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OSCILLATION AND ASYMPTOTIC BEHAVIOR OF SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS

ABSTRACT: In this paper we will study the oscillatory and asymptotic behavior of solutions of second-order nonlinear difference equation

$$\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) + F(n, x_n, \Delta x_n) = 0, \quad n \geq 1.$$

where $\gamma > 0$ is a quotient of odd positive integers.

KEY WORDS: oscillation, difference equations.

1. INTRODUCTION

In recent years, the asymptotic behavior of second order nonlinear difference equations has been the subject of investigations by many authors, see e.g. the monographs [1, 2] and the references therein. Following this trend in this paper, we shall consider the nonlinear difference equation

$$\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) + F(n, x_n, \Delta x_n) = 0, \quad n \geq 1 \tag{1.1}$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers, $\gamma > 0$ is quotient of odd positive integers, $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $a_n > 0$ and

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\gamma} = \infty, \tag{1.2}$$

or

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\gamma} < \infty \tag{1.3}$$

for same positive integer $n_0 \geq 1$, $F : N \times R^2 \to R(-\infty, \infty)$ is continuous for each fixed n, $N = \{0,1,2,\dots\}$, and there exists a nonnegative sequence $\{q_n\}$ such that $\{q_n\}$ has a positive subsequence and

$$\frac{F(n,u,v)}{u^{\beta}} \geq q_n, \quad \text{for } u \neq 0 \tag{1.4}$$

where β is quotient of odd positive integers.

By a solution of (1.1) we mean a nontrivial sequence $\{x_n\}$ satisfying equation (1.1) for $n \geq 0$. A solution $\{x_n\}$ of (1.1) is said to be oscillatory if for every $n_1 \geq n_0 \geq 1$ there exists $n \geq n_1$ such that $x_n x_{n+1} \leq 0$, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

A number of dynamical behavior of solutions of second order difference equations are possible; here we will only be concerned with conditions which are sufficient for all solutions of (1.1) to be oscillatory or tends to zero as $n \to \infty$. Our concern is motivated by several papers, especially those by Arul and Thandapani [3], Thandapani et al. [9-12] and Wong and Agarwal [13] which studied equation (1.1) or special cases of it.

Equation (1.1) is the discrete analogue of the equation

$$(p(t)(y'(t))^{\gamma})' + f(t, y, y') = 0. \tag{1.5}$$

In [5-7], the authors used the results obtained for the equation (1.5) to establish criteria for the existence of oscillatory solution, or positive symmetric solution, either bounded or unbounded of particular class of quasilinear partial differential equations of the forms

$$\sum_{i=1}^{N} D_i(|Du|^{p-2} D_i u) + g(x,u,Du) = 0$$

and

$$\sum_{i=1}^{N} D_i(|D_i u|^{p-2} D_i u) + g(x,u,Du) = 0$$

in exterior domains in R^n. Hence the study of Eq. (1.1) is important and useful.

Therefore, our aim in this paper, by using the Reccati technique we present same new oscillation criteria for Eq. (1.1).

2. Main Results

First, we consider the case when (1.2) holds and $\beta \geq 1$.

Theorem 2.1. *Assume that* (1.2) *and* (1.4) *hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=1}^{\infty}$ *such that for every positive constant M,*

$$\limsup_{n\to\infty} \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{(a_l)^{\frac{\beta}{\gamma}} (\Delta\rho_l)^2}{2^{3-\beta} (M)^{\frac{\gamma-\beta}{\gamma}} \rho_l} \right] = \infty, \tag{2.1}$$

for some $n_0 > 0$. *Then every solution of Eq.* (1.1) *oscillates.*

PROOF. Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_n > 0$ for all $n \geq n_0 \geq 1$. We shall consider only this case, since the substitution $y_n = -x_n$ transforms equation (1.1) into an equation of the same form. In view of (1.1) and (1.4) we have

$$\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) \leq -q_n x_n^{\beta} \leq 0 \tag{2.2}$$

for all large *n*, and so $\{a_{n-1}(\Delta x_{n-1})^{\gamma}\}$ is an eventually nonincreasing sequence. We first show that $\{a_n(\Delta x_n)^{\gamma}\}$ is eventually positive. Indeed, since $\{q_n\}$ has a positive subsequence, the nondecreasing sequence $\{a_{n-1}(\Delta x_{n-1})^{\gamma}\}$ is either eventually positive or eventually negative. Suppose there exists an integer $n_1 \geq n_0$, such that $a_n(\Delta x_n)^{\gamma} \leq a_{n_1}(\Delta x_{n_1})^{\gamma} \leq c$ for $n \geq n_1$, then

