FASCICULI MATHEMATICI

Nr 34 2004

DAQING JIANG[1], XIAOJIE XU[1], DONAL O'REGAN[2] AND RAVI P. AGARWAL[3]

MULTIPLE POSITIVE SOLUTIONS TO SEMIPOSITONE DIRICHLET BOUNDARY VALUE PROBLEMS WITH SINGULAR DEPENDENT NONLINEARITIES*

ABSTRACT: In this paper we establish the existence of multiple positive solutions to semipositone Dirichlet boundary value problem

$$\begin{cases} y'' + \mu q(t) f(y(t)) = 0, & 0 < t < 1, \\ y(0) = 0, & y(1) = 0 \end{cases}$$

by using the upper and lower solutions method with the existence theory in [1], where $\mu > 0$ is a constant. Here our nonlinearity f may be singular at $y = 0$.

KEY WORDS: multiple positive solutions, singular boundary value problem, semipositone problem, upper and lower solution method.

1. INTRODUCTION

This paper discusses existance of multiple solution for semipositione singular boundary value problems. In particular our nonlinear term $f(y)$ may be singular at $y = 0$, and f may take on negative values. Problems of this type are referred to as semipositone problems in the literature. Almost all papers in the literature [2, 3, 6, 7, 10] are devoted to the study of singular and nonsingular positone problems (i.e. problems where f takes nonnegative values), and only recently (see for example [4, 9]) have papers appeared which discuss the semipositone nonsingular problem. Very recently,R.P. Agarwal and D. O'Regan [1] discussed the semipositone singular problem where a very general existence theory is presented and for example it is shown that the boundary value problem

$$(1.1) \qquad \begin{cases} y'' + \mu(y^{-\alpha} + y^{\beta} - 1) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0, \quad \alpha > 0, \quad \beta > 1, & \mu > 0 \text{ small}, \end{cases}$$

has a positive solution $y \in C[0,1] \cap C^2(0,1)$ with $y(t) > 0$ for $t \in (0,1)$. Existence is established in [1] by using a general cone fixed point theorem in [2, 5]. However no paper to date has discussed multiplicity for the semipositone singular problem. This paper attempts to fill this gap in the literature.

Some general existence theorems will be presented in Section 3 and there we

* The work was supported by NNSF of China.

will show, for example, that the boundary value problem (1.1) has two positive solutions under the same conditions as in [1]. Existence in this paper will be established using the upper and lawer solution method with the existence result established in [1].

In this paper we only consider Dirichlet boundary data. It is worth remarking here that we could consider Sturm Liouville boundary data also; however since the arguments are essentially the same (in fact easier if not Dirichlet data) we will leave the details to the reader.

2. Some preliminary results

In this section we present some results from literature which will be needed in Section 3.

Lemma 2.1.[2] *If* $y\in C[0,1]\cap C^2(0,1)$ *is a nonnegative concave function on* $[0,1]$, *then*

$$y(t) \geq t(1-t)|y|_0 \quad for \quad t\in[0,1];$$

here $|y|_0=\sup_{t\in[0,1]}|y(t)|$.

Lemma 2.2.[1] *Suppose* $q\in L^1[0,1]$ *with* $q>0$ *on* $(0,1)$. *Then the boundary value problem*

$$\begin{cases} y''+q(t)=0, & 0<t<1, \\ y(0)=0, & y(1)=0; \end{cases}$$

has a solution w with

$$w(t) \leq t(1-t)C_0 \quad for \quad t\in[0,1];$$

here

$$C_0 = \max_{t\in[0,1]}\left\{\frac{1}{1-t}\int_t^1 (1-x)q(x)dx+\frac{1}{t}\int_0^t xq(x)dx\right\}.$$

Finally in this section we state the existence result established in [1] for the problem

$$\begin{cases} y''+\mu q(t)f(y(t))=0, & 0<t<1, \\ y(0)=0, & y(1)=0; \end{cases} \tag{2.1}$$

here $\mu>0$ is a constant.

