $m,n\in N$, are well defined on the space $C_{p,q}$. Moreover we have

(2.11) $$L_{m,n}^{\{i\}}(1;p,q;x,y) = 1 \quad \text{for } (x,y)\in R_0^2,\quad m,n\in N,\quad 1\le i\le 4.$$

If $f\in C_{p,q}$, $p,q>0$, and $f(x,y)=f_1(x)f_2(y)$ for $x,y\in R_0$ then

(2.12) $$L_{m,n}^{\{i\}}(f(t,z);p,q;x,y) = L_m^{\{i\}}(f_1(t);p,x)L_n^{\{i\}}(f_2(z);q,y)$$

for $x,y\in R_0$ and $m,n\in N$, where $L_n^{\{i\}}$, $i=1,2,3,4$, are defined by (1.8)-(1.11).

Similarly as in Section I we shall denote by $M_k(a,b)$, $k=1,2,...,$ the suitable positive constants depending only on indicated parameters.

2. Main results

2.1. Applying Lemma 2 we shall prove

Lemma 4. *Let* $p,q>0$ *be fixed numbers. Then*

$$\left\|L_{m,n}^{\{i\}}(1/v_{p,q}(t,z);p,q;\cdot,\cdot)\right\|_{p,q}\le M_i\ , \tag{2.13}$$

for $m,n\in N$ *and* $1\le i\le 4$, *where* $M_1=4$, $M_2=4e^4$, $M_3=9$, $M_4=25e^4$. *Moreover for every* $f\in C_{p,q}$ *we have*

$$\left\|L_{m,n}^{\{i\}}(f;p,q;\cdot,\cdot)\right\|_{p,q}\le M_i\|f\|_{p,q}\ ,\qquad m,n\in N \text{ and } 1\le i\le 4, \tag{2.14}$$

where M_i *are numbers as in* (2.13).

The inequality (2.14) *and definitions* (2.7)-(2.10) *show that* $L_{m,n}^{\{i\}}$, $m,n\in N$ *and* $1\le i\le 4$, *are positive linear operators from the space* $C_{p,q}$ *into* $C_{p,q}$.

Proof. We shall prove only (2.13) and (2.14). From (2.7)-(2.12) and by (2.3) we get

$$v_{p,q}(x,y)L_{m,n}^{\{i\}}(1/v_{p,q}(t,z);p,q;x,y)=\left[e^{-px}L_m^{\{i\}}(e^{pt};p;x)\right]\left[e^{-qy}L_n^{\{i\}}(e^{qz};q;y)\right]$$

for $x,y\in R_0$, $m,n\in N$ and $1\le i\le 4$, which by (2.4) and Lemma 2 implies (2.13).

From (2.7)-(2.12) and by (2.3) and (2.4) we obtain

$$\left\|L_{m,n}^{\{i\}}(f;p,q;\cdot,\cdot)\right\|_{p,q}\le \|f\|_{p,q}\left\|L_{m,n}^{\{i\}}(1/v_{p,q};p,q)\right\|_{p,q}$$

for $f\in C_{p,q}$, $m,n\in N$ and $1\le i\le 4$, which by (2.13) yields (2.14).

2.2. Now we shall prove theorems on the degree of approximation of functions $f\in C_{p,q}$ by operators $L_{m,n}^{\{i\}}(f;p,q)$. We shall apply the modulus of continuity of $f\in C_{p,q}$:

$$(2.15)\qquad \omega_1(f;C_{p,q};t,s) := \sup_{\substack{0\le h\le t\\ 0\le\delta\le s}} \| \Delta_{h,\delta} f(\cdot,\cdot)\|_{p,q}, \qquad t,s\ge 0,$$

where $\Delta_{h,\delta} f(x,y) := f(x+h,y+\delta)-f(x,y)$ for $(x,y)\in R_0^2$, $h,\delta\in R_0$. This modulus of continuity have the properties

(a) $\omega_1(f;C_{p,q};t,s) \ge 0$ for $t,s\ge 0$, $\omega_1(f;C_{p,q};0,0) = 0$,