$$\Delta x_n \leq c^{\frac{1}{\gamma}} \left(\frac{1}{a_n} \right)^{\frac{1}{\gamma}},$$

which implies that

$$x_n \leq x_{n_1} + c^{\frac{1}{\gamma}} \sum_{i=n_1}^{n-1} \left(\frac{1}{a_n} \right)^{\frac{1}{\gamma}} \to -\infty \quad \text{as} \quad n \to \infty,$$

which contradicts the fact that $x_n > 0$ for all large *n*. Hence $\{a_n(\Delta x_n)^{\gamma}\}$ is eventually positive. Therefore, we see that there is same n_0 such that

$$x_n > 0, \quad \Delta x_n > 0, \quad \Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) \leq 0, \quad n \geq n_0. \tag{2.3}$$

Define the sequence $\{w_n\}$ by

$$w_n = \rho_n \frac{a_{n-1}(\Delta x_{n-1})^{\gamma}}{x_n^{\beta}}, \tag{2.6}$$

then $w_n > 0$ and

$$\Delta w_n = a_n(\Delta x_n)^{\gamma} \Delta \left[\frac{\rho_n}{x_n^{\beta}} \right] + \frac{\rho_n \Delta(a_{n-1}(\Delta x_{n-1})^{\gamma})}{x_n^{\beta}},$$

in view of (2.2)

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n a_n (\Delta x_n)^{\gamma} \Delta(x_n^{\beta})}{x_n^{\beta} x_{n+1}^{\beta}}. \tag{2.7}$$

But, from (2.3) we have $x_{n+1} > x_n$, then tram (2.7) leads to

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n a_n (\Delta x_n)^\gamma \Delta(x_n^\beta)}{(x_{n+1}^\beta)^2}. \tag{2.8}$$

Now, by using the inequality

$$x^\beta - y^\beta \geq 2^{1-\beta}(x-y)^\beta \quad \text{for all} \quad x \geq y > 0 \quad \text{and} \quad \beta \geq 1,$$

then, we have

$$\Delta(x_n^\beta) = x_{n+1}^\beta - x_n^\beta \geq 2^{1-\beta}(x_{n+1}-x_n)^\beta = 2^{1-\beta}(\Delta x_n)^\beta, \quad \beta \geq 1. \tag{2.9}$$

Substitute from (2.9) in (2.8), we obtain

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \rho_n a_n \frac{2^{1-\beta}(\Delta x_n)^\beta (\Delta x_n)^\gamma}{(x_{n+1}^\beta)^2}, \tag{2.10}$$

hence

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \rho_n a_n \frac{2^{1-\beta}(\Delta x_n)^{2\gamma}}{(x_{n+1}^\beta)^2 (\Delta x_n)^{\gamma-\beta}},$$

or

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{2^{1-\beta}\rho_n}{a_n(\rho_{n+1})^2} w_{n+1}^2 \frac{1}{(\Delta x_n)^{\rho-\beta}}. \tag{2.11}$$

Now, from the fact that $\{a_{n-1}(\Delta x_{n-1})^\gamma\}$ is a positive and nonincreasing sequence, there exists a $n_1 \geq n_0$ sufficiently large such that $a_{n-1}(\Delta x_{n-1})^\gamma \leq 1/M$ for some positive constant M and $n \geq n_2$, and hence by (2.3) we have $a_n(\Delta x_n)^\gamma \leq 1/M$, so that

$$\frac{1}{(\Delta x_n)^{\gamma-\beta}} \geq (M a_n)^{\frac{\gamma-\beta}{\gamma}}, \tag{2.12}$$

then, from (2.11) and (2.12), we have

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{2^{1-\beta}\rho_n}{a_n(\rho_{n+1})^2}(M a_n)^{\frac{\gamma-\beta}{\gamma}} w_{n+1}^2 \tag{2.13}$$

hence,

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{2^{1-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_n}{(a_n)^{\frac{\beta}{\gamma}}(\rho_{n+1})^2} w_{n+1}^2 = -\rho_n[q_n - p_n] +$$

$$+\frac{(a_n)^{\frac{\beta}{\gamma}}(\Delta\rho_n)^2}{2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_n} - \left[\frac{\sqrt{2^{1-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_n}}{\rho_{n+1}\sqrt{(a_n)^{\frac{\beta}{\gamma}}}} w_{n+1} - \frac{\sqrt{(a_n)^{\frac{\beta}{\gamma}}}\Delta\rho_n}{2\sqrt{2^{1-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_n}}\right]^2 <$$

$$< -\left[\rho_n q_n - \frac{(a_n)^{\frac{\beta}{\gamma}}(\Delta\rho_n)^2}{2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_n}\right].$$