Theorem 2.1. *Suppose the following conditions are satisfied*

$$(2.2)\qquad \begin{cases} f:(0,\infty)\to R \text{ is continuous and there exists} \\ \text{a constant } M>0 \text{ with } f(u)+M\geq 0 \\ \text{for } u\in(0,\infty) \end{cases}$$

$$(2.3)\qquad \begin{cases} f(u)+M = g(u)+h(u) \text{ on } (0,\infty) \text{ with } g>0 \\ \text{constinuous and nonincreasing on } (0,\infty),\, h\geq 0 \\ \text{continuous on } [0,\infty) \text{ and } \frac{h}{g} \text{ nondecreasin g on } (0,\infty) \end{cases}$$

$$(2.4)\qquad y\in C[0,1]\cap L^1[0,1] \text{ with } q>0 \text{ on } (0,1)$$

$$(2.5)\qquad \exists K_0 \text{ with } g(ab)\leq K_0 g(a)g(b)\ \forall a>0,\ b>0$$

$$(2.6)\qquad \begin{cases} \exists r>\mu M C_0 \text{ with } \dfrac{1}{g\left(1-\frac{\mu M C_0}{r}\right)\left\{1+\frac{h(r)}{g(r)}\right\}}\displaystyle\int_0^r \frac{du}{g(u)} > \mu K_0 b_0\,; \\ \text{here } b_0 = \max\left\{2\displaystyle\int_0^{1/2} t(1-t)q(t)dt,\ 2\int_{1/2}^1 t(1-t)q(t)dt\right\} \end{cases}$$

$$(2.7)\qquad \begin{cases} \text{there exists } a\in(0,\tfrac{1}{2}) \text{ (choose and fix it) and } \exists R>r \text{ with} \\ \dfrac{R\,g(ea(1-a)R)}{g(R)a(ea(1-a)R)+g(R)h(ea(1-a)R)} \leq \mu \displaystyle\int_a^{1-a} q(s)G(\sigma,s)ds; \end{cases}$$

here $e>0$ *is any constant (choose and fix it) so that* $1-\frac{\mu M C_0}{R}\geq e$ *(note e exists since* $R>r>\mu M C_0$*) and* $G(t,s)$ *is the Green's function for*

$$\begin{cases} y''=0 \text{ on } (0,1) \\ y(0)=y(1)=0, \end{cases}$$

and $0\leq\sigma\leq 1$ *is such that*

$$\int_a^{1-a} q(s)G(\sigma,s)ds = \sup_{t\in[0,1]}\int_a^{1-a} q(s)G(t,s)ds.$$

Then (2.1) *has a solution* $y\in C[0,1]\cap C^2(0,1)$ *with* $y(t)>0$ *for* $t\in(0,1)$, $r<|y+\phi|_0\leq R$, *here* $\phi(t)=\mu M w(t)$ *(w is as in Lemma* 2.2*).*

3. SEMIPOSITIONE SINGULAR PROBLEM

In this section we examine the singular Dirichlet problem

$$(3.1)\qquad \begin{cases} y''+\mu q(t)f(y(t))=0, & 0<t<1, \\ y(0)=0, & y(1)=0; \end{cases}$$

here $\mu>0$ is a constant.

THEOREM 3.1. *Suppose the conditions* (2.2)-(2.6) *hold. In addition assume that* (*H*) *there exist constants* $L > M$ *and* $\varepsilon_0 > 0$ *such that*

$$g(u) > L, \quad \text{for all } 0 < u < \varepsilon_0 .$$

Then (3.1) *has a solution* $y \in C[0,1] \cap C^2(0,1)$ *with* $y(t) > 0$ *for* $t \in (0,1)$, $|y+\phi|_0 < r$, *here* $\phi(t) = \mu M w(t)$ (*w is as in Lemma* 2.2).

PROOF. Choose $\delta > 0$ and $\delta < r$ with

$$(3.2) \qquad \frac{1}{g\left(1-\frac{\mu M C_0}{r}\right)\left\{1+\frac{h(r)}{g(r)}\right\}} \int_\delta^r \frac{du}{g(u)} > \mu K_0 b_0 .$$

Let $m_0 \in \{1,2,\ldots\}$ be chosen so that $\frac{1}{m_0} < \min\{\frac{\delta}{2}, \varepsilon_0\}$, $N_0 = \{m_0, m_0+1, \ldots\}$.

