FASCICULI MATHEMATICI

Nr 33 2002

ALEKSANDER WASZAK

ON THE STRONG CONVERGENCE IN SOME SEQUENCE SPACES

ABSTRACT: The purpose of this paper is to introduce and study an idea of lacunary strong (A,φ)-convergence with respect to a modulus function. In coures of these investigations we study some connections between (A,φ)-strong summability of sequences and lacunary strong convergence with respect to a modulus or lacunary statistical convergence.

KEY WORDS: sequence spaces, modular spaces.

1. INTRODUCTION

In papers of J. Musielak [9], J. Musielak and W. Orlicz [12], W. Orlicz [15] and myself [18] there are considered and investigated some modular spaces connected with strong (A,φ)-summability of sequences.

The spacess N_Θ of lacunary strong convergence of sequences have been introduced by A. Freedman, J. Somberg and M. Raphel [4], where

$$N_\Theta = \left\{ x=(t_\nu) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{\nu\in I_r} |t_\nu - s| = 0 \text{ for some } s \right\}.$$

and $\Theta = (k_r)$ is a given lacunary sequence.

If f is a given modulus function (which have been introduced by H. Nakano [14]) and $A=(a_{n\nu})$ is a given matrix, then applying the concept of T. Bilging [1] we may define the sequence space (compare e.g. [2], [3] or [8])

$$N_\Theta(A,f) = \left\{ x=(t_\nu) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f\left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} t_\nu - s \right| \right) = 0 \text{ for some } s \right\}.$$

Throughout this paper it will be supposed that $s=0$ and that we take the sequence (σ_n^φ), where $\sigma_n^\varphi(x) = \sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)$ instead of the sequence $\left(\sum_{\nu=1}^{\infty} a_{n\nu} t_\nu \right)$.

Finally, the space $T_\Theta((A,\varphi),f)$ of lacunary strongly convergent sequences is defined by the formula

$$T_{\Theta}((A,\varphi),f) = \left\{ x=(t_{\nu}) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f(|\sigma_n^{\varphi}(x)|) = 0 \right\}.$$

2. Preliminaries

Let T, T_b, T_0, T_f denote spaces of all real sequences, bounded real sequences, real sequences convergent to zero and sequences with a finite number of elements different from zero, respectively. Sequences belonging to T will be denoted by $x=(t_{\nu})$, $y=(s_{\nu})$, $x_m=(t_{\nu}^m)$, $|x|=(|t_{\nu}|)$, $0=(0)$ and x^q will mean the sequence $t_1, t_2, \ldots, t_q, 0, 0, \ldots$. Moreover, we shall write e_p, e^q, e_p^q for the following sequences: 0, 0, ..., 1, 0, ... (with 1 at the p th place); 1, 1, ..., 1, 0, ... (with 1 at the first q places); 0, ... 0, 1, ... 1, 0, ... (with 1 at the p th, $(p+1)$ st, ..., $(p+q-1)$ st place), respectively.

A sequence of positive integers $\Theta=(k_r)$ is called lacunary if $k_0=0$, $k_r<k_{r+1}$ for all r and if $I_r=(k_{r-1},k_r]$ then $h_r=k_r-k_{r-1}\to 0$ as $r\to\infty$. In the following the qoutient k_r/k_{r-1} will be denoted by q_r, (compare [4]).

Let $A=(a_{n\nu})$ be an infinite matrix such that:

a) is nonnegativ i.e. $a_{n\nu}\geq 0$ for $n,\nu=1,2,\ldots$,

b) for an arbitrary positive integer n (or ν) there exists a positive integer ν_0 (or n_0) such that $a_{n\nu_0}\neq 0$ (or $a_{n_0\nu}\neq 0$), respectively,

c) there exist $\lim_{n\to\infty} a_{n\nu}=0$ for $\nu=1,2,\ldots$,

d) $\sup_n \sum_{\nu=1}^{\infty} a_{n\nu} = K < \infty$,

e) $\sup_n a_{n\nu}\to 0$ as $\nu\to\infty$.