Then, we have

$$\Delta w_n < -\left[\rho_n q_n - \frac{(a_l)^{\frac{\beta}{\gamma}}(\Delta\rho_n)^2}{2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_n}\right]. \tag{2.14}$$

Summing (2.14) from n_2 to n, we obtain

$$-w_{n_1} < w_{n+1} - w_{n_1} < -\sum_{l=n_1}^{n}\left[\rho_l q_l - \frac{(a_l)^{\frac{\beta}{\gamma}}(\Delta\rho_l)^2}{2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_l}\right]$$

which yields

$$\sum_{l=n_1}^{n}\left[\rho_l q_l - \frac{(a_l)^{\frac{\beta}{\gamma}}(\Delta\rho_l)^2}{2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}\rho_l}\right] < c_1, \tag{2.15}$$

for all large n. This is contrary to (2.1). The proof is complete.

From Theorem 2.1, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$. For instance, let $\rho_n = n^\lambda$, $n \geq n_0$ and $\lambda > 1$ is a constant, or $\rho_n = R(n, n_0) = \sum_{s=n_0}^{n-1} \frac{1}{a_s}$. Hence we have the following results.

COROLLARY 2.1. *Assume that all the assumptions of Theorem* 2.1 *hold, except the condition* (2.1) *is replaced by*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n}\left[s^\lambda q_s - \frac{(a_s)^{\frac{\beta}{\gamma}}((s+1)^\lambda - s^\lambda)^2}{2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}s^\lambda}\right] = \infty. \tag{2.16}$$

or

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n}\left[R(s,n_0)q_s - \frac{(a_s)^{\frac{\beta}{\gamma}}(\Delta R(s,n_0))^2}{2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}}R(s,n_0)}\right] = \infty. \tag{2.17}$$

Then every solution oj Eq. (1.1) *oscillates.*

As a variant of the Riccati transformation technique used above, we will derive a Kamenev type oscillation criteria which can be considered as a discrete analogy of Philos' condition for oscillation of second order differential equations [8].

THEOREM 2.2. *Assume that* (1.2) *and* (1.4) *hold. Let* $\{\rho_n\}_{n=0}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *such that (i)* $H_{m,n} = 0$ *for* $m \geq 0$, *(ii)* $H_{m,n} > 0$ *for* $m > n \geq 0$, *(iii)* $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *for* $m \geq n \geq 0$. *If*

$$\limsup_{n\to\infty} \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \left[H_{m,n}\rho_n q_{n+1} - \frac{\rho_{n+1}^2}{\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}} \right)^2 \right] = \infty, \tag{2.18}$$

where

$$\overline{\rho}_n = 2^{3-\beta}(M)^{\frac{\gamma-\beta}{\gamma}} \rho_n \Big/ (a_n)^{\frac{\beta}{\gamma}}, \qquad h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}, \quad m > n \geq 0,$$

for every positive number M, then every solution of (1.1) *oscillates.*

PROOF. We proceed as in the proof of Theorem 2.1, we may assume that (1.1) has a nonoscillatory solution $\{x_n\}_{n=0}^{\infty}$ such that $x_n > 0$, $\Delta x_n > 0$, $\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) \leq 0$ for $n \geq n_1 \geq 0$. Define $\{w_n\}$ by (2.6) as before, then we have $w_n > 0$ and there is some $M > 0$ such that (2.13) holds. From (2.13), we have for $n \geq n_1$

$$\rho_n q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2. \tag{2.19}$$

Therefore, we have

$$\sum_{n=n_1}^{m-1} H_{m,n}\rho_n q_n \leq -\sum_{n=n_1}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2, \tag{2.20}$$

which yields, after summing by parts

$$\sum_{n=n_1}^{m-1} H_{m,n}\rho_n q_n \leq H_{m,n_1} w_{n_1} + \sum_{n=n_1}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - vH_{m,n}\frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 =$$

$$= H_{m,n_1} w_{n_1} - \sum_{n=n_1}^{m-1} h_{m,n}\sqrt{H_{m,n}}\, w_{n+1} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} -$$

$$-\sum_{n=n_1}^{m-1} H_{m,n}\frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 =$$

$$= H_{m,k} w_k - \sum_{n=n_1}^{m-1}\left[\frac{\sqrt{H_{m,n}\overline{\rho}_n}}{\rho_{n+1}} w_{n+1} + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\overline{\rho}_n}}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}} H_{m,n}\right)\right]^2 +$$

$$+\frac{1}{4}\sum_{n=k}^{m-1}\frac{(\rho_{n+1})^2}{\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2.$$

Then

$$\sum_{n=n_1}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,n_2} \le H_{m,0} w_{n_2}$$

which implies that

$$\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,0}\left(w_{n_1} + \sum_{n=0}^{n_1-1}\rho_n q_n\right).$$

Hence

$$\text{(2.21)}\qquad \limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] <$$

$$< \left(w_{n_1} + \sum_{n=0}^{n_1-1}\rho_n q_n\right) < \infty$$

and this contradicts (2.18). The proof is complete.