To show (3.1) has a nonnegative solution $y \in C[0,1] \cap C^2(0,1)$ with $|y+\phi|_0 < r$, we will show

$$(3.3) \qquad \begin{cases} y'' + \mu q(t) f^*(y(t) - \phi(t)) = 0, & 0 < t < 1, \\ y(0) = 0, & y(1) = 0; \end{cases}$$

has a solution $y \in C[0,1] \cap C^2(0,1)$ with $y_1(t) > \phi(t)$ for $t \in (0,1)$, and $|y_1|_0 < r$; here

$$f^*(v) = f(v) + M = g(v) + h(v), \quad v > 0.$$

If this is true, then $u(t) - y_1(t) - \phi(t)$ is a nonnegative solution positive on $(0,1)$ of (3.1) and $|u+\phi|_0 < r$, since

$$\begin{aligned} u''(t) = y_1''(t) - \phi''(t) &= -\mu q(t) f^*(y_1(t) - \phi(t)) + \mu M q(t) = \\ &= -\mu q(t)[f(y_1(t) - \phi(t)) + M] + \mu M q(t) = \\ &= -\mu q(t) f(y_1(t) - \phi(t)) = -\mu q(t) f(u(t)), \end{aligned}$$

for $t \in (0,1)$. As a result we will concentrate our study on (3.3)

The idea is to first show that

$$(3.4)^m \qquad \begin{cases} y'' + \mu q(t) f_m^*(y(t) - \phi(t)) = 0, & 0 < t < 1, \\ y(0) = \frac{1}{m}, \quad y(1) = \frac{1}{m}, & m \in N_0, \end{cases}$$

has a solution y_m for each $m \in N_0$ with $y_m(t) \geq \frac{1}{m}$, $y_m(t) \geq \phi(t)$ for $t \in [0,1]$ and $|y_m|_0 < r$; here $f_m^*(v) = g_m^*(v) + h(v)$, and

$$g_m^*(v) = \begin{cases} g(v), & v \geq \frac{1}{m}, \\ g(\frac{1}{m}), & 0 \leq v \leq \frac{1}{m}. \end{cases}$$

We have the following claim

CLAIM 1. $\alpha_m(t) = \frac{1}{m} + lw(t) + \phi(t) = \frac{1}{m} + (l + \mu M)w(t)$ *is a (strict) lower solution for problem* $(3.4)^m$, *here* $0 < l < \min\{\mu(L-M), (\varepsilon_0 - \frac{1}{m_0})/|w|_0\}$, $m \in N_0$.

PROOF. Notice $\alpha_m(0) = \alpha_m(1) = \frac{1}{m}$, and

$$\begin{aligned} \alpha_m''(t) + \mu q(t) f_m^*(\alpha_m(t) - \phi(t)) &= lw''(t) + \phi''(t) + \mu q(t) f_m^*(lw(t) + \tfrac{1}{m}) = \\ &= -lq(t) - \mu M q(t) + \mu q(t)[g(lw(t) + \tfrac{1}{m}) + h(lw(t) + \tfrac{1}{m})] = \\ &= \mu q(t)[g(lw(t) + \tfrac{1}{m}) + h(lw(t) + \tfrac{1}{m}) - M - \tfrac{l}{\mu}] \geq \\ &\geq \mu q(t)[g(lw(t) + \tfrac{1}{m}) - M - \tfrac{l}{\mu}] \geq \\ &\geq \mu q(t)[L - M - \tfrac{l}{\mu}] > 0, \qquad 0 < t < 1, \end{aligned}$$

since $lw(t) + \frac{1}{m} \leq l|w|_0 + \frac{1}{m_0} < \varepsilon_0$, and $l < \mu(L-M)$.

In order to seek upper solutions of $(3.4)^m$, we consider the following boundary value problem

$$(3.5)^m \qquad \begin{cases} y'' + \mu q(t) g_m^*(y(t) - \phi(t))(1 + \frac{h(r)}{g(r)}) = 0, & 0 < t < 1, \\ y(0) = \frac{1}{m}, \quad y(1) = \frac{1}{m}, & m \in N_0. \end{cases}$$

In the same way as in Claim 1, we can easily prove $\alpha_m(t) = \frac{1}{m} + lw(t) + \phi(t) = \frac{1}{m} + (l + \mu M)w(t)$ is also a (strict) lower solution of $(3.5)^m$.