By a φ-function we understood a cvontinuous non-decreasing function $\varphi(u)$ defined for $u\geq 0$ and such that $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$ and $\varphi(u)\to\infty$ as $u\to\infty$. The symbol $\varphi(|x|)$ means the function $\varphi(|x(t)|)$.

A φ-function φ is called non weaker then a φ-function ψ and we write $\psi\prec\varphi$ if there are constants $c,b,k,l>0$ such that $c\psi(lu)\leq b\varphi(ku)$, (for all, large or small u, respectively).

φ-functions φ and ψ are called equivalent and we write $\varphi\sim\psi$ if there are positive constants b_1, b_2, c, k_1, k_2, l such that $b_1\varphi(k_1u)\leq c\psi(lu)\leq b_2\varphi(k_2u)$, (for all, large or small u, respectively).

A φ-function φ is said to satisfy the condition (Δ_2), (for all, large or small u, respectively) if for some constant $k>1$ there is satisfied the inequality $\varphi(2u)\le k\varphi(u)$. For more properties of φ-function see e.g. [7], [10], [11].

By a modulus function we understood the increasing function f from $[0,\infty)$ to $[0,\infty)$ such that: $f(x)=0$ if and only if $x=0$, $f(x+y)\le f(x)+f(y)$ for $x,y\ge 0$ and is continuous from the right at 0, (compare [14]).

3. SPACES OF STRONGLY (A,φ) – SUMMABLE SEQUENCES

For a given φ-function $\varphi(u)$ and the matrix $A=(a_{n\nu})$ we adopt the following notation:

$$\sigma_n^{\varphi}(x)=\sum_{\nu=1}^{\infty}a_{n\nu}\varphi(|t_\nu|) \text{ for } n=1,2,...,$$

$$T_\varphi^0 = \left\{x\in T:\sigma_n^{\varphi}(x)<\infty \text{ for } n=1,2,... \text{ and } \lim_{n\to\infty}\sigma_n^{\varphi}(x)=0\right\},$$

$$T_\varphi = \left\{x\in T:\ \lambda x\in T_\varphi^0 \text{ for an arbitrary } \lambda>0\right\},$$

$$T_\varphi^* = \left\{x\in T\ :\ \lambda x\in T_\varphi^0 \text{ for a certain } \lambda>0\right\}.$$

Sequences x belonging to T_φ^* are called strongly (A,φ) – summable to zero.

A list of the most interesting properties concerning the space T_φ^* is presented below, (compare also [9], [12], [15] or [18]).

(1) $T_f\subset T_\varphi^*$, $T_\varphi\subset T_\varphi^0\subset T_\varphi^*$.

(2) If $a_{n\nu}\to 0$ as $n\to\infty$ for all ν, then e_p, e^q, $e_p^q\in T_\varphi$.

(3) For an arbitrary φ-function φ we have $T_b\cap T_\varphi=T_b\cap T_\varphi^*$.

(4) For arbitrary two φ-function φ and ψ the following identity holds $T_b\cap T_\varphi^*=T_b\cap T_\psi^*$.

(5) If $\psi\prec\varphi$ then $T_\varphi\subset T_\psi$ and $T_\varphi^*\subset T_\psi^*$.

(6) If the φ-function $\varphi(u)$ satisfies the condition (Δ_2) then $T_\varphi=T_\varphi^*$.

4. SPACES OF LACUNARY STRONGLY CONVERGENT SEQUENCES

Let φ and f be given φ-function and modulus function, respectively. Moreover, let a matrix A and a lacunary sequence Θ be given. We introduce sequence space $T_\Theta((A,\varphi),f)$ by the formula:

$$T_\Theta((A,\varphi),f) = \left\{ x=(t_v) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f\left(\left|\sum_{v=1}^{\infty} a_{nv}\varphi(|t_v|)\right|\right) = 0 \right\}.$$

The sequence x is said to be lacunary strong (A,φ) – convergent to zero with respect to a modulus f, if $x \in T_\Theta((A,\varphi),f)$.