(b) $\omega_1(f;C_{p,q};t_1,s) \le \omega_1(f;C_{p,q};t_2,s)$ for $0\le t_1<t_2$ and $s\ge 0$,

$\omega_1(f;C_{p,q};t,s_1) \le \omega_1(f;C_{p,q};t,s_2)$ for $0\le s_1<s_2$ and $t\ge 0$,

(c) $\lim_{t,s\to 0^+} \omega_1(f;C_{p,q};t,s) = 0.$

THEOREM 8. *Suppose that $f\in C_{p,q}$, $p,q>0$, and*

$$F_{m,n}(f;x,y) := L_{m,n}^{\{2\}}(f;p,q;x,y)-L_{m,n}^{\{1\}}(f;p,q;x,y),$$

$$G_{m,n}(f;x,y) := L_{m,n}^{\{4\}}(f;p,q;x,y)-L_{m,n}^{\{3\}}(f;p,q;x,y),$$

for $(x,y)\in R_0^2$ and $m,n\in N$. Then

$$(2.16)\qquad \| F_{m,n}(f)\|_{p,q} \le 4\omega_1\left(f;C_{p,q};\frac{2}{m+p},\frac{2}{n+q}\right),$$

$$(2.17)\qquad \| G_{m,n}(f)\|_{p,q} \le 9\omega_1\left(f;C_{p,q};\frac{2}{m+p},\frac{2}{n+q}\right),$$

for all $m,n\in N$.

PROOF. We shall prove only (2.16) because the proof (2.17) is analogous. From (2.7), (2.8) and (2.11) it follows that

$$F_{m,n}(f;x,y) =$$

$$= \frac{(m+p)(n+q)}{4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j(mx)a_k(ny) \iint_{D_{jk}} \left[f(t,z)-f\left(\frac{2j}{m+p},\frac{2k}{n+q}\right)\right]dtdz$$

for $(x,y)\in R_0^2$ and $m,n\in N$. But for fixed j, k, m, n and $(t,z)\in D_{jk}$ we have by (2.15):

$$\left|f(t,z)-f\left(\frac{2j}{m+p},\frac{2k}{n+q}\right)\right| = \left|\Delta_{t-\frac{2j}{m+p},z-\frac{2k}{n+q}} f\left(\frac{2j}{m+p},\frac{2k}{n+q}\right)\right| \le$$

$$\le \omega_1\left(f;C_{p,q};t-\frac{2j}{m+p},z-\frac{2k}{n+q}\right)\left(v_{p,q}\left(\frac{2j}{m+p},\frac{2k}{n+q}\right)\right)^{-1}.$$

From this and by property (b) of modulus of continuity we get

$$|F_{m,n}(f;x,y)| \le \omega_1\left(f;C_{p,q};\frac{2}{m+p},\frac{2}{n+q}\right) L_{m,n}^{\{1\}}(1/v_{p,q};p,q;x,y)$$

for $(x,y)\in R_0^2$ and $m,n\in N$, wich by (2.1), (2.2) and (2.13) implies

$$\|F_{m,n}(f)\|_{p,q} \le \omega_1\left(f;C_{p,q};\frac{2}{m+p},\frac{2}{n+q}\right)\left\|L_{m,n}^{\{1\}}(1/v_{p,q};p,q)\right\|_{p,q} \le$$

$$\le 4\omega_1\left(f;C_{p,q};\frac{2}{m+p},\frac{2}{n+q}\right), \quad \text{for } m,n\in N.$$

THEOREM 9. *Suppose that* $f\in C_{p,q}^1$, $p,q>0$. *Then there exists a positive constant* $M_5 \equiv M_5(p,q)$ *such that*

(2.18) $$v_{p,q}(x,y)\left|L_{m,n}^{\{i\}}(f;p,q;x,y)-f(x,y)\right| \le$$
$$\le M_5\{\|f_x'\|_p\ \Phi_m(x;p) + \|f_y'\|_q\ \Phi_m(x;q)\}$$

for all $(x,y)\in R_0^2$, $m,n\in N$ *and* $i=1,3$, *where*

(2.19) $$\Phi_m(x;p) := \left\{\frac{x^2}{(m+p)^2}+\frac{x}{m+p}\right\}^{1/2}.$$

