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# AN APPLICATION OF MODULAR SPACES TO APPROXIMATION PROBLEMS, IX

ABSTRACT: By means of terms of a sequence $(\rho_n)$, where $\rho_n$, $n=1,2,...$, are pseudomodulars, and by means of an infinite matrix $A=[a_{mn}]$ of non-negative numbers we shall construct the modular spaces $X\rho_o^{A,s}$ and $X\rho_{os}^{A}$. Then we shall approximate elements of these spaces by means of terms of a sequence $(\widetilde{\rho}_i)$, where $\widetilde{\rho}_i$, $i=1,2,...$, are pseudomodulars. In particular, we will investigate the special cases when $\rho_n$ and $\widetilde{\rho}_i$ are singular integrals.

KEY WORDS: modular space, Lebesgue measure, singular integral.

## 1. INTRODUCTION

Let $(\Omega,\Sigma,\mu)$ denote a space with a finite measure $\mu$, defined on a $\sigma-$algebra $\Sigma$ of subsets of the set $\Omega$, $\rho_n(t,f):\Omega\times X\to<0,\infty>$ for $n=1,2,...$ and $f\in X$ - the space of functions $f:\Omega<-\infty,\infty>$ which are $\Sigma-$measurable and almost everywhere finite, with equality $\mu-$almost everywhere (a.e.).

Let us assume:

$1^0$. $\rho_n(\cdot,f)$ is a pseudomodular in X for almost all $t$ and for every $n=1,2,...$

$2^0$. $\rho_n(\cdot,f)$ is measurable and a.e. finite for every $f\in X$ and every $n=1,2,...$.

$3^0$. If for $n=1,2,...$ $\rho_n(t,f)=0$ for almost all $t$, then $f=0$.

Let us denote by $A=[a_{mn}]$ an infinite matrix of non-negative numbers such that none of the columns of the matrix $A$ consists only of zeros.

Let

$$\rho_{mo}^{A}(t,f)=\sup_n a_{mn}\,\rho_n(t,f),\qquad \rho_{mo}^{A}(f)=\int_\Omega \sup_n a_{mn}\,\rho_n(t,f)d\mu$$

for $m=1,2,...$ By means of terms of a sequence $(\rho_n)$ and by means of a matrix $A$ we shall construct the following modulars in X:

$$\rho_o^{A,s}(f)=\sum_{m=1}^{\infty}\frac{1}{2^m}\frac{\rho_{mo}^{A}(f)}{1+\rho_{mo}^{A}(f)},$$

$$\rho_{os}^{A}(f) = \int_{\Omega} \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\rho_{mo}^{A}(t,f)}{1+\rho_{mo}^{A}(t,f)} d\mu.$$

Let us denote by: $X\rho_o^{A,s}$, $X\rho_{os}^{A}$ the respective modular spaces.

**THEOREM 1.1.** *Let* $f \in X$ *and assume that there exists a number* $\lambda_0 = \lambda_0(f) > 0$ *such that for every* $m = 1,2,\ldots$

$$(*) \qquad \int_{\Omega} \sup_n a_{mn} \rho_n(t, \lambda_0 f) d\mu < \infty.$$

*Then the following conditions are equivalent:*

(a) $f \in X\rho_o^{A,s}$,

(b) $f \in X\rho_{os}^{A}$,

(c) *there exists a set* $A \in \Sigma$, $\mu(A) = 0$, *such that for every* $t \in \Omega \setminus A$ *and for every* $m = 1,2,\ldots$ *we have*

$$\lim_{\lambda \to 0} \sup_n a_{mn} \rho_n(t, \lambda f) = 0.$$

**PROOF.** We prove for example that the conditions (a) and (c) are equivalent. (a) $\Rightarrow$ (c) We assume that $f \in X\rho_o^{A,s}$. Hence for every $m = 1,2,\ldots$