Let $\beta_m^0 \in C[0,1] \cap C^2(0,1)$ be the unique solution of the boundary value problem

$$\begin{cases} y'' + \mu q(t) g(\alpha_m(t) - \phi(t))(1 + \frac{h(r)}{g(r)}) = 0, & 0 < t < 1, \\ y(0) = \frac{1}{m}, \quad y(1) = \frac{1}{m}, & m \in N_0, \end{cases}$$

then we have

$$\beta_m^0 = \frac{1}{m} + \mu \int_0^1 G(t,s) q(s) g(\alpha_m(s) - \phi(s)(1 + \frac{h(r)}{g(r)}) ds =$$

$$= \tfrac{1}{m} + \mu \int_0^1 G(t,s)q(s)g(\tfrac{1}{m} + l\,w(s))(1 + \tfrac{h(r)}{g(r)})ds \geq$$

$$\geq \tfrac{1}{m} + \mu\, L(1 + \tfrac{h(r)}{g(r)}) \int_0^1 G(t,s)q(s)ds =$$

$$= \tfrac{1}{m} + \mu\, L(1 + \tfrac{h(r)}{g(r)})\, w(t) \geq \tfrac{1}{m} + \mu L w(t) \geq$$

$$\geq \tfrac{1}{m} + l\,w(t) + \phi(t) = \alpha_m(t),$$

where $G(t,s)$ is the Green's function to the problem $-y'' = 0,\;\; y(0) = y(1) = 0$.

Thus we obtain that

$$(\beta_m^0(t))'' + \mu q(t) g_m^*(\beta_m^0(t) - \phi(t))(1 + \tfrac{h(r)}{g(r)}) =$$

$$= \mu q(t)(1 + \tfrac{h(r)}{g(r)})[g(\beta_m^0(t) - \phi(t)) - g(\alpha_m(t) - \phi(t))] \leq 0, \qquad 0 < t < 1,$$

so that β_m^0 is an upper solution for problem $(3.5)^m$.

If we now take $\alpha_m^0 \equiv \alpha_m$, we have that α_m^0 and β_m^0 are, respectively, lower and upper solutions for $(3.5)^m$ with $\alpha_m^0(t) \leq \beta_m^0(t)$, for all $t \in [0,1]$. So we know (from classical upper and lower solution theory) that there exists a solution $\beta_m \in C[0,1] \cap C^2(0,1)$ of $(3.5)^m$ such that

$$\alpha_m(t) = \alpha_m^0(t) \leq \beta_m(t) \leq \beta_m^0(t), \qquad \forall t \in [0,1].$$

Since g_m^* is nonincreasing on $[0,\infty)$, it is also standard to conclude that β_m is the unique solution of $(3.5)^m$.

Now we claim that $|\beta_m|_0 < r$. Suppose this is false, i.e., suppose $|\beta_m|_0 \geq r$. Since $(\beta_m)'' \leq 0$ on $(0,1)$, then $\beta_m(t) \geq t(1-t)|\beta_m|_0 \geq t(1-t)r$ (from Lemma 2.1) and there exists $\sigma_m \in (0,1)$ with $\beta_m' \geq 0$ on $(0, \sigma_m]$, $\beta_m' \leq 0$ on $[\sigma_m, 1)$ and $\beta_m(\sigma_m) = |\beta_m|_0 \geq r$. Notice also for $t \in [0,1]$ that

$$\beta_m(t) - \phi(t) = \beta_m(t)\left[1 - \frac{\mu M\, w(t)}{\beta_m(t)}\right] \geq \beta_m(t)\left[1 - \frac{\mu M\, C_0}{r}\right],$$

since $\beta_m(t) \geq t(1-t)|\beta_m|_0 \geq t(1-t)r$, and $w(t) \leq t(1-t)C_0$ for $t \in [0,1]$. Thus

$$\beta_m(t) - \phi(t) \geq \beta_m(t)\left[1 - \frac{\mu M\, C_0}{r}\right] > 0, \;\; \text{for } \; t \in (0,1), \tag{3.6}$$

since $r > \mu M C_0$. Also notice $\beta_m(t) - \phi(t) \ge \alpha_m(t) - \phi(t) \ge \frac{1}{m}$, so we have $g_m^*(\beta_m(t) - \phi(t)) = g(\beta_m(t) - \phi(t))$. Thus for $x \in (0,1)$, we have

$$-\beta_m''(x) = \mu q(x) g(\beta_m(x) - \phi(x)) \left\{1 + \frac{h(r)}{g(r)}\right\}$$

and this together with (3.6) yields

$$-\beta_m''(x) \le \mu K_0 q\left(1 - \frac{\mu M C_0}{r}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} g(\beta_m(x)) q(x) \tag{3.7}$$

for $x \in (0,1)$. Integrate from t $(t \le \sigma_m)$ to σ_m to obtain

$$\beta_m'(x) \le g(\beta_m(x)) \mu K_0 q\left(1 - \frac{\mu M C_0}{r}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \int_t^{\sigma_m} q(x)dx,$$

and then integrate from 0 to σ_m to obtain

$$\int_{\frac{1}{m}}^{r} \frac{du}{g(u)} \le \int_{\frac{1}{m}}^{\beta_m(\sigma_m)} \frac{du}{g(u)} \le \mu K_0 q\left(1 - \frac{\mu M C_0}{r}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \int_t^{\sigma_m} x q(x) dx,$$

since $\beta_m(\sigma_m) \ge r$.