PROOF. Let $i=1$ and let $(x,y)\in R_0^2$ be a fixed point. Then for $f\in C_{p,q}^1$ we have

$$f(t,z)-f(x,y) = \int_x^t f_u'(u,z)du + \int_y^z f_w'(x,w)dw, \qquad (t,z)\in R_0^2.$$

From this and by (2.11) we get

(2.20) $$L_{m,n}^{\{1\}}(f;p,q;x,y)-f(x,y) = L_{m,n}^{\{1\}}\left(\int_x^t f_u'(u,z)du;p,q;x,y\right)+$$

$$+ L_{m,n}^{\{1\}}\left(\int_y^z f'_w(x,w)dw; p,q;x,y\right), \qquad m,n \in N.$$

By (2.3) and (2.4) we have

$$\left|\int_x^t f_u(u,z)du\right| \le \| f'_x \|_{p,q} \left|\int_x^t \frac{du}{v_{p,q}(u,z)}\right| \le \| f'_x \|_{p,q} (e^{pt+qz} + e^{px+qz}) \,|\, t-x \,|$$

and from this it follows that

$$v_{p,q}(x,y)\left|L_{m,n}^{\{1\}}\left(\int_x^t f'_u(u,z)du; p,q;x,y\right)\right| \le$$

$$\le v_{p,q}(x,y) L_{m,n}^{\{1\}}\left(\left|\int_x^t f'_u(u,z)du\right|; p,q;x,y\right) \le$$

$$\le \| f'_x \|_{p,q}\, e^{-qy} L_n^{\{1\}}(e^{qz};q;y)\{e^{-px} L_m^{\{1\}}(|\, t-x \,| e^{pt};p;x) + L_m^{\{1\}}(|\, t-x \,|;p;x)\},$$

$m,n \in N$. By the Hölder inequality and by Lemma 2 and Lemma 3 we get

$$e^{-px} L_m^{\{1\}}(|\, t-x \,| e^{pt};p;x) \le$$

$$\le \{e^{-px} L_m^{\{1\}}((t-x)^2 e^{pt};p;x)\}^{1/2} \{e^{-px} L_m^{\{1\}}(e^{pt};p;x)\}^{1/2} \le$$

$$\le \left\{2\left(\frac{9p^4x^2}{(m+p)^4} + \frac{4x}{m+p}\right)\right\}^{1/2},$$

$$v_p(x) L_m^{\{1\}}(|\, t-x \,|;p;x) \le \{L_m^{\{1\}}((t-x)^2;p;x)\}^{1/2} \le \left\{\frac{p^2x^2}{(m+p)^2} + \frac{3x}{m+p}\right\}^{1/2},$$

for $m \in N$. Conseguently,

$$v_{p,q}(x,y)\left|L_{m,n}^{\{1\}}\left(\int_x^t f'_u(u,z)du; p,q;x,y\right)\right| \le M_6(p) \| f'_x \|_{p,q}\, \Phi_m(x;p)$$

for $m,n \in N$, where $\Phi_m(\cdot\,;p)$ is defined by (2.19). Analogously we obtain

$$v_{p,q}(x,y)\left|L_{m,n}^{\{1\}}\left(\int_y^z f'_w(x,w)dw; p,q;x,y\right)\right| \le M_7(q) \| f'_y \|_{p,q}\, \Phi_n(y;q)$$

for $m,n \in N$.

The last inequalities and (2.20) immediately yield the estimation (2.18) for $i=1$. The proof of (2.18) for $i=3$ is analogous. Finaly, we remark that for $f \in C_{p,q}$, $p,q>0$,

(2.21) $$L_{m,n}^{\{i\}}(f;p,q;0,0)=f(0,0), \qquad m,n\in N, \quad i=1,3.$$

THEOREM 10. *Let $f \in C_{p,q}$, $p,q>0$. Then there exists a positive constant $M_8 \equiv M_8(p,q)$ such that*

(2.22) $$v_{p,q}(x,y)\left| L_{m,n}^{\{i\}}(f;p,q;x,y)-f(x,y)\right| \le$$
$$\le M_8\omega_1(f;C_{p,q};\Phi_m(x;p),\Phi_n(y;q))$$

for all $(x,y)\in R_0^2$, $m,n\in N$ and $i=1,3$, where $\Phi_m(\cdot;p)$ is defined by (2.19).