Let us remark that in particular we have:

1^0 If $\varphi(u)=u$ for all u, then we obtain the space

$$N_\Theta^0(A,f) \equiv \left\{ x=(t_v) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f\left(\left|\sum_{v=1}^{\infty} a_{nv}\, t_v\right|\right) = 0 \right\}$$

which was defined and considered in [1].

2^0 If $f(v)=v$ then $T_\Theta((A,\varphi),v) = T_\Theta((A,\varphi))$, where

$$T_\Theta((A,\varphi)) = \left\{ x=(t_v) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} \sum_{v=1}^{\infty} a_{nv}\varphi(|t_v|) = 0 \right\}.$$

3^0 If $A=I$ then we obtain the following sequence space

$$T_\Theta((I,\varphi),f) = \left\{ x=(t_v) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f(\varphi(|t_v|)) = 0 \right\}.$$

4^0 If $A=I$ and moreover $\varphi(u)=u$ and $f(v)=v$ for all u and v, respectively, then we have the sequence space

$$N_\Theta^0 = T_\Theta((I,u),v) = \left\{ x=(t_v) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} |t_v| = 0 \right\},$$

(compare [1]).

5^0 If the matrix $A=(a_{nv})$ is defined by the formula:

$$a_{nv} = \frac{1}{n} \quad \text{for } n \ge v \quad \text{and} \quad a_{nv} = 0 \quad \text{for } n < v,$$

then applying the properties of Θ and f we obtain the sequence space

$$T_\Theta((A,\varphi),f) = \left\{ x=(t_v) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f\left(\left|\frac{1}{n}\sum_{v=1}^{n} \varphi(|t_v|)\right|\right) = 0 \right\}.$$

Moreover, we have the following inequalities

$$\frac{1}{h_r} f\left(\frac{1}{q_r} \min_{1 \le \nu \le k_{r-1}-1} \{\varphi(|t_\nu|)\}\right) \le \frac{1}{h_r} \sum_{n \in I_r} f\left(\frac{1}{n} \sum_{\nu=1}^{n} \varphi(|t_\nu|)\right) \le$$

$$\le f\left(\frac{1}{k_{r-1}+1} \sum_{\nu=1}^{k_r} \varphi(|t_\nu|)\right).$$

THEOREM 1. *Let f be a any modulus function and let φ-function φ, the matrix A and the sequence Θ be given. If*

$$w((A,\varphi),f) = \left\{ x = (t_\nu) : \lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} f\left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} \varphi(|t_\nu|) \right| \right) = 0 \right\}$$

then the following relations are true:

(a) *If* $\liminf_r q_r > 1$, *then we have* $w((A,\varphi),f) \subseteq T_\Theta((A,\varphi),f)$.

(b) *If* $\limsup_r q_r < \infty$, *then we have* $T_\Theta((A,\varphi),f) \subseteq w((A,\varphi),f)$.

(c) *If* $1 < \liminf_r q_r \le \limsup_r q_r < \infty$, *then* $T_\Theta((A,\varphi),f) = w((A,\varphi),f)$.

PROOF. (a). Let us suppose that $x \in w((A,\varphi),f)$. There exists $\delta > 0$ such that $q_r > 1+\delta$ for sufficiently large r and we have $h_r/k_r \ge \delta/(1+\delta)$ for sufficiently large r. Consequently,

$$\frac{1}{k_r} \sum_{n=1}^{k_r} f\left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} \varphi(|t_\nu|) \right| \right) \ge \frac{1}{k_r} \sum_{n \in I_r} f\left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} \varphi(|t_\nu|) \right| \right) =$$

$$= \frac{h_r}{k_r} \frac{1}{h_r} \sum_{n \in I_r} f\left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} \varphi(|t_\nu|) \right| \right) \ge \frac{\delta}{1+\delta} \frac{1}{h_r} \sum_{n \in I_r} f\left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} \varphi(|t_\nu|) \right| \right).$$

Finally, $x \in T_\Theta((A,\varphi),f)$.