As an immediate consequence of Theorem 2.2, we get the following:

COROLLARY 2.2. *Assume that all the assumptions oj Theorem* 2.2 *hold, except the condition* (2.21) *is replaced by*

$$\text{(2.22)}\qquad \limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1} H_{m,n}\rho_n q_n = \infty,$$

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1}\frac{\rho_{n+1}^2}{\overline{\rho}_n}\left(h_{m,n}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2<\infty. \tag{2.23}$$

Then every solution of Eq. (1.1) *oscillates.*

By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\begin{cases} H_{m,n}=(m-n)^{\lambda}, & \lambda\geq 1,\ m\geq n\geq 0,\\ H_{m,n}=\left(\log\frac{m+1}{n+1}\right)^{\lambda}, & \lambda\geq 1,\ m\geq n\geq 0,\\ H_{m,n}=(m-n)^{(\lambda)}, & \lambda>2,\ m\geq n\geq 0.\end{cases} \tag{2.24}$$

where $(m-n)^{(\lambda)}=(m-n)(m-n+1)\ldots(m-n+\lambda-1)$ and

$$\Delta_2(m-n)^{(\lambda)}=(m-n-1)^{(\lambda)}-(m-n)^{(\lambda)}=-\lambda(m-n)^{(\lambda-1)}$$

Then $H_{m,m}=0$ for $m\geq 0$ and $H_{m,m}>0$ and $\Delta_2 H_{m,n}\leq 0$ for $m>n>\geq 0$. By Theorem 2.2 we get the following oscillation criteria for Eq. (1.1).

COROLLARY 2.3. *Assume that all the assumptions of Theorem* 2.2 *hold, except the condition* (2.18) *is replaced by*

$$\limsup_{m\to\infty}\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}\left[(m-n)^{\lambda}\rho_n q_n-\right.$$
$$\left.-\frac{\rho_{n+1}^2}{4\overline{\rho}_n}\left(\lambda(m-n)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{(m-n)^{\lambda}}\right)^2\right]=\infty. \tag{2.25}$$

Then, every solution of Eq. (1.1) *oscillates.*

COROLLARY 2.4. *Assume that all the assumptions of Theorem* 2.2 *hold, except the condition* (2.18) *is replaced by*

$$\limsup_{m\to\infty}\frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m-1}\left[\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_n q_n-\right.$$
$$\left.-\frac{\rho_{n+1}^2}{4\overline{\rho}_n}\left(\frac{\lambda}{n+1}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{\left(\log\frac{m+1}{n+1}\right)^{\lambda}}\right)^2\right]=\infty. \tag{2.26}$$

Then, every solution of Eq. (1.1) *oscillates.*

COROLLARY 2.5. *Assume that all the assumptions of Theorem* 2.2 *hold, except the condition* (2.18) *is replaced by*

$$(2.27)\qquad \limsup_{m\to\infty}\frac{1}{m^{(\lambda)}}\sum_{n=0}^{m-1}(m-n)^{(\lambda)}\left[\rho_n q_n-\frac{\rho_{n+1}^2}{4\overline{\rho}_n}\left(\frac{\lambda}{n-n+\lambda-1}-\frac{\Delta\rho_n}{\rho_{n+1}}\right)^2\right]=\infty.$$

Then, every solution of Eq. (1.1) *oscillates.*

Next, we consider the case when (1.3) holds, and $\beta\geq 1$.

THEOREM 2.3. *Assume that* (1.3) *and* (1.4) *hold. Furthermore, we assume that there exist positive sequences* $\{\rho_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$. *such that for every positive constant M* (2.1) *holds, and*

$$(2.28)\qquad \Delta\delta_n\leq 0,\qquad \Delta(a_n\Delta\delta_n)\leq 0,\qquad \sum_{n=n_0}^{\infty}\delta_{n+1}q_n=\infty$$

$$\text{and}\quad \sum_{n=n_0}^{\infty}\left(\frac{1}{a_n\delta_n}\sum_{n=n_0}^{n-1}\delta_{i+1}q_i\right)^{\frac{1}{\gamma}}=\infty$$

for some $n_0>0$. *Then every solution of Eq.(l.l) oscillates or* $\lim_{n\to\infty}x_n=0$.

