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APPROXIMATION BY MEANS OF BOREL TYPE OF FOURIER SERIES

ABSTRACT: In this paper we introduce and study approximation properties of means of the Borel type of Fourier series of functions belonging to the space $C_{2\pi}$.

KEY WORDS: Fourier series, Borel summability, degree of approximation.

1. INTRODUCTION

1.1. Let S_n, $n \in N := \{0,1,2,...\}$, be the n-th partial sum of real numerical series

(1) $$\sum_{k=0}^{\infty} a_k.$$

As is know ([1]) the series (1) is summable to s $(|s| < \infty)$ by the Borel method (B), if the series $\sum_{k=0}^{\infty} \frac{r^k}{k!} S_k$ is convergent on $R_+ := (0,+\infty)$ and if

$$\lim_{r \to +\infty} B(r) = s,$$

where

$$B(r) := e^{-r} \sum_{k=0}^{\infty} \frac{r^k}{k!} S_k.$$

In this paper we define methods (B_i), $i = 1,2$, of the Borel type of summability of the series (1).

DEFINITION. *The sereies* (1) *is summable to* s $(|s| < \infty)$ *by the method* (B_1), *if the series* $\sum_{k=0}^{\infty} r^{2k}/(2k)! S_k$ *is convergent on* R_+ *and if means*

$$B_1(r) := \frac{1}{\cosh r} \sum_{k=0}^{\infty} \frac{r^{2k}}{(2k)!} S_k, \quad r \in R_+$$

satisfies the condition

$$\lim_{r \to +\infty} B_1(r) = s.$$

The series (1) *is summable to* s ($|s|<\infty$) *by the method* (B_2), *if the series* $\sum_{k=0}^{\infty} r^{2k+1}S_k/(2k+1)!$ *is convergent on* R_+ *and if*

$$\lim_{r\to+\infty} B_2(r) = s,$$

where

$$B_2(r) := \frac{1}{\sinh r}\sum_{k=0}^{\infty}\frac{r^{2k+1}}{(2k+1)!}S_k, \qquad r\in R_+$$

($\sinh$, $\cosh$ *are elementary hyperbolic functions*).

By verification of Toeplitz conditions ([1], p. 43) we obtain the following theorem.

THEOREM 1. *The methods* (B_i), $i=1,2$, *of summability of series are regular.*

It is easily verified that the divergent series $\sum_{k=0}^{\infty}(-1)^k$ is summable to $s=1/2$ by the Borel method (B) and by methods (B_i), $i=1,2$.

2. (B_i) – MEANS OF FOURIER SERIES

2.1. Let $C_{2\pi}$ be the space of 2π-periodic real-valued functions f, continuous on $Q=[-\pi,\pi]$ with the norm

$$\|f\| := \max_{x\in Q}|f(x)|. \tag{2}$$

For the function $f\in C_{2\pi}$ we define the modulus of smoothness ([2])

$$\omega_2(t;f) := \sup_{|h|\le t}\|\Delta_h^2 f(\cdot)\|, \qquad t\ge 0, \tag{3}$$

where

$$\Delta_h^2 f(x)) := f(x+h) + f(x-h) - 2f(x). \tag{4}$$

Denote by $Lip^2\alpha$, with a fixed $0<\alpha\le 2$, the class of all functions $f\in C_{2\pi}$ for which $\omega_2(t;f)=O(t^\alpha)$.

2.2. Let $f \in C_{2\pi}$ and let

(5) $$f(x) \sim \frac{1}{2}a_0(f) + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin(kx)$$

be its Fourier series. Let $S_n(\cdot;f)$, $n \in N$, be the n-th partial sum of the series (5). It is easily proved that for every $f \in C_{2\pi}$ there exist means $B_i(\cdot;\cdot;f)$, $i = 1,2$, of the Fourier series (5)

(6) $$B_i(r;x;f) := \sum_{k=0}^{\infty} p_{i,k}(r) S_k(x;f)$$

for $x \in R = (-\infty,+\infty)$, $r \in R_+$ and $i = 1,2$, where

(7) $$p_{1,k}(r) := \frac{1}{\cosh r}\frac{r^{2k}}{(2k)!}, \qquad p_{2,k}(r) := \frac{1}{\sinh r}\frac{r^{2k+1}}{(2k+1)!},$$

for $r \in R_+$ and $k \in N$.

In this section we shall give some approximation properties of means $B_i(t)$ for $f \in C_{2\pi}$.

In this paper we shall denote by $M_k(a,b)$, $k = 1,2,...$, suitable positive constants depending only on indicated parametrs a, b.