PROOF. (b). The condition $\limsup_r q_r < \infty$ implies that there exists a constant $M > 0$ such that $q_r < M$ for every r. If $x \in T_\Theta((A,\varphi),f)$ and ε is an arbitrar positive number, then there exists an index m_0 such that

$$H_m = \frac{1}{h_m} \sum_{n \in I_m} f\left(\left| \sum_{\nu=1}^{\infty} a_{n\nu}\, \varphi(|t_\nu|) \right| \right) < \varepsilon$$

for every $m \geq m_0$. Thus, we can find a positive constant L such that $H_m \leq L$ for all m. In the following choosing an integer α such that $k_{r-1} < \alpha > k_r$ we obtain

$$I = \frac{1}{\alpha}\sum_{n=1}^{\alpha} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) \leq \frac{1}{k_{r-1}}\sum_{n=1}^{k_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) = I_1 + I_2$$

where

$$I_1 = \frac{1}{k_{r-1}}\sum_{m=1}^{m_0}\sum_{n\in I_m} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right),$$

$$I_2 = \frac{1}{k_{r-1}}\sum_{m=m_0+1}^{\alpha}\sum_{n\in I_m} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right).$$

It is easily verified that

$$I_1 = \frac{1}{k_{r-1}}\sum_{m=1}^{m_0}\sum_{n\in I_m} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) =$$

$$= \frac{1}{k_{r-1}}\left(\sum_{n\in I_1} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) + \ldots + \sum_{n\in I_{m_0}} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right)\right) \leq$$

$$\leq \frac{1}{k_{r-1}}\left(h_1 H_1 + \ldots + h_{m_0} H_{m_0}\right) \leq \frac{1}{k_{r-1}} m_0 k_{m_0} \sup_{1\leq i\leq m_0} H_i \leq \frac{m_0 k_{m_0}}{k_{r-1}} L .$$

Moreover, we have

$$I_2 = \frac{1}{k_{r-1}}\sum_{m=m_0+1}^{\alpha}\sum_{n\in I_m} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) =$$

$$= \frac{1}{k_{r-1}}\sum_{m=m_0+1}^{\alpha}\left(\frac{1}{h_m}\sum_{n\in I_m} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right)\right) h_m \leq$$

$$\leq \varepsilon\frac{1}{k_{r-1}}\sum_{m=m_0+1}^{\alpha} h_m \leq \varepsilon\frac{k_r}{k_{r-1}} = \varepsilon q_r < \varepsilon M .$$

Thus $I \leq (m_0 k_{m_0} / k_{\nu-1})L + \varepsilon M$. Finally, $x \in w((A,\varphi), f)$.

5. Properties and theorems

Let the sequence Θ, the modulus function f be given and let φ and ψ are two φ-functions.

Theorem 2. *Let us suppose that the matrix A satisfies the condition*

$$a_{n1} + a_{n2} + \ldots \leq K \text{ for } n = 1, 2, \ldots$$

and let φ-functions φ and ψ satisfy the condition (Δ_2) for large u.

(a) If $\psi \overset{l}{\prec} \varphi$ then $T_\Theta((A,\varphi), f) \subset T_\Theta((A,\psi), f)$.

(b) If φ-function φ and ψ are equivalent for large u, then $T_\Theta((A,\varphi), f) = T_\Theta((A,\psi), f)$.

Proof. Let $x = (t_\nu) \in T_\Theta((A,\varphi), f)$. By assumption we have

$$\psi(|t_\nu|) \leq b\varphi(c|t_\nu|) \tag{+}$$

for b, c, $u_0 > 0$ and $|t_\nu| > u_0$. Let us denote $x = x^1 + x^2$, where $x^1 = (t_\nu^1)$ and $t_\nu^1 = t_\nu$ for $|t_\nu| < u_0$ and $t_\nu^1 = 0$ for remaining values of ν. It is easily seen that $x^1 \in T_\Theta((A,\varphi), f)$. Moreover, by the assumptions and the inequality (+) we get

$$\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\psi(|t_\nu^2|)\right|\right) \leq \frac{1}{h_r}\sum_{n\in I_r} f\left(b\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(c|t_\nu^2|)\right|\right) \leq$$
$$\leq \frac{L}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu^2|)\right|\right),$$

where the constant L is connected with properties of f and φ.

