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ON A RADIAL PERIODIC SOLUTION TO THE DIFFUSION EQUATION FOR THE EXTERIOR OF A BALL

ABSTRACT: The subject of the paper is the construction of a periodic radial solution to the diffusion parabolic differential equation (I) $\Delta u(x,t) - D_t u(x,t) = f(x,t)$, where $\delta = D^2_{x_1} + D^2_{x_2} + D^2_{x_3}$, $x = (x_1, x_2, x_3)$, $t \in (-\infty, \infty)$, in radial coordinates (r,t), $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, in the exterion of the ball $D_1 = \{(r,t) : r > R, t \in (-\infty, \infty)\}$. Equation (I) is of the from (II) $D_r^2(W(r,t)) - D_t(W(r,t)) = 0$, $W(r,t) = rU(r,t)$. By the suitable Green function $(r,t,p,s) \to G(r,t,p,s)$, we construct the periodic solution with respect to the variable t of equation (II) as the potential $W(r,t)$ of the double layer.

KEY WORDS: a boundary-value problem, a periodic solution, Green function, Green potentials.

1. INTRODUCTION

The subject of the paper is the construction of a periodic solution to the equation

(1) $$PW(r,t) = D_r^2 W(r,t) - D_t W(r,t) = 0$$

in the domain

$$D = \{(r,t) : r \in (R, \infty),\ t \in (-\infty, \infty)\}.$$

To the construction of the solution we apply the suitable Green function G. We suppose that solution is the potential of the double layer:

(2) $$W_1(r,t) = \int_{-\infty}^{t} H(s)(t-s)^{-3/2}(r-p)\exp\left(-\frac{(r-p)^2}{4(t-s)}\right) ds.$$

In [1], the similar problem for the equation $(D_r^2 - D_t)u(r,t) = 0$ and for the strip is treated. In [3], periodic solution of a parabolic problem is studied.

2. GREEN FUNCTION G

Let

$$U(r,t,p,s) = A(t-s)^{-1/2}\exp(B(t,s)(r-p)^2),$$

where

$$A = (2\sqrt{\pi})^{-1} \quad \text{and} \quad B(t,s) = (-4(t - s))^{-1},$$

denote the fundamental solution to the equation $PU = D_r^2 U - D_t U = 0$.

By [2], the function

$$(r,t,p,s) \to G(r,t,p,s) = U(r,t,p,s) - U(2R - r,t,p,s), \quad (r,t,p,s) \in D_1,$$

where $D_1 = \{(r,t,p,s): -\infty < s < t,\ t \in (-\infty,\infty),\ r > R,\ p > R,\ r \neq p\}$, is the Green function to the equation

$$PG(r,t,p,s) = 0,$$

to the half space $r > R$ and to the Dirichlet boundary-value conditions:

$$G(R,t,p,s) = G(\infty,t,p,s) = 0.$$

3. Green potentials

Let us consider the Green potentail

$$W(r,t) = \int_{-\infty}^{t} H(s) D_p G(r,t,R,s) ds$$

and the potential of the double layer

$$W_1(r,t) = \int_{-\infty}^{t} H(s)(t-s)^{-3/2}(r-p)\exp\left(-\frac{(r-p)^2}{4(t-s)}\right) ds.$$

4. Motivation of the problem

Consider the solution $(r,t,T) \to W(r,t,T)$ of tyhe Cauchy problem to the equation $PW(r,t) = 0$. By [1], the solution is of the from

$$W(r,t,T) = \int_{R}^{\infty} W(p,T) G(r,t,p,T) dp,$$

where

$$PU = D_r^2 U - D_t U = 0.$$

Let $M(T) = \sup_{p \in (R,\infty)} |W(p,T)|$ and let (K) denote the class of all functions W for which $M(T)$ is bounded.

LEMMA. *If* $W \in (K)$ *then*:

1° *the inequality*

$$|W(r,t,T)| \leq M(T)\int_R^{\infty} (t-T)^{-1/2} \exp\left(-\frac{(r-p)^2}{4(t-T)}\right)dp$$

holds,

2° $W(r,t) \to 0$ *as* $T \to -\infty$ *for every* $t \in (-\infty, \infty)$.

PROOF. 1°: Since $G \geq 0$ thus $G \leq U(r,t,r,T)$ and we obtain 1°. 2° is a consequence of 1°.

5. PROPERTIES OF THE PONTENTIAL W

Let us consider the Green potential

$$W_1(r,t) = \int_{-\infty}^{t} H(s)(t-s)^{-3/2}(r-R)\exp\left(-\frac{(r-R)^2}{4(t-s)}\right)ds.$$

Denote by (K_w) the class of all functions $t \to H(t)$ continuous and bounded for $t \in (-\infty, \infty)$ with the period w.

THEOREM. *If* $H \in (K_w)$ *then:*

1° $PW_1(r,t) = 0, \quad (r,t) \in D$,

2° $W_1(r,t) \to H(t)$ as $(r,t) \to (R,t)$,

3° $W_1(r,t) \to 0$ as $(r,t) \to (\infty, t)$ uniformly for every $t \in (-\infty, \infty)$,

4° $(r,t) \to W_1(r,t)$ is the periodic function with respect to t with the period w.

PROOF. 1°: BY [2], we obtain 1°, 2°.

3°: We have

$$W_1(r,t) = \int_{-\infty}^{t} H(s)\frac{(r-R)(r-R)^3}{(t-s)^{3/2}(r-R)^3}\exp(B(t,s)(r-R)^2)ds.$$

By the last formula, we get the inequality

$$|W_1(r,t)| \leq c(r-R)^{-2}\int_{-\infty}^{t} H(s)\frac{(r-R)^3}{(t-s)^{3/2}}\exp(B(t,s)(r-R)^2)ds.$$

Supplying in the last integral the change of the integral variable

$$z = \frac{r-R}{(t-s)^{1/2}}, \qquad dz = -\frac{2(r-R)}{(t-s)^{3/2}}ds,$$

we obtain

$$|W_1(r,t)| \le \frac{c}{(R-r)^2}\int_0^\infty z^3 \exp(-z^2)dz \to 0 \quad \text{as} \quad r\to\infty$$

with $c = \sup_{s\in(-\infty,\infty)} |H(s)|$.

4°: We have

$$|W_1(r,t+w)| = \int_{-\infty}^{t+w} H(s)D_pG(r-R,t+w-s)ds.$$

Applying in the last integral the change of the integral variables

$$s = w+z, \qquad ds = dz, \qquad z\in(-\infty,\infty),$$

we obatin

$$W_1(r,t+w) = \int_{-\infty}^{t} H(w+z)D_pG(r-R,t-z)\,dz = W_1(r,w).$$

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