Consequently,

$$\int_{\delta}^{r} \frac{du}{g(u)} \le \mu K_0 q\left(1 - \frac{\mu M C_0}{r}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \frac{1}{1 - \sigma_m} \int_0^{\sigma_m} x(1-x) q(x) dx. \tag{3.8}$$

Similarly if we integrate (3.7) from σ_m to t $(t \ge \sigma_m)$ and then from σ_m to 1 we obtain

$$\int_{\delta}^{r} \frac{du}{g(u)} \le \mu K_0 q\left(1 - \frac{\mu M C_0}{r}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \frac{1}{\sigma_m} \int_{\sigma_m}^{1} x(1-x) q(x) dx. \tag{3.9}$$

Now, (3.8) and (3.9) imply

$$\int_{\delta}^{r} \frac{du}{g(u)} \le \mu K_0 b_0 q\left(1 - \frac{\mu M C_0}{r}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \tag{3.10}$$

where b_0 is as defined in (2.6). This contradicts (3.2) and consequently $|\beta_m|_0 < r$.

Observe that

$$f_m^*(\beta_m(t) - \phi(t)) = f^*(\beta_m(t) - \phi(t)) = g(\beta_m(t) - \phi(t)) + h(\beta_m(t) - \phi(t)) =$$

$$= g(\beta_m(t)-\phi(t))\left(1+\tfrac{h(\beta_m(t)-\phi(t))}{g(\beta_m(t)-\phi(t))}\right) \le$$
$$= g(\beta_m(t)-\phi(t))\left(1+\tfrac{h(r)}{g(r)}\right), \qquad t\in(0,1).$$

So we have

$$\beta_m''(t)+\mu q(t)f_m^*(\beta_m(t)-\phi(t)) = -\mu q(t)g_m^*(\beta_m(t)-\phi(t))\{1+\tfrac{h(r)}{g(r)}\}+$$
$$+\mu q(t)f_m^*(\beta_m(t)-\phi(t)) \le 0,\ \ t\in(0,1),$$

so that β_m is an upper solution for $(3.4)^m$. This together with Claim 1, yields that α_m and β_m are, respectively, lower and upper solutions for $(3.4)^m$ with $\alpha_m(t)\le\beta_m(t)$, for all $t\in[0,1]$. So we conclude (from classical upper and lower solution theory) that $(3.4)^m$ has a solution $y_m\in C[0,1]\cap\ C^2(0,1)$ such that

$$\alpha_m(t)\ \le\ y_m(t)\ \le\ \beta_m(t), \qquad t\in[0,1].$$

Thus we have

$$|\,y_m\,|_0 < r,\quad y_m(t)-\phi(t) \ge \tfrac{1}{m} + lw(t) > lw(t), \qquad t\in[0,1].$$

Next we show

$$\{y_m\}_{m\in N_0}\ \textit{is a bounded, equicontinuous family on } [0,1].$$

Now since

$$f^*(y_m(t)-\phi(t)) = g(y_m(t)-\phi(t)) + h(y_m(t)-\phi(t))$$

for $t\in(0,1)$ we have

$$-\,y_m''(x)\ \le\ \mu q(x)g(y_m(x)-\phi(x))\left\{1+\tfrac{h(y_m(t)-\phi(x))}{g(y_m(t)-\phi(x))}\right\}.$$