Finally, we obtain $x^2 = (t_\nu^2) \in T_\Theta((A,\psi), f)$ and in consequence $x \in T_\Theta((A,\psi), f)$.

The identity $T_\Theta((A,\varphi), f) = T_\Theta((A,\psi), f)$ is proved, analogously.

Theorem 3. *Let the φ-function $\varphi(u)$ satisfies the condition (Δ_2) and let the matrix A has the property $a_{n1} + a_{n2} + \ldots \leq K$ for $n = 1, 2, \ldots$. The following conditions are true.*

(a) If $x = (t_\nu) \in T_\Theta((A,\varphi), f)$ and α is an arbitrary number, then $\alpha x \in T_\Theta((A,\varphi), f)$.

(b) *If* $x, y = (t_\nu) \in T_\Theta((A,\varphi), f)$ *where* $x = (t_\nu)$, $y = (s_\nu)$ *and* α, β *are given numbers, then* $\alpha x + \beta y \in T_\Theta((A,\varphi), f)$.

(c) $T_\Theta((A,\varphi), f)$ *is a linear space.*

PROOF. Let $x \in T_\Theta((A,\varphi), f)$. First let us remark that for $0 < \alpha < 1$ we get

$$\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(\alpha\,|\,t_\nu\,|)\right|\right) \le \frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|\,t_\nu\,|)\right|\right).$$

Moreover, if $\alpha > 1$ then we may find a positive number s such that $\alpha < 2^s$ and we obtain

$$\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(\alpha\,|\,t_\nu\,|)\right|\right) \le \frac{1}{h_r}\sum_{n\in I_r} f\left(d^s\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|\,t_\nu\,|)\right|\right) \le$$

$$\le \frac{L}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|\,t_\nu\,|)\right|\right),$$

where d and L are constants connected with the properties of φ and f. Hence we obtain the condition (a).

In the following let the numbers α, β and the elements $x, y \in T_\Theta((A,\varphi), f)$ be given. From the part (a) it follows that the following inequality is true

$$\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|\,\alpha t_\nu + \beta s_\nu\,|)\right|\right) \le L_1\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|\,t_\nu\,|)\right|\right) +$$

$$+ L_2\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|\,s_\nu\,|)\right|\right),$$

where the constants L_1 and L_2 are defined as in (a). In consequence $x + y \in T_\Theta((A,\varphi), f)$.

REMARK. Let us remark that the modulus function f is continuous in the interval $[0,\infty)$. Moreover, it is easily verified that by the assumptions of matrix A, the sums of elements in n-th row of the matrix A

$$S^n_{pq} = a_{n,p} + a_{n,p+1} + \ldots + a_{n,p+q-1} \quad \textit{and} \quad \sum_{n\in I_r} f(S^n_{pq})$$

are bounded and tend to zero as $n \to \infty$, (compare [13], [19]).

In consequence we have $e,\ e^q,\ e^q_p \in T_\Theta((A,\varphi),f)$.

THEOREM 5. $T_\Theta((A,\varphi)) \subseteq T_\Theta((A,\varphi),f)$.

PROOF. Let $x \in T_\Theta((A,\varphi))$. For a given $\varepsilon > 0$ we choose $0 < \delta < 1$ such that $f(v) < \varepsilon$ for every $v \in [0,\delta]$. We can write

$$\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) = S_1 + S_2,$$

where $S_1 = \frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right)$ and this sum is taken over $\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|) \le \delta$ and $S_2 = \frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right)$ and this sum is taken over $\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|) > \delta$.