PROOF. Suppose that $\{x_n\}$ is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. From (1.1) and (1.4) we have

$$(2.29)\qquad \Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}\leq -q_n x_n^{\beta}$$

and so $\{a_{n-1}(\Delta x_{n-1})^{\gamma}\}$ is an eventually nonincreasing sequence. Since $\{q_n\}$ has a positive subsequence, either $\{\Delta x_n\}$ is eventually negative or eventually positive.

If $\{\Delta x_n\}$ is eventually positive, we are then back to the case where (2.3) holds. Thus the proof of Theorem 2.1 goes through, and we may conclude that $\{x_n\}$ cannot be eventually positive, which is not possible.

If $\{\Delta x_n\}$ is eventually negative. Then $\lim_{n\to\infty}x_n=b\geq 0$. We assert that $b=0$. If not, then $x_n^{\beta}\to b^{\beta}>0$ as $n\to\infty$, and hence there exists $n_2\geq n_1$ such that $x_n^{\beta}\geq b^{\beta}$. Therefore from (2.29) we have

$$(2.30)\qquad \Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}\leq -q_n b^{\beta}$$

Define the sequence $u_n=\delta_n(a_{n-1}(\Delta x_{n-1})^{\gamma})$ for $n\geq n_2$. Then we have

$$\Delta u_n \le -b^\beta \delta_{n+1} q_n + \Delta\delta_n (a_{n-1}(\Delta x_{n-1})^\gamma.$$

Summing the last inequality from n_2 to $n-1$, we have

$$u_n \le u_{n_2} - b^\beta \sum_{s=n_2}^{n-1} \delta_{s+1} q_s + \sum_{s=n_2}^{n-1} (a_{n-1}\Delta\delta_s)(\Delta x_{s-1})^\gamma$$

and then

$$u_n \le u_{n_2} - b^\beta \sum_{s=n_2}^{n-1} \delta_{s+1} q_s + a_{s-1}\Delta\delta_s (\Delta x_{s-1})^\gamma |_{s=n_2}^{n} - \sum_{s=n_2}^{n-1} (a_{s-1}\Delta\delta_s)(\Delta x_s)^\gamma$$

In view of (2.28) we have

$$u_n \le M - b^\beta \sum_{s=n_2}^{n-1} \delta_{s+1} q_s$$

where $M = u_{n_2} - a_{n_2-1}\Delta\delta_{n_2-1}\Delta_{x_{n_2}-1}$. In view of (2.28), since $\sum_{n=n_2}^{\infty} \delta_{n+1} q_n = \infty$ it is possible to choose integer n_3 sufficiently large such that for all

$$u_n \le -\frac{b^\beta}{2} \sum_{s=n_2}^{n-1} \delta_{s+1} q_s .$$

Summing the last inequality from n_3 to n we obtain

$$x_{n+1} \le x_{n_3} - \left(\frac{b^\beta}{2}\right)^{\frac{1}{\gamma}} \sum_{s=n_3}^{n} \left(\frac{1}{a_s \delta_s} \sum_{i=n_2}^{s-1} \delta_{i+1} q_i\right)^{\frac{1}{\gamma}} .$$

Condition (2.28) implies that $\{x_n\}$ is eventually negative, which is a contradiction. The proof is complete.

By choosing $\{\rho_n\}_{n=1}^{\infty}$ in appropriate manners, we may obtain different oscillation criteria. For instance, let $\rho_n = n^\lambda$ for $n \ge 0$ and $\lambda > 1$. Then we have the following result.

COROLLARY 2.6. *Assume that all the assumptions of Theorem* 2.3 *hold, except the condition* (2.1) *is replaced by* (2.16) *or* (2.17). *Then, every solution of Eq.* (1.1) *oscillates or* $\lim_{n\to\infty} x_n = 0$.

THEOREM 2.4. *Assume that* (1.2) *and* (1.4) *hold, and let* $\{\rho_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be two positive sequences such that* (2.28) *holds. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \ge n \ge 0\}$ *as defined in Theorem* 2.2 *and* (2.18) *holds. Then every solution of Eq.* (1.1) *oscillates or* $\lim_{n\to\infty} x_n = 0$.

