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# THE RATE OF CONVERGENCE BY A NEW TYPE OF MEYER-KÖNIG AND ZELLER OPERATORS

ABSTRACT: In the present paper we introduce a simple integral modification of Meyer-König and Zeller operators and study their rate of convergence, for functions of bounded variation.

KEY WORDS: approximation by positive operators, rate of convergence, degree of approximation, functions of bounded variation, total variation.

## 1. INTRODUCTION

The Meyer-König and Zeller operators (see [1]) associate to each function $f$ bounded on $[0,1]$ the power-series

$$M_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x)\, f\left(\frac{k}{n+k}\right), \quad x\in[0,1), \quad n\in N,$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n.$$

In order to approximate Lebesgue integrable functions on $[0,1]$, we define the integral modification of the operators $M_n$ as follows

$$\tilde{M}_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 b_{n,k}(t)\, f(t)\,dt, \quad x\in[0,1], \quad n\in N, \tag{1}$$

where

$$b_{n,k}(t) = \frac{(n+k)!}{k!(n-1)!} t^k (1-t)^{n-1}.$$

Guo [3], [4] estimated the rate of convergence of some other integral modification of the operators $M_n$. Love [2] et al., Gupta and Ahmad [6] and Gupta [5] gave sharp estimates over the results of Guo [4], [3]. Recently, Zeng [7] also gave the exact bound for the term $p_{n,k}(x)$ which improves the results of Guo [3], [4] and Gupta [5].

Actually the operators (1) represent a new type of integral modification of Meyer-König and Zeller operators. The main purpose to introduce these operators is that some approximation formulas for $\tilde{M}_n$ become simpler than the corresponding results for other modifications considered in [2], [3], [4], [5] and [7], In the present paper we study the behaviour of $\tilde{M}_n$ for functions of bounded variation arid give an estimate of the rate of convergence of $\tilde{M}_n$ $(n \in N)$.

## 2. AUXILIARY RESULTS

In order to prove our main result we shall need the following lemmas.

**LEMMA 1** [7, Theorem 2]. *For all $n, k \in N$, and $x \in (0,1]$, we have*

$$p_{n,k}(x) < \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{nx}},$$

*where the constant $1/\sqrt{2e}$ is the best possible.*

**LEMMA 2.** *For all $n, k \in N$, and $x \in [0,1]$, we have*

$$\int_x^1 b_{n,k}(t)\,dt = \sum_{j=0}^{k} p_{n,j}(x).$$

*The proof is left to the reader.*

**LEMMA 3.** *For all $n, k \in N$, and $x \in [0,1]$ $n \geq 3$, the second central moment of the operators $\tilde{M}_n$ satisfies the estimate*

$$\tilde{M}_n((t-x)^2; x) \leq \frac{6x(1-x)}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}.$$

**PROOF.** First, we note

$$\sum_{k=0}^{\infty} p_{n,k}(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1-x)^n = 1, \text{ for all } x \in (0,1).$$

Now

$$\tilde{M}_n((t-x)^2; x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 b_{n,k}(t)(t^2 - 2tx + x^2)\,dt =$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 \frac{(n+k)!}{k!(n-1)!} t^{k+2}(1-t)^{n-1} dt -$$

$$- 2x \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 \frac{(n+k)!}{k!(n-1)!} t^{k+1}(1-t)^{n-1} dt + x^2 =$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \frac{(k+2)(k+1)}{(n+k+2)(n+k+1)} - 2x \sum_{k=0}^{\infty} p_{n,k}(x) \frac{k+1}{n+k+1} + x^2 =$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \frac{[k(k-1)+4k+2]}{(n+k+2)(n+k+1)} + 2x \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n}{n+k+1} 2x + x^2 =$$

$$= S_1 + S_2 - 2x + x^2,$$

where

$$S_1 = \sum_{k=0}^{\infty} p_{n,k}(x) \frac{[k(k-1)+4k+2]}{(n+k+2)(n+k+1)} \le$$