$$\lim_{\lambda \to 0} \int_{\Omega} \rho_{mo}^{A}(t, \lambda f) d\mu = 0.$$

Putting $\lambda = 1/k$, $k = 1,2,\ldots,$ we obtain that for every $m = 1,2,\ldots$ $\rho_{mo}^{A}(\cdot, f/k) \to 0$ as $k \to \infty$, in measure in $\Omega$, and so for every $m = 1,2,\ldots$ there exists the esquence $(k_p)$ such that for every $m = 1,2,\ldots$ we have $\rho_{mo}^{A}(t, f/k_p) \to 0$ a.e. in $\Omega$ as $p \to \infty$. Since $\rho_{mo}^{A}(t, \lambda f)$ is a non-decreasing function with respect to $\lambda$, so for every $m = 1,2,\ldots$ we obtain that $\rho_{mo}(t, \lambda f) \to 0$ a.e. in $\Omega$ as $\lambda \to 0$. Let us put $A_m = \{t \in \Omega : \rho_{mo}^{A}(t, \lambda f) \nrightarrow 0 \text{ as } \lambda \to 0\}$ and $A = \bigcup_{m=1}^{\infty} A_m$. Then $\mu(A) = 0$ and for every $t \in \Omega \setminus A$ and for $m = 1,2,\ldots$ we have

$$\lim_{\lambda \to 0} \sup_n a_{mn} \rho_n(t, \lambda f) = 0.$$

(c) $\Rightarrow$ (a) By the conditions (*) and (c), using the Lebesgue Dominated Convergence Theorem, we obtain for every $m = 1,2,\ldots$ that

$$\int_{\Omega} \sup_{n} a_{mn} \rho_n(t, \lambda f) d\mu \to 0 \quad \text{as} \quad \lambda \to 0.$$

Therefore $\rho_o^{As}(\lambda f) \to 0$ with $\lambda \to 0$, and so $f \in X\rho_o^{As}$.

The remaing relations we prove in the same way as in [2] (see Theorem 16.2 pp. 119-121).

Because the condition (*) we use only for the proof of the implication (c) $\Rightarrow$ (a), so we obtain:

**COROLLARY.** *The following inclusion*

$$X\rho_o^{A,s} \subset X\rho_{os}^{A}$$

*holds.*

**2.** Approximation of elements of the spaces $X\rho_o^{A,s}$ and $X\rho_{os}^{A}$. We will approximate elements of the modular spaces $X\rho_o^{A,s}$ and $X\rho_{os}^{A}$ by terms of a sequence $(\widetilde{\rho}_i)$, where $\rho_i : \Omega \times \mathrm{X} \to <0, \infty>$ for $i = 1,2,...,$ which can be different from a sequence $(\rho_n)$ generating these spaces.

We shall give conditions under which

$$\rho(\lambda(f(\cdot) - \widetilde{\rho}_i(\cdot, f))) \to 0 \quad \text{as} \quad i \to \infty \quad \text{for every} \quad \lambda > 0.$$

where $\rho = \rho_o^{A,s}$ or $\rho = \rho_{os}^{A}$, holds.

In what follows we shall suppose that besides conditions $^{\circ} - 3^{\circ}$ the following condition is satisfied:

$4^0$. If $f, g \in \mathrm{X}$, $|f(t)| \leq |g(t)|$ a.e. in $\Omega$, then for $n = 1,2,...$ we have $\rho_n(t, f) \leq \rho_n(t, g)$ a.e. in $\Omega$.

Consider a sequence $(\widetilde{\rho}_i)$ such that:

1) $\widetilde{\rho}_i(t, f)$ is a pseudomodular in X for almost all $t$ and $i = 1,2,...$
2) $\widetilde{\rho}_i(\cdot, f)$ and $\widetilde{\rho}_i(\cdot, f - f(\cdot))$ are measurable and a.e. finite for every $f \in \mathrm{X}$ and $i = 1,2,...$
3) The sequence $(\widetilde{\rho}_i)$ preserves constants, i.e. $\widetilde{\rho}_i(t, c) = c$ for every $t \in \Omega$ and for each constant $c \geq 0$, $i = 1,2,...$

The sequence $(\widetilde{\rho}_i)$ is called singular at the point $f \in X\rho_o^{A,s}$ iff for any two positive numbers $a$, $b$ and for $m = 1,2,...$

$$S_i^m(f) = \int\limits_{\Omega} \sup_n a_{mn} \rho_n(t, a\widetilde{\rho}_i(\cdot, b(f - f(\cdot))))d\mu \to 0 \quad \text{as} \quad i \to \infty.$$

**THEOREM 2.1.** *If the sequence* $(\widetilde{\rho}_i)$ *is singular at the point* $f \in X\rho_o^{A,s}$, $f \geq 0$, *then for every* $\lambda > 0$

$$\rho_o^{A,s}(\lambda(f(\cdot) - \widetilde{\rho}_i(\cdot, f))) \to 0 \quad \text{as} \quad i \to \infty.$$