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1.1). Then as seen in the proof of Theorem 2.1, either $\{\Delta x_n\}$ is eventually positive or is eventually negative. In the farmer case, if $\{\Delta x_n\}$ is eventually positive we may follow the proof of Theorem 2.2 and obtain a contradiction. If $\{\Delta x_n\}$ is eventually negative, then we may follow the proof of Theorem 2.3 to show that $\{x_n\}$ converges to zero.

By choosing $\{H_{m,n}\}$ in appropriate manners as in (2.24), we derive some criteria for (1.1) when (1.3) holds which are sufficient for oscillation or imply that $\lim_{n\to\infty} x_n = 0$ when (1.3) holds and $\beta \geq 1$. The details are left to the reader.

Now, we consider the case when (1.2) holds, $0 < \beta < 1$ and $\Delta a_n \geq 0$.

THEOREM 2.5. *Assume that* (1.2), *and* (1.4) *hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=1}^{\infty}$ *such that for every* $b \geq 1$

(2.31)
$$\limsup_{n\to\infty} \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{(a_l)^{\frac{1}{\gamma}} b^{1-\beta} (l+1)^{1-\beta} (\Delta\rho_n)^2}{4\beta (M)^{\frac{\gamma-1}{\gamma}} \rho_l} \right] = \infty$$

for some $n_0 \geq 1$. *Then every solution of Eq.* (1.1) *oscillates.*

PROOF. Proceeding as in the proof of Theorem 2.1 we will assume that Eq. (1.1) has a nonoscillatory solution $x_n > 0$ all $n \geq n_0$. Defining again w_n by (2.6), then we obtain

(2.8)
$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n a_n (\Delta x_n)^\gamma \Delta(x_n^\beta)}{(x_{n+1}^\beta)^2}$$

Now we claim $\Delta^2 x_n \leq 0$. If not there exists $n_1 \geq n_0$ such that $\Delta^2 x_n \leq 0$ and this implies that $\Delta x_n > \Delta x_{n-1}$, so that since $\Delta a_n \geq 0$, $a_n(\Delta x_n)^\gamma > a_n(\Delta x_{n-1})^\gamma \geq a_{n-1}(\Delta x_{n-1})^\gamma$ and this contradicts the fact that $\{a_{n-1}(\Delta x_{n-1})^\gamma\}$ is nonincreasing sequence, then $\Delta^2 x_n \leq 0$, and therefore we have

(2.32)
$$x_n \geq 0, \qquad \Delta x_n \geq 0 \text{ and } \Delta^2 x_n \leq 0 \text{ for } n \geq n_0$$

Now, by using the inequality (cf. [4, p. 39])

$$x^\beta - y^\beta \geq \beta x^{\beta-1}(x-y) \text{ for all } x \neq y > 0 \text{ and } 0 < \beta \leq 1$$

Then, we have

(2.33) $$\Delta(x_n^\beta)=x_{n+1}^\beta-x_n^\beta\geq\beta(x_{n+1})^{\beta-1}(x_{n+1}-x_n)=\beta(x_{n+1})^{\beta-1}(\Delta x_n).$$

Substitute from (2.33) in (2.8), we have

(2.34) $$\Delta w_n\leq-\rho_nq_n+\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\rho_na_n\frac{\beta(x_{n+1})^{\beta-1}(\Delta x_n)(\Delta x_n)^\gamma}{(x_{n+1}^\beta)^2},$$

or

$$\Delta w_n\leq-\rho_nq_n+\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\frac{\beta\rho_n}{a_n(\rho_{n+1})^2(\Delta x_n)^{\gamma-1}(x_{n+1})^{1-\beta}}\frac{(\rho_{n+1}a_n)^2(\Delta x_n)^{2\gamma}}{(x_{n+1}^\beta)^2},$$

hence,

(2.35) $$\Delta w_n\leq-\rho_nq_n+\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\frac{\beta\rho_n}{a_n(\rho_{n+1})^2(\Delta x_n)^{\gamma-1}(x_{n+1})^{1-\beta}}w_{n+1}^2,$$

From (2.32), we conclude that

$$x_n\leq x_{n_0}+\Delta x_{n_0}(n-n_0),\qquad n\geq n_0,$$

and consequently there exists a $n_1\geq n_0$ and appropriate constant $b\geq1$ such that

$$x_n\leq bn\quad\text{for}\quad n\geq n_1,$$

and this implies that

$$x_{n+1}\leq b(n+1)\quad\text{for}\quad n\geq n_2=n_1-1$$

and, hence

(2.36) $$\frac{1}{(x_{n+1})^{1-\beta}}\geq\frac{1}{b^{1-\beta}(n+1)^{1-\beta}}$$

then from (2.12), (2.35) and (2.36) we obtain

(2.37) $$\Delta w_n\leq-\rho_nq_n+\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\frac{\beta(M)^{\frac{\gamma-1}{\gamma}}}{(\rho_{n+1})^2(a_n)^{\frac{1}{\gamma}}b^{1-\beta}(n+1)^{1-\beta}}w_{n+1}^2.$$

The remainder of the proof is similar to that of Theorem 2.1 and hence is omitted.