As a result

$$(3.12)\qquad -\,y_m''\ \le\ \mu K_0\, g\left(1-\frac{\mu M}{\mu M+l}\right)\left\{1+\frac{h(r)}{g(r)}\right\}q(x)g(y_m(x)),$$

since $y_m(x)\ge\phi(x)+lw(x)=(\mu M+l)w(x)$, $|\,y_m\,|_0< r$ and

$$y_m(x)-\phi(x) = y_m(x)\left(1-\frac{\mu M\, w(x)}{y_m(x)}\right) \ge y_m(x)\left(1-\frac{\mu M}{\mu M+l}\right)$$

for $x\in[0,1]$. Also, as before, there exists $t_m\in(0,1)$ with $y_m'\ge 0$ on $(0,t_m)$ and $y_m'\le 0$ on $(t_m,1)$. Integrate (3.12) from t $(t<t_m)$ to t_m to obtain

$$(3.13) \qquad \frac{y_m'(t)}{g(y_m(t))} \leq \mu K_0\, g\left(1-\frac{\mu M}{\mu M+l}\right)\left\{1+\frac{h(r)}{g(r)}\right\}\int_t^{t_m} q(x)dx.$$

On the other hand integrate (3.12) from t_m to t $(t < t_m)$ to obtain

$$(3.14) \qquad \frac{y_m'(t)}{g(y_m(t))} \leq \mu K_0\, g\left(1-\frac{\mu M}{\mu M+l}\right)\left\{1+\frac{h(r)}{g(r)}\right\}\int_{t_m}^{t} q(x)dx.$$

We now claim that there exists a_0 and a_1 with $a_0 > 0,\ a_1 < 1,\ a_0 < a_1$ with

$$(3.15) \qquad a_0 < \inf\{t_m : m\in N_0\} \leq \sup\{t_m : m\in N_0\} < a_1.$$

REMARK 3.1. Here t_m (as before) is the unique point in (0,1) with $y_m'(t_m) = 0$. We now show $\inf\{t_m : m\in N_0\} > 0$. If this is not true then there is a subsequence S of N_0 with $t_m \to 0$ as $m\to\infty$ in S. Now integrate (3.13) from 0 to t_m to obtain

$$(3.16) \qquad \int_0^{y_m(t_m)} \frac{du}{g(u)} \leq \mu K_0\, g\left(1-\frac{\mu M}{\mu M+l}\right)\left\{1+\frac{h(r)}{g(r)}\right\}\int_0^{t_m} xq(x)dx + \int_0^{\frac{1}{m}} \frac{du}{g(u)}$$

for $m\in S$. Since $t_m \to 0$ as $m\to\infty$ in S, we have from (3.16) that $y_m(t_m)\to 0$ as $m\to\infty$ in S. However since the maximum of y_m on [0,1] occurs at t_m we have $y_m \to 0$ in $C[0,1]$ as $m\to\infty$ i S. This contradicts, the fact that $y_m(t) \geq \phi(t) + l w(t)$ for $t\in[0,1]$. Consequently $\inf\{t_m : m\in N_0\} > 0$. A similar argument shows $\sup\{t_m : m\in N_0\} < 1$. Let a_0 and a_1 be chosen as in (3.15). Now (3.13), (3.14) and (3,1.5) imply

$$(3.17) \qquad \frac{|y_m'(t)|}{g(y_m(t))} \leq \mu K_0\, g\left(1-\frac{\mu M}{\mu M+l}\right)\left\{1+\frac{h(r)}{g(r)}\right\} v(t) \quad \text{for } t\in(0,1)$$

where

$$v(t) = \int_{\min\{t,a_0\}}^{\max\{t,a_1\}} q(x)dx.$$

Notice $v\in L^1[0,1]$. Let $I:[0,\infty)\to[0,\infty)$ be defined by

$$I(z) = \int_0^z \frac{du}{g(u)}.$$

Note I is an increasing map from $[0,\infty)$ on to $[0,\infty)$ (notice $I(\infty)=\infty$ since $g>0$ is nonincreasing on $(0,\infty)$) with I continuous on $[0,A]$ for any $A>0$.

Notice

(3.18) $\{I(y_m)\}_{m\in N_0}$ *is a bounded, equicontinuous family on* $[0,1]$.

The equicontinuity follows from (here $t, s\in[0,1]$)

$$\left|I(y_m(t))-I(y_m(s))\right| = \left|\int_s^t \frac{y_m'(x)}{g(y_m(x))}dx\right| \le$$

$$\le \mu K_0 g\left(1-\frac{\mu M}{\mu M+l}\right)\left\{1+\frac{h(r)}{g(r)}\right\}\left|\int_s^t v(x)dx\right|.$$

This inequality, the uniform continuity of I^{-1} on $[0, I(r)]$, and

$$|y_m(t)-y_m(s)| = |I^{-1}(I(y_m(t)))-I^{-1}(I(y_m(s)))|$$

naw establishes (3.11).