**PROOF.** Let $f \in X\rho_o^{A,s}$, $f \geq 0$. Since the sequence $(\widetilde{\rho}_i)$ preserves constants, so for a.a. $t \in \Omega$ we have

$$\widetilde{\rho}_i(t, f(t) = f(t) \quad \text{for} \quad i = 1,2,...$$

Thus, similarly as in [1] – [3], for a.e. $t \in \Omega$ and $i = 1,2,...,$ $\alpha, \beta > 0$, $\alpha + \beta = 1$, we obtain

$$|\widetilde{\rho}_i(t, f) - f(t)| \leq \widetilde{\rho}_i\left(t, \frac{f - f(t)}{\beta}\right) + \frac{\beta}{\alpha} f(t).$$

Hence, for an arbitrary $\lambda > 0$ and $i = 1,2,...$ we have

$$\rho_0^{A,s}(\lambda(\widetilde{\rho}_i(\cdot, f)f(\cdot))) \leq \rho_0^{A,s}\left(2\lambda\widetilde{\rho}_i\left(\cdot, \frac{f - f(\cdot)}{\beta}\right)\right) + \rho_0^{A,s}\left(2\lambda\frac{\beta}{\alpha}f(\cdot)\right).$$

Next, for an arbitrary $\varepsilon > 0$ there exists $\beta > 0$ such that

$$\rho_0^{A,s}\left(2\lambda\frac{\beta}{\alpha}f(\cdot)\right) < \frac{\varepsilon}{2}. \tag{1}$$

By the singularity of the sequence $(\widetilde{\rho}_i)$ at the point *f* for such $\beta$ we obtain

$$\rho_0^{A,s}\left(2\lambda\widetilde{\rho}_i\left(\cdot, \frac{f - f(\cdot)}{\beta}\right)\right) < \frac{\varepsilon}{2} \tag{2}$$

for $i > i_0 = i_0(\varepsilon, \lambda) > 0$. Consequently, using estimations (1) and (2), for every $\lambda > 0$ the inequality

$$\rho_0^{A,s}(\lambda(f(\cdot) - \widetilde{\rho}_i(\cdot, f))) < \varepsilon \quad \text{for} \quad i > i_0$$

holds. The proof is complete.

We say that the sequence $(\widetilde{\rho}_i)$ is singular at the point $f \in X\rho_{os}^A$ iff for any two positive numbers *a*, *b* and for $m = 1,2,...$

$$S_m^i(f) = \sup_n a_{mn} \rho_n(\cdot, a\widetilde{\rho}_i(\cdot, b(f - f(\cdot)))) \to 0 \quad \text{as} \quad i \to \infty$$

in measure in $\Omega$.

Using the Lebesgue Dominated Convergence Theorem, we immediately obtain the following:

**THEOREM 2.2.** *If the sequence* $(\widetilde{\rho}_i)$ *is singular at the point* $f \in X\rho^A_{os}$, $f \geq 0$, *then for every* $\lambda > 0$ *the relation*

$$\rho^A_{os}(\lambda(f(\cdot) - \widetilde{\rho}_i(\cdot, f))) \to 0 \quad as \quad i \to \infty$$

*holds.*

## 3. SPECIAL CASES

In the sequel we will consider the following particular cases.

Let $\Omega = <0,1>$, $\Sigma$ be $\sigma$-algebra of Lebesgue measurable subsets of $<0,1>$ and $\mu$-the Lebesgue measure. Moreover, let X denote the set of measurable and a.e. finite functions in $<0,1)$ extended periodically, with the period 1, outside $<0,1)$ such that $f = g$ iff $f(t) = g(t)$ for a.a. $t \in \Omega$.

By $K_n$, $\widetilde{K}_i$, $n$, $i = 1,2,\ldots$, we denote functions which are measurable and positive a.e. in the closed interval $<0,1>$ and such, respectively

$$\int_0^1 K_n(u)du < \infty \quad \text{for} \quad n = 1,2,\ldots$$

and

$$\int_0^1 \widetilde{K}_i(u)du = 1 \quad \text{for} \quad i = 1,2,\ldots$$

We define the following sequence of operators

$$\text{(A)} \qquad \rho_n(t,f) = \varphi^{-1}\left(\int_0^1 K_n(u)\varphi(|f(u+t)|)du\right),$$

$$\text{(B)} \qquad \widetilde{\rho}_i(t,f) = \varphi^{-1}\left(\int_0^1 \widetilde{K}_i(u)\varphi(|f(u+t)|)du\right)$$

for $n$, $i = 1,2,\ldots$ and for $t \in <0,1>$, where $\varphi$ is a convex $\varphi$-function and $\varphi^{-1}$ is the inverse function to $\varphi$. The sequence (A) generates in particular the modular spaces $X\rho^{A,s}_o$ and $X\rho^A_{os}$ and its elements satisfy conditions $1^\circ. - 4^\circ$. Elements of these spaces we shall approximate by terms of the sequence (B) which satisfy conditions 1) – 3).