PROOF. Similarly as [2] we shall use the Stieklov function $f_{h,\delta}$ of $f\in C_{p,q}$:

$$f_{h,\delta}(x,y):=\frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+w)dw, \qquad (x,y)\in R_0^2, \quad h,\delta>0.$$

From this we get

$$f_{h,\delta}(x,y)-f(x,y)=\frac{1}{h\delta}\int_0^h du\int_0^\delta \Delta_{u,w}f(x,y)dw, \qquad (x,y)\in R_0^2, \quad h,\delta>0.$$

$$(f_{h,\delta}(x,y))'_x=\frac{1}{h\delta}\int_0^\delta (\Delta_{h,w}f(x,y)-\Delta_{0,w}f(x,y))dw,$$

$$(f_{h,\delta}(x,y))'_y=\frac{1}{h\delta}\int_0^h (\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y))du,$$

and by (2.4) and (2.15) we obtain

(2.23) $$\left\|f_{h,\delta}-f\right\|_{p,q}\le \omega_1(f;C_{p,q};h,\delta),$$

(2.24) $$\left\|(f_{h,\delta})'_x\right\|_{p,q}\le 2h^{-1}\omega_1(f;C_{p,q};h,\delta),$$

(2.25) $$\left\|(f_{h,\delta})'_y\right\|_{p,q}\le 2\delta^{-1}\omega_1(f;C_{p,q};h,\delta),$$

for $h,\delta>0$, which prove that $f_{h,\delta}\in C_{p,q}^1$ if $f\in C_{p,q}$. Hence for $(x,y)\in R_0^2$, $m,n\in N$ and $i=1,3$ and for $h,\delta>0$ we have

$$\left|L_{m,n}^{\{i\}}(f;p,q;x,y)-f(x,y)\right| \le \left|L_{m,n}^{\{i\}}(f-f_{h,\delta};p,q;x,y)\right| +$$

$$+\left|L_{m,n}^{\{i\}}(f_{h,\delta};p,q;x,y)-f_{h,\delta}(x,y)\right|+\left|f_{h,\delta}(x,y)-f(x,y)\right| := S_1+S_2+S_3.$$

By (2.4), (2.14) and (2.23) it follows that

$$v_{p,q}(x,y)S_1 \le M_i \,\| f-f_{h,\delta}\|_{p,q} \le 3\omega_1(f;C_{p,q};h,\delta),$$

$$v_{p,q}(x,y)S_3 \le \omega_1(f;C_{p,q};h,\delta).$$

Applying Theorem 9 and (2.24) and (2.25), we get

$$v_{p,q}(x,y)S_2 \le M_5\{\|(f_{h,\delta})'_x\|_{p,q}\,\Phi_m(x;p)+\|(f_{h,\delta})'_y\|_{p,q}\,\Phi_n(y;q)\}\le$$

$$\le 2M_5\omega_1(f;C_{p,q};h,\delta)\{h^{-1}\Phi_m(x;p)+\delta^{-1}\Phi_n(y;q)\}.$$

Combining these, we obtain

$$v_{p,q}(x,y)\left|L_{m,n}^{\{i\}}(f;p,q;x,y)-f(x,y)\right| \le$$

$$\le M_8(p,q)\omega_1(f;C_{p,q};h,\delta)\{1+h^{-1}\Phi_m(x;p)+\delta^{-1}\Phi_n(y;q)\}$$

for $(x,y)\in R_0^2$, $m,n\in N$, $i=1,2$ and $h,\delta>0$. Now, for fixed $x,y>0$, $m,n\in N$ and $i=1,2$, choosing $h=\Phi_m(x;p)$, $\delta=\Phi_n(y;q)$, we obtain (2.22) for $x,y>0$.