By definition of the modulus f we have $S_1 = \frac{1}{h_r}\sum_{n\in I_r} f(\delta) = f(\delta) < \varepsilon$ and moreover $S_2 = f(1)\frac{1}{\delta}\frac{1}{h_r}\sum_{n\in I_r}\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)$. Finally, we get $x \in T_\Theta((A,\varphi),f)$.

6. SOME REMARKS ON LACUNARY (A,φ) – STATISTICAL CONVERGENCE

Let Θ be a lacunary sequence, and let the matrix $A = (a_{n\nu})$, the sequence $x = (t_\nu)$, the φ-function $\varphi(u)$ and a positive number ε be given. We adopt the following notation

$$K^{\gamma}_{\Theta}((A,\varphi),\varepsilon) = \left\{ n \in I_r \ :\ \sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|) \ge \varepsilon \right\}.$$

The sequence x is said to be lacunary (A,φ) – statistically convergent to a number zero if for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{k_r}\mu(K^{r}_{\Theta}((A,\varphi),\varepsilon)) = 0,$$

where $\mu(K^r_\Theta((A,\varphi),\varepsilon))$ denotes the number of elements belonging to $K^r_\Theta((A,\varphi),\varepsilon)$. The set of all lacunary (A,φ) – statistical convergent sequebces is denoted by $S_\Theta((A,\varphi))$,

$$S_\Theta((A,\varphi)) = \left\{ x=(t_\nu) : \lim_{r\to\infty} \frac{1}{h_r} \mu(K^r_\Theta((A,\varphi),\varepsilon)) = 0 \right\},$$

(compare [2], [4], [5], [6], and [17]).

THEOREM 6. *If* $\psi \prec \varphi$ then $S_\Theta((A,\psi)) \subset S_\Theta((A,\varphi))$.

PROOF. By assumptions we have $\psi(|t_\nu|) \le b\varphi(c|t_\nu|)$ and we have

$$\sum_{\nu=1}^{\infty} a_{n\nu}\psi(|t_\nu|) \le b\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(c|t_\nu|) \le L\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\ ,$$

for $b,c>0$, where the constant L is connected with properties of φ. Thus, the condition $\sum_{\nu=1}^{\infty} a_{n\nu}\psi(|t_\nu|) \ge \varepsilon$ implies the condition $\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|) \ge \varepsilon$ and in consequence we obtain

$$\mu(K^r_\Theta((A,\varphi),\varepsilon)) \le \mu(K^r_\Theta((A,\psi),\varepsilon))$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \mu\left(K^r_\Theta((A,\varphi),\varepsilon)) \le \lim_{r\to\infty} \frac{1}{h_r} \mu(K^r_\Theta((A,\psi),\varepsilon)) \right).$$

THEOREM 7. *If* $\psi \sim \varphi$ *then* $S_\Theta((A,\varphi)) = S_\Theta((A,\psi))$.

THEOREM 8.

(a) *If the matrix A, the sequebce* Θ *and functions f and* φ *be given, then*

$$T_\Theta((A,\varphi),f) \subset S_\Theta((A,\varphi)).$$

(b) *If the* φ*-function* $\varphi(u)$ *and the matrix A are given, and if the modulus function f is bounded, then*

$$S_\Theta((A,\varphi)) \subset T_\Theta((A,\varphi),f).$$

(c) *If the* φ*-function* $\varphi(u)$ *and the matrix A are given, and if the modulus function f is bounded, then*

$$S_\Theta((A,\varphi)) = T_\Theta((A,\varphi),f).$$

PROOF. (a) Let f be a modulus function and let ε be a positive number. We have the following inequalities

$$\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) \geq \frac{1}{h_r}\sum_{n\in I_r^1} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) \geq \frac{1}{h_r} f(\varepsilon)\sum_{n\in I_r^1} 1 \geq$$

$$\geq \frac{1}{h_r} f(\varepsilon)\mu(K_\Theta^r((A,\varphi),\varepsilon),$$

where $I_r^1 = \left\{ n\in I_r : \left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right| \geq \varepsilon \right\}$. Finally, if $x\in T_\Theta((A,\varphi),f)$ then $x\in S_\Theta((A,\varphi))$.