$$\le (1-x)^n \sum_{k=0}^{\infty} \frac{(n+k-1)!}{k!(n-1)!} x^2 \frac{[k(k-1)+4k+2]}{(n+k-1)(n+k-2)} \le$$

$$\le (1-x)^n \left\{ \sum_{k=0}^{\infty} \frac{(n+k-3)!}{(k-2)!(n-1)!} x^k + 4 \sum_{k=0}^{\infty} \frac{(n+k-3)!}{(k-1)!(n-1)!} x^k + \right.$$

$$\left. + 2 \sum_{k=0}^{\infty} \frac{(n+k-3)!}{k!(n-1)!} x^k \right\} \le$$

$$\le (1-x)^n \left\{ x^2 \sum_{k=0}^{\infty} \frac{(n+k-1)!}{k!(n-1)!} x^k + \frac{4x}{n-1} \sum_{k=0}^{\infty} \frac{(n+k-2)!}{k!(n-2)!} x^k + \right.$$

$$\left. + \frac{2}{(n-1)(n-2)} \sum_{k=0}^{\infty} \frac{(n+k-3)!}{k!(n-3)!} x^k \right\} =$$

$$= x^2 + \frac{4x(1-x)}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}$$

and

$$S_2 = 2x \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1-x)^n \frac{n}{n+k+1} \le$$

$$\leq 2nx\sum_{k=0}^{\infty}\frac{(n+k-2)!}{k!(n-1)!}x^k(1-x)^n =$$

$$= \frac{2nx}{n-1}\sum_{k=0}^{\infty}\binom{n+k-2}{k}x^k(1-x)^n = \frac{2nx(1-x)}{n-1}.$$

Collecting the above estimates, we obtain the required result.

Note that Lemma 3 implies that, given any $\lambda > 6$ and any $x \in (0,1)$, there is an integer $N(\lambda,x)$ such that, for all $n \in N(\lambda,x)$,

$$\tilde{M}_n((t-x)^2;x) \leq \frac{\lambda x(1-x)}{n}.$$

Throughout the paper let

$$a_n(x,t) = \sum_{k=0}^{\infty} p_{n,k}(x)b_{n,k}(t)$$

and

$$A_n(x,t) = \int_0^t a_n(x,u)du.$$

With this definition, for each function $f$ bounded on $[0,1]$, there holds

$$\tilde{M}_n(f;x) = \int_0^1 a_n(x,t)f(t)dt. \tag{2}$$

Note that, in particular,

$$A_n(x,1) = \int_0^1 a_n(x,u)du = 1.$$

As a corollary of Lemma 3 we get

**Lemma 4.** *Let* $x \in (0,1)$. *For each* $\lambda > 6$, *there is an integer* $N(\lambda,x)$ *such that, for all* $n \in N(\lambda,x)$,

(i) $$A_n(x,y) = \int_0^y a_n(x,t)dt \leq \frac{\lambda x(1-x)}{n(x-y)^2} \qquad (0 \leq y < x),$$

(ii) $$1 - A_n(x,z) = \int_z^1 a_n(x,t)dt \leq \frac{\lambda x(1-x)}{n(z-x)^2} \qquad (x < z \leq 1).$$

## 3. MAIN RESULT

In this section we state and prove as our main result the following theorem,

**THEOREM 5.** *Let f be a function of bounded variation on* $[0,1]$. *Then, for each* $x\in(0,1)$, $K>12.25$ *there exists a constant N independent of f and n, such that for all* $n>N$, *the operators* (1) *satisfy the estimate*

$$\left|\tilde{M}_n(f;x)-\frac{1}{2}\{f(x+)+f(x-)\}\right|\leq \tag{3}$$

$$\leq \frac{1}{\sqrt{8enx}}|f(x+)-f(x-)|+\frac{K}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x),$$

*where*

$$g_x(t)=\begin{cases} f(t)-f(x-) & (0\leq t<x),\\ 0 & (t=x),\\ f(t)-f(x+) & (x<t\leq 1),\end{cases}$$

*and* $\bigvee_a^b(g_x)$ *is the total variation of* $g_x$ *on* $[a,b]$.