2.3. First we shall give some auxiliary results. Applying the integral formula for $S_n(\cdot;f)$:

$$S_n(x;f) = \frac{1}{n}\int_{-\pi}^{\pi} f(x+t)D_n(t)dt, \qquad x \in R, \qquad n \in N,$$

$$D_0(t) = \frac{1}{2}, \quad D_n(t) = \frac{1}{2} + \sum_{j=1}^{n} \cos jt \quad \text{for} \quad n \geq 1,$$

we get from (6) and (7)

(8) $$B_i(r;x;f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_i(r;t)dt,$$

for $x \in R$, $r \in R_+$ and $i = 1,2$, where

(9) $$K_i(r;t) := \sum_{n=0}^{\infty} p_{i,n}(r) D_n(t).$$

Using the formula

(10) $$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}}, \qquad |t| \in (0,\pi], \quad n \in N,$$

we shall prove the following

Lemma 1. *For all* $r \in R$, *and* $|t| \in (0, \pi]$ *and* $i = 1,2$ *we have*

(11) $$K_1(r;t) = \frac{1}{2\cosh r \cdot \sin\frac{t}{2}}\left[\cos\tfrac{t}{2}\cdot\sin(r\sin\tfrac{t}{2})\cdot\sinh(\cos\tfrac{t}{2}) + \right.$$
$$\left. + \sin\tfrac{t}{2}\cos(r\sin\tfrac{t}{2})\cosh(r\cos\tfrac{t}{2})\right],$$

(12) $$K_2(r;t) = \frac{\sin(r\sin\frac{t}{2})(\cosh(r\cos\frac{t}{2})}{2\sinh r \cdot \sin\frac{t}{2}}.$$

Proof. Let $i = 1$. Then by (7), (9) and (10) we get

(13) $$K_1(r;t) = \frac{1}{2\cosh r \cdot \sin\frac{t}{2}}\left\{\cos\tfrac{t}{2}\sum_{n=0}^{\infty}\frac{r^{2n}}{(2n)!}\cdot\sin(\tfrac{2nt}{2}) + \right.$$
$$\left. + \sin\tfrac{t}{2}\sum_{n=0}^{\infty}\frac{r^{2n}}{(2n)!}\cos(\tfrac{2nt}{2})\right\}.$$

It is know that

$$\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} = \cosh z,$$

for all complex numbers z. Setting $z = r[\cos\frac{t}{2} + i\sin\frac{t}{2}]$, we obtain

$$\sum_{n=0}^{\infty}\frac{r^{2n}\left[\cos(\frac{2nt}{2}) + i\sin(\frac{2nt}{2})\right]}{(2n)!} = \cosh\left[r(\cos\tfrac{t}{2} + i\sin\tfrac{t}{2}\right]$$

and by elementary calculations we have

(14) $$\sum_{n=0}^{\infty}\frac{r^{2n}}{(2n)!}\cos(\tfrac{2nt}{2}) = \cosh(r\sin\tfrac{t}{2})\cosh(r\cos\tfrac{t}{2}),$$

(15) $$\sum_{n=0}^{\infty}\frac{r^{2n}}{(2n)!}\sin(\tfrac{2nt}{2}) = i\sin(r\sin\tfrac{t}{2})\sinh(r\cos\tfrac{t}{2}).$$

From (13) - (15) immediately follows (11). The proof for $i = 2$ is similar.

By

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)dt = 1 \quad \text{for} \quad n \in N,$$

and by (9) and (7) we get

$$(16)\qquad \frac{1}{\pi}\int_{-\pi}^{\pi} K_i(r;t)dt = 1 \qquad \text{for} \qquad r\in R_+, \qquad i=1,2.$$

From (8), (16) and (4) it follows that

$$(17)\qquad B_i(r;x;f)-f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+t)-f(x)]K_i(r;t)dt = \\ = \frac{1}{\pi}\int_{0}^{\pi}[\Delta_t^2 f(x)]K_i(r;t)dt$$

for every $f\in C_{2\pi}$, $x\in R$, $r\in R_+$ and $i=1,2$.

Now we shall prove two auxiliary inequalities.