If $x=0$ or $y=0$ we obtain (2.22) similarly as in [2] or [5].

THEOREM 11. *Suppose that $f\in C_{p,q}$, $p,q>0$. Then where exists a positive constant $M_9\equiv M_9(p,q)$ such that*

$$\text{(2.26)}\qquad v_{p,q}(x,y)\left|L_{m,n}^{\{i\}}(f;p,q;x,y)-f(x,y)\right| \le$$

$$\le M_9\left\{\omega_1\left(f;C_{p,q};\frac{2}{m+p},\frac{2}{n+q}\right)+\omega_1(f;C_{p,q};\Phi_m(x;p),\Phi_n(y;q)\right\}$$

for all $(x,y)\in R_0^2$, $m,n\in N$ and $i=2,4$, where Φ_m is defined by (2.19).

PROOF. The estimation (2.26) follows by the inequality

$$\left|L_{m,n}^{\{i\}}(f;p,q;x,y)-f(x,y)\right| \le \left|L_{m,n}^{\{i\}}(f;p,q;x,y)-L_{m,n}^{\{i-1\}}(f;p,q;x,y)\right| +$$

$$+\left|L_{m,n}^{\{i-1\}}(f;p,q;x,y)-f(x,y)\right| \qquad (x,y)\in R_0^2,\; m,n\in N,\; i=2,4,$$

and by Theorem 8 and Theorem 10.

Theorem 10 and Theorem 11 imply the following

COROLLARY 5. *For every* $f\in C_{p,q}$, $p,q>0$, *and* $(x,y)\in R_0^2$ *we have*

$$\lim_{m,n\to\infty} L_{m,n}^{\{i\}}(f;p,q;x,y)=f(x,y), \quad 1\le i\le 4.$$

This convergence is uniform on every rectangle $P=\{(x,y): x_1\le x\le x_2,$ $y_1\le y\le y_2\}$, $x_1,y_1\ge 0$.

2.3 In this section we shall give the Voronovskaya type theorems for operators $L_{n,n}^{\{i\}}$ defined by (2.7)-(2.10) if $m=n$.

THEOREM 12. *Let* $f\in C_{p,q}^2$, $p,q>0$. Then

$$\lim_{n\to\infty} n\left[L_{n,n}^{\{i\}}(f;p,q;x,y)-L_{n,n}^{\{i-1\}}(f;p,q;x,y)\right]=f_x'(x,y)+f_y'(x,y), \tag{2.27}$$

for $(x,y)\in R_0^2$ *and* $i=2,4$.

PROOF. Let $i=2$. Similarly as in Theorem 8 we have

$$F_{n,n}(f;x,y) := L_{n,n}^{\{2\}}(f;p,q;x,y)-L_{n,n}^{\{1\}}(f;p,q;x,y) = \tag{2.28}$$

$$=\frac{(n+p)(n+q)}{4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_j(nx)a_k(ny)\iint_{D_{jk}}\left[f(t,z)-f\left(\frac{2j}{n+p},\frac{2k}{n+q}\right)\right]dtdz,$$

for $(x,y)\in R_0^2$ and $n\in N$, where $Q_{jk}=\left\{(t,z): \frac{2j}{n+p}\le t\le\frac{2j+2}{n+q},\right.$ $\left.\frac{2k}{n+p}\le z\le\frac{2k+2}{n+q}\right\}$. For fixed j, k, n we apply the Taylor formula for $f\in C_{p,q}^2$:

$$f(t,z)=f\left(\frac{2j}{n+p},\frac{2k}{n+q}\right)+f'_x\left(\frac{2j}{n+p},\frac{2k}{n+q}\right)\left(t-\frac{2j}{n+p}\right)+$$

$$+f'_y\left(\frac{2j}{n+p},\frac{2k}{n+q}\right)\left(z-\frac{2k}{n+p}\right)+$$