From Theorem 2.5, we can obtain different conditions for oscillation of all solutions of Eq. (1.1) when (1.2) holds by different choices of $\{\rho_n\}$. Let $\rho_n=n^\lambda n\geq n_0$ and $\lambda>1$ is a constant, or $\rho_n=R(n,n_0)=\sum_{s=n_0}^{n-1}\frac{1}{a_s}$. By Theorem 2.5 we have the following results.

COROLLARY 2.7. *Assume that all the assumptions of Theorem 2.5 hold,* *except the condition* (2.31) *is replaced by*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[s^{\lambda}q_s-\frac{(a_l)^{\frac{1}{\gamma}}b^{1-\beta}(l+1)^{1-\beta}((s+1)^{\lambda}-s^{\lambda})^2}{4\beta(M)^{\frac{\gamma-1}{\gamma}}s^{\lambda}}\right]=\infty. \tag{2.38}$$

or

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[R(s,n_0)q_s-\frac{(a_l)^{\frac{1}{\gamma}}b^{1-\beta}(l+1)^{1-\beta}(\Delta R(s,n_0))^2}{4\beta(M)^{\frac{\gamma-1}{\gamma}}R(s,n_0)}\right]=\infty. \tag{2.39}$$

Then, every solution oj Eq. (1.1) *osciltates.*

THEOREM 2.6. *Assume that* (1.2) *and* (1.4) *hold. Let* $\{\rho_n\}_{n=0}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H_{m,n}: m\geq n\geq 0\}$ *such that (i)* $H_{m,n}=0$ *for* $m\geq 0$, *(ii)* $H_{m,n}>0$ *for* $m>n\geq 0$, *(iii)* $\Delta_2 H_{m,n}=H_{m,n+1}-H_{m,n}\leq 0$ *for* $m\geq n\geq 0$. *If*

$$\limsup_{n\to\infty}\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1}\left[H_{m,n}\rho_n q_n-\frac{\rho_{n+1}^2}{4P_n}\left(h_{m,n}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]=\infty \tag{2.41}$$

where $P_n=\dfrac{\beta(M)^{\frac{\gamma-1}{\gamma}}}{(a_n)^{\frac{1}{\gamma}}b^{1-\beta}(n+1)^{1-\beta}}$. *Then every solution of Eq.* (1.1) *osciltates.*

PROOF: Proceeding as in Theorem 2.5, we assume that Eq. (1.1) has a nonoscillatary solution, say $x_n>0$ and for all $n\geq n_0$. From the proof of Theorem 2.5 we obtain (2.37) for all $n\geq n_2$. From (2.37), we have for $n\geq n_2$.

$$\Delta w_n\leq -\rho_n q_n+\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\frac{P_n}{(\rho_{n+1})^2}w_{n+1}^2 \tag{2.42}$$

the remainder of the proof is similar to that of Theorem 2.2 and hence is omitted.

As an immediate from Theorem 2.6 and (2.24) we have the following results.

COROLLARY 2.8. *Assume that all the assumptions of Theorem 2.6 hold, except the condition* (2.41) *is replaced by*

$$\limsup_{n\to\infty}\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}\left[(m-n)^{\lambda}\rho_n q_n-\right. .$$
$$\left.-\frac{\rho_{n+1}^2}{4P_n}\left(\lambda(m-n)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{(m-n)^{\lambda}}\right)^2\right]=\infty. \tag{2.4}$$

Then, every solution of Eq. (1.1) *osciltates.*

COROLLARY 2.9. *Assume that all the assumptions of Theorem* 2.6 *hold, except the condition* (2.41) *is replaced by*

$$(2.44)\qquad \limsup_{n\to\infty}\frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m-1}\left[\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_n q_n-\frac{\rho_{n+1}^2}{4P_n}B_{m,n}\right]=\infty$$

where

$$B_{m,n}=\left(\frac{\lambda}{n+1}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{\left(\log\frac{m+1}{n+1}\right)^{\lambda}}\right)^2$$

Then, every solution of Eq. (1.1) *osciltates.*

COROLLARY 2.10. *Assume that all the assumptions of Theorem* 2.6 *hold, the condition* (2.41) *is replaced by*

$$(2.45)\qquad \limsup_{n\to\infty}\frac{1}{m^{(\lambda)}}\sum_{n=0}^{m-1}(m-n)^{(\lambda)}\left[\rho_n q_n-\frac{\rho_{n+1}^2}{4P_n}\left(\frac{\lambda}{m-n+\lambda-1}-\frac{\Delta\rho_n}{\rho_{n+1}}\right)^2\right]=\infty$$

Then, every solution of Eq. (l.l) *oscillates.*

Next, we consider the case when (1.3) holds, $0<\beta<1$ and $\Delta a_n\geq 0$.