PROOF. (b) Let us suppose that $x\in S_\Theta((A,\varphi))$. If the modulus function f is a bounded function, then there exists an integer L such that $f(v)\leq L$ for all $v\geq 0$. In the following let

$$I_r^2 = \left\{ n\in I_r : \left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right| < \varepsilon \right\}.$$

Thus, we have

$$\frac{1}{h_r}\sum_{n\in I_r} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) \leq \frac{1}{h_r}\sum_{n\in I_r^1} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) +$$

$$+\frac{1}{h_r}\sum_{n\in I_r^2} f\left(\left|\sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)\right|\right) \leq \frac{1}{h_r} L\mu(K_\Theta^r((A,\varphi),\varepsilon)) + f(\varepsilon).$$

Taking the limit as $\varepsilon\to 0$, we obtain that $x\in T_\Theta((A,\varphi),f)$.

PROOF of the part (c) follows from (a) and (b).

THEOREM 9. *Let us suppose that the matrix A is regular and that the modulus function f is bounded. Then the condition $x\in T_0$ implies $x\in S_\Theta((A,\varphi))$.*

PROOF. If $t_\nu\to 0$ as $\nu\to\infty$ then be regularity of A and by the definition of statistical (A,φ) – convergence we have

$$\lim_{n\to\infty} \sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|) = 0.$$

Thus

$$\lim_{r\to\infty} \frac{1}{h_r} \mu(K_\Theta^r((A,\varphi),\varepsilon)) = 0.$$

Finally, we obtain $x \in T_\Theta((A,\varphi), f) \subset S_\Theta((A,\varphi))$.

REFERENCES

[1] T. Bilgin, Lacunary strong A-convergence with respact to a modulus, (preprint).

[2] J.S. Connor, On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull.* 32(1989), 194-198.

[3] A. Esi, The A-statyistyical and strongly (A-p)-Cesaro convergence of sequences, *Pure and Appl. Math. Sci.* 43(1996), 89-93.

[4] A.R. Freedman, J.J. Sember, M. Raphel, Some Cesaro-type summability spaces, *Proc. London Math. Soc.* 37(1978), 508-520.

[5] J. Frydy, On statistical convergence, *Analysis* 5(1985), 301-313.

[6] J. Frydy, C. Orhan, Lakunary statistical convergence, *Pacific J. Math.* 160(1993), 45-51.

[7] A. Kufner, O. John, S. Fučik, *Function spaces*, Prague 1977.

[8] I.J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Camb. Phil. Soc.* 100(1986), 161-166.

[9] J. Musielak, On some modular spaces connected with strong summability, *Math. Student* 27(1959), 129-136.

[10] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. 1034, Springer Verlag, Berlin-Heidelberg-New York-Tokyo 1983.

[11] J. Musielak, W. Orlicz, On modular spaces, *Studia Math.* 18(1959), 49-65.

[12] J. Musielak, W. Orlicz, On modular spaces of strongly summable sequences, *Studia Math.* 22(1962), 127-144.

[13] J. Musielak, A.Waszak, On a property of some methods of summability, *Publ. Elektr. Facult. Univ. u Beogradu* 280(1969), 27-32.

[14] H. Nakano, Concave modulars, *J. Math. Soc. Japan* 5(1953), 29-49.

[15] W. Orlicz, On some spaces of strongly summable sequences, *Studia Math.* 22(1963), 331-336.

[16] S. Pehlivan, B. Fisher, On some sequence spaces, *Indian J. Pure Appl. Math.* 25(1994), 1067-1071.

[17] T. Salat, On statistically convergent sequences of real numbers, *Math. Slovaca* 2(1980), 139-150.

[18] A. Waszak, On spaces of strongly summable sequences with an Orlicz metric, *Comment. Math.* 11(1968), 229-246.

[19] A. Waszak, Orlicz spaces connected with strng summability II, *Fasc. Math.* 5(1970), 85-91.

(Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland)

Received on 12.02.2001 and, in revised form, on 30.05.2001.