LEMMA 2. *Let* $i=1$ *or* $i=2$*. Then there exists a positive constant* M_1 *such that*

$$(18)\qquad \int_{-\pi}^{\pi}|K_i(r;t)|\,dt = 2\int_{0}^{\pi}|K_i(r;t)|\,dt \le M_1(\ln r+1) \qquad \textit{for} \qquad r>1.$$

PROOF. It is obvious that

$$\int_{-\pi}^{\pi}|K_i(r;t)|\,dt = 2\int_{0}^{\pi}|K_i(r;t)|\,dt = 2\int_{0}^{1/r}|K_i(r;t)|\,dt + 2\int_{1/r}^{\pi}|K_i(r;t)|\,dt,$$

for $r>1$ and $i=1,2$. By the inequality $|\sin y|\le|y|$ for $y\in R$, we get from (11) and (12)

$$|K_1(r;t)| \le \frac{r[\sinh(r\cos\frac{t}{2})+\cosh(r\cos\frac{t}{2})]}{2\cosh r} \le r\exp(-2r\sin^2\tfrac{t}{4}) \le r,$$

$$|K_2(r;t)| \le \frac{r\cosh(r\cos\frac{t}{2})}{2\sinh r} \le \frac{r}{(1-e^{-2})}\exp(-2r\sin^2\tfrac{t}{4}) \le \\ \le \frac{r}{(1-e^{-2})} \le \frac{2r}{2(1-e^{-2})} < 2r,$$

for all $r>1$ and $t\in[0;\pi]$. Hence for $r>1$ we have

$$\int_{0}^{1/r}|K_i(r;t)|\,dt \le 2, \qquad i=1,2.$$

Moreover, by $\sin y\ge(2/\pi)y$ for $y\in[0;\pi/2]$, we drive from (11) and (12):

$$(19)\quad |K_1(r;t)| \le \frac{\sinh(r\cos\frac{t}{2})+\cosh(r\cos\frac{t}{2})}{2\cosh r\sin\frac{t}{2}} \le \frac{1}{\sin\frac{t}{2}}\exp(-2r\sin^2\tfrac{t}{4}) \le$$

$$\le \frac{\pi}{t}\exp\left(-\frac{rt^2}{2\pi^2}\right),$$

$$(20)\quad |K_2(r;t)| \le \frac{1}{(1-e^{-2})\sin\frac{t}{2}}\exp(-2r\sin^2\tfrac{t}{4}) \le \frac{\pi}{(1-e^{-2})t}\exp\left(-\frac{rt^2}{2\pi^2}\right) <$$

$$< \frac{2\pi}{t}\exp\left(-\frac{rt^2}{2\pi^2}\right),$$

for $r>1$ and $t\in(0;\pi]$. Setting $rt^2/2\pi^2 = u$, we get

$$\int_{1/r}^{\pi} t^{-1}\exp\left(-\frac{rt^2}{2\pi^2}\right)dt = \frac{1}{2}\int_{1/(2\pi^2 r)}^{r/2} u^{-1}e^{-u}\,du \le \frac{1}{2}\int_{1/(2\pi^2 r)}^{r/2} u^{-1}\,du = \ln(\pi r),$$

for $r>1$. Consequently we have

$$\int_{1/r}^{\pi} |K_i(r;t)|\,dt \le M_2(\ln r+1) \quad \text{for} \quad r>1.$$

Combining these, we obtain the desired inequality (18).

LEMMA 3. *Suppose that $0<\alpha\le 2$ be a fixed number. Then there exists a positive constant $M_3(\alpha,i)$ such that*

$$(21)\qquad \int_0^{\pi} t^{\alpha}\,|K_i(r;t)|\,dt \le M_3(\alpha,i)r^{-\alpha/2}$$

for all $r>1$ and $i=1,2$.

PROOF. Let $i=1$. Applying (19), we have

$$I_\alpha := \int_0^{\pi} t^{\alpha}\,|K_1(r;t)|\,dt \le \pi\int_0^{\pi} t^{\alpha-1}\exp\left(-\frac{rt^2}{2\pi^2}\right)dt$$

for $r>1$, which by $(t/\pi)\sqrt{r/2} = u$ implies that

$$I_\alpha \le \pi^{\alpha+1}\left(\frac{2}{r}\right)^{\alpha/2}\int_0^{\sqrt{r/2}} u^{\alpha-1}e^{-u^2}\,du \le \pi^{\alpha+1}\left(\frac{2}{r}\right)^{\alpha/2}\left\{\int_0^1 u^{\alpha-1}e^{-u^2}\,du + \int_1^{\infty} u^{\alpha-1}e^{-u^2}\,du\right\}.$$

If $0<\alpha\leq 2$, then

$$\int_0^1 u^{\alpha-1}e^{-u^2}du \leq \int_0^1 u^{\alpha-1}du = \frac{1}{\alpha}.$$

If $0<\alpha\leq 1$, then

$$\int_1^{+\infty} u^{\alpha-1}e^{-u^2}du \leq \int_1^{+\infty} u^{-u^2}du < \frac{\sqrt{\pi}}{2}.$$

For $1<\alpha\leq 2$ we have

$$\int_1^{+\infty} u^{\alpha-1}e^{-u^2}du \leq \int_1^{+\infty} ue^{-u^2}du \leq \frac{1}{2}\int_0^{+\infty} e^{-t}dt = \frac{1}{2}.$$

From these we get

$$I_\alpha \leq \pi^{\alpha+1}\left(\frac{2}{r}\right)^{\alpha/2}\left(\frac{1}{\alpha}+\frac{\sqrt{\pi}}{2}\right) \quad \text{for} \quad r>1.$$

Thus the estimation (21) is proved for $i=1$. The proof of (21) for $i=2$ is analogous by (20).