$$+\frac{1}{2}\left\{f''_{xx}(\alpha_{j,t},\beta_{k,z})\left(t-\frac{2j}{n+p}\right)^2+2f''_{xy}(\alpha_{j,t},\beta_{k,z})\left(t-\frac{2j}{n+p}\right)\left(z-\frac{2k}{n+q}\right)+\right.$$

$$\left.+f''_{yy}(\alpha_{j,t},\beta_{k,z})\left(z-\frac{2k}{n+p}\right)^2\right\},\qquad (t,z)\in D_{jk},$$

where $\frac{2j}{n+p}<\alpha_{j,t}<t$, $\frac{2k}{n+q}<\beta_{k,z}<z$. Hence we can write

$$F_{n,n}(f;x,y)=\sum_{r=1}^{3}H_{n;r}(x,y),\qquad (x,y)\in R_0^2,\quad n\in N, \tag{2.29}$$

where

$$H_{n;1}(x,y):=$$

$$:=\frac{(n+p)(n+q)}{4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_j(nx)a_k(ny)f'_x\left(\frac{2j}{n+p},\frac{2k}{n+q}\right)\iint_{D_{jk}}\left(t-\frac{2j}{n+p}\right)dtdz=$$

$$=\frac{1}{n+p}L_{n,n}^{\{1\}}(f'_x(t,z);p,q;x,y),$$

$$H_{n;2}(x,y):=$$

$$:=\frac{(n+p)(n+q)}{4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_j(nx)a_k(ny)f'_y\left(\frac{2j}{n+p},\frac{2k}{n+q}\right)\iint_{D_{jk}}\left(z-\frac{2k}{n+q}\right)dtdz=$$

$$=\frac{1}{n+q}L_{n,n}^{\{1\}}(f'_y(t,z);p,q;x,y),$$

$$H_{n;3}(x,y):=$$

$$:=\frac{(n+p)(n+q)}{8}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_j(nx)a_k(ny)\iint_{D_{jk}}\left[f''_{xx}(\alpha_{j,t},\beta_{k,z})\left(t-\frac{2j}{n+p}\right)^2+\right.$$

$$+2f''_{xy}(\alpha_{j,t},\beta_{k,z})\left(t-\frac{2j}{n+p}\right)\left(z-\frac{2k}{n+q}\right)+f''_{yy}(\alpha_{j,t},\beta_{k,z})\left(z-\frac{2k}{n+p}\right)^2\Bigg]dtdz.$$

We observe that

$$|H_{n;3}(x,y)|\le\frac{(n+p)(n+q)}{8}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_j(nx)a_k(ny)e^{\frac{(2j+2)p}{n+p}+\frac{(2k+2)q}{n+q}}\times$$

$$\times\left\{\|f''_{xx}\|_{p,q}\iint_{D_{jk}}\left(t-\frac{2j}{n+p}\right)^2dtdz+2\|f''_{xy}\|_{p,q}\iint_{D_{jk}}\left(t-\frac{2j}{n+p}\right)\left(z-\frac{2k}{n+q}\right)dtdz+\right.$$

$$\left.+\|f_{yy}\|_{p,q}\iint_{D_{jk}}\left(z-\frac{2k}{n+p}\right)^2dtdz\right\}=$$

$$=\frac{(n+p)(n+q)}{8}e^{\frac{2p}{n+p}+\frac{2q}{n+q}}L^{\{1\}}_{n,n}(e^{pt+qz};p,q;x,y)\left\{\|f''_{xx}\|_{p,q}\frac{1}{3}\left(\frac{2}{n+p}\right)^3\frac{2}{n+q}+\right.$$

$$\left.+\frac{2}{4}\|f''_{xy}\|_{p,q}\left(\frac{2}{n+p}\right)^2\left(\frac{2}{n+q}\right)^2+\|f''_{yy}\|_{p,q}\frac{1}{3}\frac{2}{n+p}\left(\frac{2}{n+q}\right)^3\right\}\le$$

$$\le e^4\left\|L^{\{1\}}_{n,n}(e^{pt+qz};p,q;\cdot,\cdot)\right\|_{p,q}e^{px+qy}\left[\|f''_{xx}\|_{p,q}\frac{2}{3(n+p)^2}+\right.$$