THEOREM 2.7. *Assume that* (1.3), *and* (1.4) *hold. Furthermore, we assume that there exist positive sequences* $\{\rho_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *such that* (2.28) *and* (2.31) *hold. Then every solution oj Eq.* (1.1) *oscillates or* $\lim_{n\to\infty}x_n=0$.

The proof is similar to that of Theorem 2.3 by using the inequality (2.37) and hence is omitted.

THEOREM 2.8. *Assume that* (1.3) *and* (1.4) *hold, and let* $\{\rho_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ *be two positive sequences such that* (2.28) *holds. Furthermore, we assume that there exists a double sequence* $\{H_{m,n}: m\geq n\geq 0\}$ *as defined in Theorem* 2.6 *and* (2.41) *holds. Then every solution of Eq.* (l.l) *oscillates or* $\lim_{n\to\infty}x_n=0$.

The proof is similar to that of Theorem 2.4 by using the inequality (2.37) and hence is omitted.

COROLLARY 2.10. *Assume that* (1.3) *and* (1.4) *hold, and let* $\{\rho_n\}_{n=1}^{\infty}$ *and*

$\{\delta_n\}_{n=1}^{\infty}$ *be two positive sequences such that* (2.28) *holds. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *as defined in Theorem* 2.6 *except the condition* (2.41) is *replaced by* (2.43). *Then every solution of Eq.* (l.l) *oscillates or* $\lim_{n\to\infty} x_n = 0$.

COROLLARY 2.11. *Assume that* (1.3) *and* (1.4) *hold, and let* $\{\rho_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be two positive sequences such that* (2.28) *holds. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *as defined in Theorem* 2.6 *except the condition* (2.41) is *replaced by* (2.44) *or* (2.45). *Then every solution of Eq.* (l.l) *oscillates or* $\lim_{n\to\infty} x_n = 0$.

REFERENCES

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Theory, Methods and Applications, Second Edition, Marcel Dekker, New York, 2000.

[2] R.P. Agarwal, P.J.Y. Wong, *Advanced Topics in Difference Equations*, Kluwer Academic Publishers, 1997.

[3] R. Arul, E. Thandapani, Asymptotic behavior of positive solutions of second order quasilinear difference equations, *Kyungpook Math. J.*, 40 (2000), 275-486.

[4] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, 2nd Ed. Cambridge Univ. Press 1952.

[5] H.L. Hong, W.C. Lian, C.C. Yeh, The oscillation of half-linear diflerential equations wit h oscillatory coefficients, *Mathl. Com. Modelling*, 24 (1996), 77-86.

[6] H.B. Hsu, C.C. Yeh, Oscillation theorems for second-order half-linear differential equations, *Appl. Math. Lett.*, 9 (1996), 71-77.

[7] T. Kusano, O. Akio, Existence and asymptotic behavior of positive solutions of second order quasilinear diflerential equations, *Funk. Ekva.*, 37 (1994),345 - 361.

[8] Ch.G. Philos, Oscillation theorems for linear diflerential equation of second order, Arch. Math., 53(1989), 483-492.

[9] E. Thandapani, J.R. Greaf, P.W. Spikes, On the oscillation of solutions of second order quasilinear diflerence equations, *Nonlin. World.*, 3 (1996), 545-565.

[10] E. Thandapani, M.M.S. Mannuel, R.P. Agarwal, Oscillation and nonoscillation theorems for second order quasilinear diflerence equations, *Facta. Univ. Series, Math. Inform.*, 11 (1996), 49-65.

[11] E. Thandapani, R. Arul, Oscillation and nonoscillation theorems for a class of second order quasilinear diflerence equations, *ZAA.*, 16 (1997), 749-759.

[12] E. Thandapani, R. Arul, Oscillation theory for a class of second order quasilinear diflerence equations, *Tamaking J. Math.*, 28 (1997), 229-238.

[13] P.J.Y. Wong, R.P. Agarwal, Oscillation and monotone solutions of second order quasilinear diflerence equations, *Fankcialaj Ekvacioj*, 39(1996), 491-517.

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