Lemma 4. *Let $f\in C_{2\pi}$ and $i=1,2$. Then there exists a positive constant M_4 such that*

(22) $$\| B_i(r;\cdot;f)\| \leq M_4\| f\|(\ln r+1) \quad \textit{for all} \quad r>1.$$

Proof. From (8) we get

$$B_i(r;x;f) = \frac{1}{\pi}\int_0^\pi [f(x-t)+f(x+t)]K_i(r;t)dt$$

for $r\in R_+$, $x\in R$, $i=1,2$. By (2) we have for $f\in C_{2\pi}$:

$$|f(x-t)+f(x+t)| \leq 2\| f\|, \qquad x,t\in R.$$

Hence

$$\| B_i(r;\cdot;f)\| \leq \max_{x\in Q}\left\{\frac{1}{\pi}\int_0^\pi |f(x-t)+f(x+t)\|K_i(r;t)|dt\right\} \leq$$

$$\leq \frac{2}{\pi}\| f\|\int_0^\pi |K_i(r;t)|dt$$

for $r>1$ and $i=1,2$, which by Lemma 2 implies (22).

2.4. Now we shall prove main theorems

THEOREM 2. *Let* $f \in C_{2\pi}$ *and* $i = 1,2$. *Then there exists a positive constant* $M_5(i)$ *such that*

(23) $$\| B_i(r;\cdot;f) - f(\cdot) \| \leq M_5(i)\,\omega_2(r^{-1/2};f)(\ln r + 1) \quad \text{for} \quad r > 1,$$

where $\omega_2(f)$ *is defied by* (3).

PROOF. From (17) and (2) it follows that

$$\| B_i(r;\cdot;f) - f(\cdot) \| \leq \frac{1}{\pi}\int_0^{\pi} \| \Delta_t^2 f(\cdot) \| \, | K_i(r;t) | \, dt$$

for $r > 1$ and $i = 1,2$. But by (3) and by the inequality $\omega_2(\lambda t; f) \leq (\lambda + 1)^2 \omega_2(t;f)$ for $\lambda, t \geq 0$, we have

$$\| B_i(r;\cdot;f) - f(\cdot) \| \leq \frac{1}{\pi}\omega_2(r^{-1/2};f)\left\{ r\int_0^{\pi} | K_i(r;t) | \, dt + \right.$$

$$\left. + 2\sqrt{r}\int_0^{\pi} t\, | K_i(r;t) | \, dt + \int_0^{\pi} | K_i(r;t) | \, dt \right\}$$

for $r > 1$ and $i = 1,2$. Applying Lemma 2 and Lemma 3, we immediately obtain the desired assertion (23) for $r > 1$ and $i = 1,2$.

Theorem 2 implies the following

COROLLARY. *If the modulus of smoothness of function* $f \in C_{2\pi}$ *satisfies the condition*

$$\lim_{r \to +\infty} \omega_2(t;f) \ln \tfrac{1}{t} = 0,$$

then the Fourier series (5) *is* (B_i) *summable,* $i = 1,2$, *to f in the norm of the space* $C_{2\pi}$, *i.e.*

$$\lim_{r \to +\infty} \| B_i(r;\cdot;f) - f(\cdot) \| = 0, \quad i = 1,2.$$

THEOREM 3. *Suppose that* $f \in Lip^2\alpha$ *with a fixed* $0 < \alpha \leq 2$ *and* $i = 1,2$. *Then there exists a positive constant* $M_6(\alpha, i)$ *such that*

(24) $$\| B_i(r;\cdot;f) - f(\cdot) \| \leq M_6(\alpha, i, f) r^{-\alpha/2} \quad \textit{for all} \quad r > 1.$$

PROOF. Similarly as in the proof of Theorem 2 we get for $f \in Lip^2\alpha$

$$\| B_i(r;\cdot;f) - f(\cdot) \| \le \frac{1}{\pi}\int_0^\pi \omega_2(t;f) | K_i(r;t) | \, dt \le M_7(\alpha, f) \int_0^\pi t^\alpha \, | K_i(r;t) | \, dt,$$

$r > 1$ and $i = 1,2$. Applying Lemma 3, we obtain (24).

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