**PROOF.** Taking advantage of the identity

$$f(t)=g_x(t)+\frac{1}{2}\{f(x+)+f(x-)\}+\frac{1}{2}\operatorname{sign}(t-x)\cdot\{f(x+)-f(x-)\}\quad (t\neq x)$$

we obtain

$$\left|\tilde{M}_n(t;x)-\frac{1}{2}\{f(x+)+f(x-)\}\right|\leq \tag{4}$$

$$\leq\left|\tilde{M}_n(g_x;x)\right|+\frac{1}{2}|f(x+)-f(x-)|\cdot\left|\tilde{M}_n(\operatorname{sign}(t-x);x)\right|.$$

In order to show (4) we need estimates for $\tilde{M}_n(g_x;x)$ and $\tilde{M}_n(\operatorname{sign}(t-x);x)$. We have

$$\tilde{M}_n(\operatorname{sign}(t-x);x)=\int_0^1\operatorname{sign}(t-x)a_n(x,t)dt= \tag{5}$$

$$=\int_x^1 a_n(x,t)dt-\int_0^x a_n(x,t)dt=2Q_n(x)-1$$

with $Q_n(x):=\int_x^1 a_n(x,t)dt$, since $\int_0^x a_n(x,t)dt=1$. Application of Lemma 2 yields

$$Q_n(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \sum_{j=0}^{k} p_{n,j}(x).$$

Using $\sum_{k=0}^{\infty} p_{n,k}(x) \equiv 1$, we conclude

$$Q_n(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \left\{ 1 + p_{n,k}(x) - \sum_{j=k}^{\infty} p_{n,j}(x) \right\} =$$

$$= 1 + \sum_{k=0}^{\infty} p_{n,k}^2(x) - \sum_{k=0}^{\infty} p_{n,k}(x) \sum_{j=k}^{\infty} p_{n,j}(x) =$$

$$= 1 + \sum_{k=0}^{\infty} p_{n,k}^2(x) - Q_n(x).$$

Therefore

$$2Q_n(x) - 1 = \sum_{k=0}^{\infty} p_{n,k}^2(x).$$

Combining this identity with Eq. (5) we observe, by application of Lemma 1,

$$\text{(6)} \qquad \left| \tilde{M}_n(\operatorname{sign}(t-x); x) \right| \le \frac{1}{\sqrt{2enx}} \sum_{k=0}^{\infty} p_{n,k}(x) = \frac{1}{\sqrt{2enx}}.$$

Now, we consider $\tilde{M}_n(g_x; x)$.

$$\tilde{M}_n(g_x; x) = \int_0^1 a_n(x,t) g_x(t) dt =$$

$$= \left( \int_0^{x - x/\sqrt{n}} + \int_{x - x/\sqrt{n}}^{x - (1-x)/\sqrt{n}} + \int_{x + (1-x)/\sqrt{n}}^{1} \right) a_n(x,t)\, g_x(t)\, dt =$$

$$= I_1 + I_2 + I_3,$$

say. For the sake of brevity, put $y = x - x/\sqrt{n}$. Using integration by parts, we obtain

$$I_1 = \int_0^y a_n(x,t) g_x(t) dt = \int_0^y g_x(t)\, d_t A_n(x,t) =$$

$$= g_x(y) A_n(x,y) - \int_0^y A_n(x,t) d_t\, g_x(t).$$

Since $|g_x(y)| \le \mathrm{V}_y^x(g_x)$, we have, by Lemma 4, for each $\lambda > 6$ and all $n \ge N(\lambda, x)$,

$$|I_1| \le \bigvee_y^x (g_x) A_n(x,y) + \int_0^y A_n(x,t)\, d_t\left(-\bigvee_t^x (g_x)\right) \le$$

$$\le \bigvee_y^x (g_x) \frac{\lambda x(1-x)}{n(x-y)^2} + \frac{\lambda x(1-x)}{n} \int_0^y \frac{1}{(x-t)^2} d_t\left(-\bigvee_t^x (g_x)\right).$$