We say that $(\widetilde{K}_i)$, defined as in (B), is a singualr kernel iff

$$\lim_{i\to\infty}\int_{\delta}^{1}\widetilde{K}_i(u)\,du = 0$$

for every $\delta\in(0,1)$.

**THEOREM 3.1.** *Assume that:* a) *a convex* $\varphi-$ *function* $\varphi$ *satisfies the condition* $(\Delta_2)$ *for large arguments,*

b) *for an arbitrary* $a>0$ *and for* $m=1,2,\ldots$

$$\sup_n a_{mn}k_n(a\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0$$

*and*

$$\int_0^1 \sup_n a_{mn}k_n(a\varepsilon)\overline{K}_n(u)du = 0(\varphi(\varepsilon)),$$

*where*

$$k_n(a\varepsilon) = \varphi^{-1}\left(\varphi(a\varepsilon)\int_0^1 K_n(u)du\right), \qquad \overline{K}_n(u) = K_n(u)\Big/\int_0^1 K_n(u)du,$$

c) *the sequence* $(\widetilde{K}_i)$ *is a singular kernel.*

*Then for* $f\in X\rho_o^{A,s}\cap L^{\varphi}(0,1)$, $f\ge 0$, *and for every* $\lambda>0$ *we have*

$$\rho_o^{A,s}(\lambda(f(\cdot)-\widetilde{\rho}_i(\cdot,f)))\to 0 \quad \text{as } i\to\infty.$$

**PROOF.** Because the $\varphi-$ function $\varphi$ satisfies the condition $(\Delta_2)$ for large arguments, so for every $\varepsilon>0$ and for $a>0$ there exists $a'=a'(\varepsilon,a)>0$ such that $\varphi(au)\le a'\varphi(u)$ for $u\ge\varepsilon$. Hence we obtain for $b>0$ the following estimation

$$S_i^m(f)\le\int_0^1\sup_n a_{mn}\varphi^{-1}\left\{\varphi(a\varepsilon)\int_0^1 K_n(u)du + \right.$$
$$\left. + a\int_0^1 K_n(u)\left[\int_0^1\widetilde{K}_i(v)\varphi(b\,|\,f(u+v+t)-f(u+t)\,|)dv\right]du\right\}dt.$$

Denote

$$v_\varepsilon^n = \varphi(a\varepsilon)\int_0^1 K_n(u)du, \qquad \delta_\varepsilon^n = v_\varepsilon^n\sup_{u\ge v_\varepsilon^n}\frac{\varphi^{-1}(u)}{u}, \qquad c^n = \frac{\delta_\varepsilon^n}{v_\varepsilon^n}$$

Since $\varphi$ is continuous, so for $n=1,2,\ldots$ we have $v_\varepsilon^n\to 0$ as $\varepsilon\to 0$, and hence for $n=1,2,\ldots$ $\delta_\varepsilon^n\to 0$ as $\varepsilon\to 0$. Moreover, the following inequality $\varphi^{-1}(u)\le c_\varepsilon^n u$ for every $u\ge v_\varepsilon^n$ holds. Therefore we get for $i$, $m=1,2,\ldots$ and for $b>0$ that

$$S_i^m(f) \le \sup_n a_{mn}\delta_\varepsilon^n + a\left(\int_0^1 \sup_n a_{mn}\, c_\varepsilon^n K_n(u)du\right)\times$$
$$\times\left(\int_0^1 \widetilde{K}_i(v)\left(\int_0^1 \varphi(b\,|\,f(u+s)-f(s)|)ds\right)dv\right).$$

The assumptions b) and c) imply that for every $m=1,2,\ldots$ and for $f\in\rho_o^{A,s}\cap L^\varphi(0,1)$, $f\ge 0$, we have $S_i^m(f)\to 0$ as $i\to\infty$. Thus the sequence $(\widetilde{\rho}_i)$ is singular at the point *f*. Now our claim follows from Theorem 2.1.