The Arzela-Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y\in C[0,1]$ with y_m converging uniformly on $[0,1]$ to y as $m\to\infty$ through N. Also $y(0)=y(1)=0$, $\phi(t)+lw(t)\le y(t)\le r$ for $t\in[0,1]$, that is $y(t)>\phi(t)$ for $t\in(0,1)$.

Fix $t\in(0,1)$. Without lass of generality assume $t>\frac{1}{2}$. Fix $x\in(0,1)$ with $x>t$. For $s\in[\frac{1}{2},x]$ notice

$$y(s)-\phi(s)\ge lw(s)\ge l\min\{w(\tfrac{1}{2}),w(x)\}.$$

Choose $n_1\in N$ with

$$\frac{1}{n_1}<l\min\{w(\tfrac{1}{2}),w(x)\}.$$

Let $N_1=\{m\in N, m\ge n_1\}$. Now y_m, $m\in N_1$, satisfies the equation

$$y_m(x)=y_m(\tfrac{1}{2})+y_m'(\tfrac{1}{2})(x-\tfrac{1}{2})+\mu\int_{\frac{1}{2}}^{x}(s-x)q(s)[g(y_m(s)-\phi(s))+$$

$$+h(y_m(s)-\phi(s))]ds.$$

Notice $\{y_m'(\frac{1}{2})\}$, $m\in N_1$, is a bounded sequence since $\phi(s)+lw(s)\le y_m(s)<r$ for $s\in[0,1]$. Thus $\{y_m'(\frac{1}{2})\}_{m\in N_1}$ has a convergent subsequence; for convenience let $\{y_m'(\frac{1}{2})\}_{m\in N_1}$ denote this subsequence also and let $r_0\in R$ be its limit. Let $m\to\infty$ through N_1 to obtain

$$y_m(x) = y_m(\tfrac{1}{2}) + r_0(x-\tfrac{1}{2}) + \mu\int_{\frac{1}{2}}^{x} (s-x)q(s)[g(y(s)-\phi(s)) + h(y(s)-\phi(s))]ds.$$

In particular $y''(t)+\mu q(t)[g(y(t)-\phi(t))+h(y(t)-\phi(t))]=0$. We can do this argument for each $t\in(0,1)$ and so $y''(t)+\mu q(t)f^*(y(t)-\phi(t))=0$ for $0<t<1$. Thus y is a solution of (3.3) with $y(t)>\phi(t)$ for $t\in(0,1)$. Finally it is easy to see that $|y|_0<r$ (note if $|y|_0=r$ then following essentially the argument from (3.6) - (3.10) will yield a contradiction).

THEOREM 3.2. *Assume the conditions* (2.2)-(2.7) *and* (*H*) *hold. Then* (3.1) *has two solutions* $y_1, y_2\in C[0,1]\cap C^2(0,1)$ *with* $y_1(t)>0$, $y_2(t)>0$ *for* $t\in(0,1)$, *and* $0<|y_1+\phi|_0<r<|y_2+\phi|_0$, *here* $\phi(t)=\mu M\,w(t)$ (*w is as in Lemma* 2.2).

PROOF. The existence of y_1 follows from Theorem 3.1 and the existence of y_2 follows from Theorem 2.1.

EXAMPLE 3.1. *Consider the boundary value problem*

$$\begin{cases} y''+\mu(y^{-\alpha}+y^{\beta}-1) = 0, & 0<t<1, \\ y(0)=y(1)=0, & \alpha>0,\ \beta>1 \end{cases} \tag{3.19}$$

where $\mu\in(0,\mu_0)$ *is Buch that*

$$\left(\frac{\mu_0(\alpha+1)}{3}\right)^{\frac{1}{\alpha}} + \frac{\mu_0}{2} \le 1. \tag{3.20}$$

Then (3.19) *have two solutions* y_1, y_2 *with* $y_1(t)>0$, $y_2(t)>0$ *for* $t\in(0,1)$, *and* $0<|y_1+\phi|_0<1<|y_2+\phi|_0$, *here* $\phi(t)=\frac{\mu}{2}t(1-t)$.