Integrating by parts yields

$$\int_0^y \frac{1}{(x-t)^2} d_t\left(-\bigvee_t^x (g_x)\right) =$$

$$= \frac{-\mathrm{V}_y^x(g_x)}{(x-y)^2} + \frac{\mathrm{V}_0^x(g_x)}{x^2} + 2\int_0^y \frac{\mathrm{V}_t^x(g_x)}{(x-t)^3} dt,$$

and, therefore, we have

$$|I_1| \le \bigvee_y^x (g_x) \frac{\lambda x(1-x)}{n(x-y)^2} +$$

$$+ \frac{\lambda x(1-x)}{n}\left(\frac{-\mathrm{V}_y^x(g_x)}{(x-y)^2} + \frac{\mathrm{V}_0^x(g_x)}{x^2} + 2\int_0^y \frac{\mathrm{V}_t^x(g_x)}{(x-t)^3} dt\right) =$$

$$= \frac{\lambda x(1-x)}{n}\left(\frac{\mathrm{V}_0^x(g_x)}{x^2} + 2\int_0^y \frac{\mathrm{V}_t^x(g_x)}{(x-t)^3} dt\right).$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we obtain

$$\int_0^{x-x/\sqrt{n}} \bigvee_t^x (g_x)(x-t)^{-3} dt = \sum_{k=1}^{n-1} \int_{x/\sqrt{k+1}}^{x/\sqrt{k}} \bigvee_{x-t}^x (g_x) t^{-3} dt \le$$

$$\le \frac{1}{2x^2} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^x (g_x).$$

Hence

$$|I_1| \le \frac{2\lambda(1-x)}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^x (g_x). \tag{7}$$

Next, we estimate $I_2$. For $t \in (x - x/\sqrt{n},\ x + (1-x)/\sqrt{n})$, there holds

$$|g_x(t)| = |g_x(t) - g_x(x)| \le \bigvee_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x).$$

Since $\int_a^b d_t A_n(x,t) \le 1$, for all intervals $(a,b) \subset [0,1]$, we have

$$(8) \qquad |I_2| \le \bigvee_{x-x/\sqrt{n}}^{x-(1-x)/\sqrt{n}} (g_x) \le \frac{1}{n}\sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x).$$

Using a similar method as in the estimate of $I_1$ and Lemma 4, we conclude

$$(9) \qquad |I_3| \le \frac{2\lambda x}{n(1-x)}\sum_{k=1}^{n} \bigvee_{x}^{x+(1-x)/\sqrt{k}} (g_x).$$

Collecting the estimates (7)–(9), we obtain

$$(10) \quad \left|\tilde{M}_n(g_x;x)\right| \le \frac{2\lambda}{nx(1-x)}\sum_{k=1}^{n}\left[(1-x)^2 \bigvee_{x-x/\sqrt{k}}^{x}(g_x) + x^2 \bigvee_{x}^{x+(1-x)/\sqrt{k}}(g_x)\right] + $$

$$+\frac{1}{n}\sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x) \le \frac{2\lambda + 0.25}{nx(1-x)}\sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x),$$

since z $x(1-x) \le 1/4$, for $0 \le x \le 1$. Finally, combining (4), (6), (10), we obtain (3). This completes the proof of the theorem.

**REMARK 1.** We mention that there are some misprints in the paper of Zeng [7].

(i) For the integrated Meyer-König and Zeller operators $\hat{M}_n$ it should be read as follows:

$$(11) \qquad \hat{M}_n = (n+1)\sum_{k=0}^{\infty} \hat{M}_{n,k}(x) \int_{k/(n+k)}^{(k+1)/(n+k+1)} f(t)\,dt,$$

where

$$\hat{M}_{n,k}(x) = \binom{n+k+1}{k} x^k (1-x)^n.$$

(ii) In Theorem 4.3 of [7] the second term on the right hand side should be

$$\frac{3}{\sqrt{8e}\sqrt{n}x^{3/2}}\,|f(x+) - f(x-)|.$$

**REMARK 2.** The main advantage in introducing the operators (1) is that in the analysis we do not need results of the type [3, Lemma 5] and [6, Lemma 2.6].

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