Let *f* a bounded function and denote by

$$g(x) = \sup_{|v|\le\delta}\int_0^1 \varphi(|\,f(v+u+x)-f(u+x)|)du,$$

where $x\in R$, $\delta\ge 0$ and $\varphi-$function $\varphi$ satisfies the condition $(\Delta_2)$ for large arguments. It is know (see [4]) that *g* is a measurable function. We shall define the $\varphi-$integral modulus of continuity in measure for a bounded function $f\in X$ by taking

$$\omega_\mu^\varphi(\eta,\delta;f) = \mu\left(\left\{x\in<0,1>: \sup_{|v|\le\delta}\int_0^1 \varphi(|\,f(v+u+x)-f(u+x)|)du \ge \eta\right\}\right),$$

where $\eta\ge 0$, $\delta\ge 0$. In [4] the properties of $\omega_\mu^\varphi(\eta,\delta;f)$ were shown.

For $f\in X\rho_{os}^A$, $f\ge 0$, consider the truncations

$$f_k(x) = \begin{cases} f(x) & \text{for } \ x\in\{x\in<0,1): f(x)\le k\},\\ k & \text{for the remaining } \ x\in<0,1),\end{cases}$$

where *k* is a positive integer.

We say that $f\in X\rho_{os}^A$, $f\ge 0$, is a $\mu-$regular function iff for every $k=1,2,\ldots$ and for every $\eta>0$ we have

$$\omega_\mu^\varphi(\eta,\delta;f_k)\to 0 \quad \text{as} \quad \delta\to 0.$$

The sequence $(\widetilde{\rho}_i)$ of the from (B) is called regular at the point $f\in X\rho_{os}^A$, $f\ge 0$, iff for every $m=1,2,\ldots$ and for an arbitrary $a>0$ we have

$$\sup_n a_{mn}\rho_n(t,a\,|\,\widetilde{\rho}_i(\cdot,f_k)-\widetilde{\rho}_i(\cdot,f)|\to 0 \quad \text{as} \quad k\to\infty,$$

for a.e. $t\in<0,1>$ and uniformly with respect to $i=1,2,\ldots$ .

We say that $\rho_n(\cdot, f)$, $n=1,2,\ldots$, in the form (A), are equiabsolutely continuous at the point $f \in X\rho^A_{os}$, $f \geq 0$, iff for a.e. $t \in <0,1>$ and for an arbitrary $\varepsilon > 0$ there exists a $\Delta > 0$ such that for every $n=1,2,\ldots$ and for every $A \subset <0,1>$, $A \in \Sigma$, with $\mu(A) < \Delta$, we have

$$\int_A K_n(u)\varphi(f(u+t))du < \varepsilon.$$

Let us denote by $\ell_n = \ell_n(\varepsilon)$ the least positive integer such that

$$(**) \qquad \int_{\{u\in<0,1>:\, K_n(u)>\ell_n\}} K_n(u)du < \varepsilon, \quad \text{where} \quad \varepsilon > 0.$$

**THEOREM 3.2.** *Let $f \in X\rho^A_{os}$, $f \geq 0$, be a $\mu-$ regular function. Assume that:*

a) *a convex $\varphi-$function $\varphi$ satisfies the condition $(\Delta_2)$ for large arguments,*

b) *the sequence*

$$\left(\int_0^1 K_n(u)du\right)$$

is bounded and for every $m=1,2,\ldots$ and for an arbitrary $\varepsilon > 0$ we have $\sup_n a_{mn}\ell_n(\varepsilon) < \infty$, where $\ell_n$ is defined by the condition $(**)$, in addition $\varphi^{-1}(\varepsilon)\sup_n a_{mn}\ell_n(\varepsilon) \to 0$ as $\varepsilon \to 0$,

c) $\rho_n(\cdot, f)$, $n=1,2,\ldots$, *of the from* (A), *are equiabsolutely continuous at the point f,*

d) *the sequence $(\widetilde{\rho}_n)$ of the from* (B) *is regular at the point f and $(\widetilde{K}_i)$ is a singular kernel.*

Then for every $\lambda > 0$

$$\rho^A_{os}(\lambda(f(\cdot) - \widetilde{\rho}_i(\cdot, f))) \to 0 \quad \text{as} \quad i \to \infty.$$

We prove the above result analogously as in the Theorem 5 in the paper [5], pp. 811-814.

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