To see this we will apply Theorem 3.2 with (here $R>1$ will be chosen later; in fact here we choose $R>1$ so that $e=\frac{1}{2}$ works i.e. we choose R so that $1-\frac{\mu}{2R}\ge\frac{1}{2}$)

$$M=1, \qquad w(t)=\frac{1}{2}t(1-t), \qquad \phi(t)=\frac{\mu}{2}t(1-t),$$

and

$$g(y)=y^{-\alpha},\ h(y)=y^{\beta},\ e=\frac{1}{2},\quad a=\frac{1}{4},\ C_0=\frac{1}{2},\ K_0=1.$$

Clearly (2.2), (2.3), (2.4), (2.5) and (*H*) hold. Also note

$$b_0 = \max\left\{2\int_0^{\frac{1}{2}} t(1-t)dt,\ 2\int_{\frac{1}{2}}^{1} t(1-t)dt\right\} = \frac{1}{6}$$

and

$$\frac{1}{g(1-\frac{\mu M C_0}{r})\{1+\frac{h(r)}{g(r)}\}}\int_0^r \frac{du}{g(u)} = \left(1-\frac{\mu}{2r}\right)^{\alpha}\frac{1}{1+r^{\alpha+\beta}}\frac{r^{\alpha+1}}{\alpha+1}.$$

Now (2.6) holds with $r=1$ since

$$\mu M C_0 = \frac{\mu}{2} < \frac{\mu_0}{2} \le 1 = r$$

and

$$\mu K_0 b_0 \le \frac{\mu}{2} < \frac{\mu_0}{2} \le \frac{(1-\frac{\mu_0}{2})^{\alpha}}{2(\alpha+1)} \le \frac{(1-\frac{\mu}{2})^{\alpha}}{2(\alpha+1)} =$$

$$= \frac{1}{g(1-\frac{\mu M C_0}{r})\{1+\frac{h(r)}{g(r)}\}}\int_0^r \frac{du}{g(u)}$$

from (3.20). Finally notice (2.7) is satisfied for R large since

$$\frac{Rg(ea(1-a)R)}{g(R)g(ea(1-a)R)+g(R)h(ea(1-a)R)} = \frac{(\frac{3}{32})^{-\alpha}R^{1+\alpha}}{(\frac{3}{32})^{-\alpha}+(\frac{3}{32})^{\beta}R^{\alpha+\beta}} \to 0$$

as $R\to\infty$, since $\beta>1$. Thus all the conditions of Theorem 3.2 are satisfied so existence of two positive solutions is guaranteed.

REFERENCES

[1] R.P. Agarwal, D. O'Regan, Semipositione Dirichlet boundary value problem with singular dependent nonlinearities, *To appear in Houston. J. Math.*

[2] R.P. Agarwal, D. O'Regan, Existence theorem for single and multiple solutions to singular positone boundary value problems, *Jour. Differential Equations*, 175(2001), 393-414.

[3] R.P. Agarwal, D. O'Regan, P.J.Y Wong, *Positive solution of Differentaial, Difference and Integral equations*, Kluwer Acad. Publishers, Dordrecht, 1999.

[4] L.E. Bobisud, Behaviour of solutions for a Robin problem, *Jour. Differential Equations*, 85(1990), 91-104.

[5] K. Deimling, *Nonlinear functional analysis*, Springer Verlag, 1985.

[6] P.W. Eloe, J. Henderson, Singular nonlinear $(k,n-k)$ conjugate boundary value problem, *Jour. Differential Equations*, 133(1997), 136-151.

[7] C.P. Gupta, Existence and uniqueness theorems for a bending of an elastic beam equations, *Applicable Analysis*, 26(1988), 289-304.

[8] J.W. Lee, D. O'Regan, Existence principles for differential equations and systems of equations, *Topological Methods in Differential Equations and Inclusions* (*edited by A. Grans and M. Frigon*), NATO ASI Series c, *Kluwer Acad. Pebl.*, Dordrecht, (1995), 239-289.

[9] L. Mengseng, On a fourth order eigenvalue problem, *Advances in Math.*, 29(2000), 91-93.

[10] H.B. Thompson, Second order differential equations with fully nonlinear two point boundary value problems, *Pacific Jour. Math.*, 172(1996), 255-277.

([1]Department of Mathematics, Northeast Normal University, Changchun 130024, P. R. China;

[2]Department of Mathematics, National University of Ireland, Galway, Ireland;

[3]Department of Mathematical Science, Florida Institute of Technology, Melbourne, Florida 32901-6975, USA)

Received on 13.11.2002 and, in revised form, on 31.12.2002.