$$\left.+\|f''_{xy}\|_{p,q}\frac{1}{(n+p)(n+q)}+\|f''_{yy}\|_{p,q}\frac{2}{3(n+q)^2}\right],$$

for $(x,y)\in R_0^2$ and $n\in N$. From this and by (2.13) we deduce that

(2.30) $$\lim_{n\to\infty}nH_{n;3}(x,y)=0\quad\text{for}\quad(x,y)\in R_0^2.$$

Applying Corollary 5, we get

$$\lim_{n\to\infty}L^{\{1\}}_{n,n}(f'_x(t,z);p,q;x,y)=f'_x(x,y),$$

$$\lim_{n\to\infty}L^{\{1\}}_{n,n}(f'_y(t,z);p,q;x,y)=f'_y(x,y),$$

for $(x,y)\in R_0^2$, which imply

(2.31) $$\lim_{n\to\infty}nH_{n;1}(x;y)=f'_x(x,y),$$

(2.32) $$\lim_{n\to\infty} nH_{n;2}(x;y) = f'_y(x,y),$$

for $(x,y)\in R_0^2$. Now the desired assertion (2.27) for $i=2$ follows by (2.28)-(2.32).

The proof for $i=4$ is identical.

Arguing analogously as in the proof of Theorem 12, we can prove the following

THEOREM 13. *Let* $f\in C^2_{p,q}$, $p,q>0$. *Then*

$$\lim_{n\to\infty} n\{L^{\{1\}}_{n,n}(f;p,q;x,y)-A_{n,n}(f;x,y)\} = -pxf'_x(x,y)-qyf'_y(x,y),$$

$$\lim_{n\to\infty} n\{L^{\{3\}}_{n,n}(f;p,q;x,y)-B_{n,n}(f;x,y)\} = -pxf'_x(x,y)-qyf'_y(x,y),$$

for every $(x,y)\in R_0^2$.

Applying the above theorems, we shall prove the main Voronovskaya type theorem.

THEOREM 14. *Suppose that* $f\in C^2_{p,q}$, $p,q>0$. *Then for every* $(x,y)\in R_0^2$ *and* $i=1,3$ *we have*

(2.33) $$\lim_{n\to\infty} n\{L^{\{i\}}_{n,n}(f;p,q;x,y)-f(x,y)\} = -pxf'_x(x,y)-qyf'_y(x,y)+ \\ +\frac{x}{2}f''_{xx}(x,y)+\frac{y}{2}f''_{yy}(x,y).$$

Moreover for $i=2,4$ *and* $(x,y)\in R_0^2$ *we have*

(2.34) $$\lim_{n\to\infty} n\{L^{\{i\}}_{n,n}(f;p,q;x,y)-f(x,y)\} = (1-px)f'_x(x,y)+ \\ +(1-qy)f'_y(x,y)+\frac{x}{2}f''_{xx}(x,y)+\frac{y}{2}f''_{yy}(x,y).$$

PROOF. Let $i=1$. Then for $(x,y)\in R_0^2$ and $n\in N$ we have

$$L^{\{1\}}_{n,n}(f;p,q;x,y)-f(x,y)= \\ =\left[L^{\{1\}}_{n,n}(f;p,q;x,y)-A_{n,n}(f;x,y)\right]+\left[A_{n,n}(f;x,y)-f(x,y)\right].$$

Applying Theorem 13 and Theorem II given in §1, Section II, we obtain (2.33) for $i = 1$. Analogously by Theorem 12, Theorem 13 and Theorem II we obtain (2.33) and (2.34) for $i = 2,3,4$. ■

From Theorem 14 we derive the following

COROLLARY 6. *For every* $f \in C^2_{p,q}$, $p, q > 0$, *and for every fixed* $(x, y) \in R^2_0$ *we have*

$$L^{\{i\}}_{n,n}(f; p, q; x, y) - f(x, y) = O(1/n), \qquad 1 \le i \le 4.$$

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