FASCICULI MATHEMATICI

Nr 34 2004

S. PADHI[1], P.K. KAR[1] AND S.K. NAYAK[2]

NECESSARY AND SUFFICIENT CONDITIONS FOR THE SOLUTIONS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS TO BE OSCILLATORY OR TENDING TO ZERO

ABSTRACT: In this paper, necessary and sufficient conditions have been obtained for a class of forced superlinear second order neutral differential equations of Emden-Fowler type such that every solution of the equation is either oscillatory or tends to zero as $t \to \infty$.

KEY WORDS: oscillation, nonoscillation, neutral differential equation.

1. Here we intend to study a class of neutral differential equations in which the highest order derivative of the unknown function appears simultaneously with and without delayed arguments.

In this paper, necessary and sufficient conditions are obtained such that every solution of the second order superlinear neutral delay differential equations of Emden-Fowler type

$$(NH_1) \qquad [y(t)-p(t)y(\tau(t))]'' + q(t)\,|\,y(\sigma(t))|^{\gamma} \operatorname{sgn} y(\sigma(t)) = g(t)$$

and

$$(NH_2) \qquad [y(t)+p(t)y(\tau(t))]'' + q(t)\,|\,y(\sigma(t))|^{\gamma} \operatorname{sgn} y(\sigma(t)) = g(t)$$

are either oscillatory or tend to zero as $t \to \infty$. We assume that p, q, τ and $\sigma \in C([0,\infty),[0,\infty))$, $\tau(t) \le t$, $\sigma(t) \le t$, $\tau(t)$ and $\sigma(t)$ are increasing, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$, $\gamma > 1$, $g \in C(R,R)$ and $g(t)$ is allowed to change sign.

In [2], Das bas obtained a necessary and sufficient condition for all solutions of Eq. (NH_1) to be oscillatory with $g(t) \equiv 0$ and $p(t)$ is a real constant. The oscillatory behaviour of homogeneous equations with constant coefficients is usually characterised by the nonexistence of real roots of the associated characteristic equations [1, 4]. If $p(t) \equiv 0$, then Eq. (NH_1) and (NH_2) reduces to the forced superlinear second order differential equation with delayed argument of Emden-Fowler type

$$(1) \qquad y''(t) + g(t)\,|\,y(\sigma(t))|^{\gamma} \operatorname{sgn} y(\sigma(t)) = g(t).$$

The motivation of this work has come from the recent work due to Nasr [6]. In

[6], necessary and sufficient conditions has been obtained for the oscillation of all solutions of Eq. (1). In this paper, we extend the results in [6] to Equations (NH_1) and (NH_2) by repeated use of the following lemma.

LEMMA 1. *Let* $\sigma'(t) \geq \alpha$, $\alpha > 0$ *is a constant. If* $\int_0^\infty tq(t)dt = \infty$, *then all solutions of*

$$y''(t) + g(t)\,|\,y(\sigma(t))|^{\gamma} \operatorname{sgn} y(\sigma(t)) = 0$$

are oscillatory.

The proof of Lemma 1.1 follows from Theorem 2 due to Nasr [6].

In [8], Parhi and Mohanty, has obtained sufficient conditions for the oscillation of the equation

$$[x(t) + p(t)x(\tau(t))]'' + q(t)h(x(\sigma(t))) = f(t)$$

where p, q, τ, σ and f are as given above and $h \in C(R,R)$ such that $uh(u) > 0$ for $u \neq 0$. Our results improves few of their results.

By a solution of (NH_1) (or (NH_2)) we mean a real-valued continuous function y on $[T_y, \infty)$ for some $T_y \geq 0$ such that $\{y(t) - p(t)y(\tau)\}$ (or $\{y(t) + p(t)y(\tau(t))\}$) is twice continuously differentiable and (NH_1) (or (NH_2)) is satisfied for $t \in [T_y, \infty)$.

As is customary, a solution of Eq. (NH_1) or (NH_2) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory.

Section 2 is devoted to the study of Eq. (NH_1) and in Section 3 we consider Eq. (NH_2).

2. This section contains the results concerning necessary and sufficient conditions for every solution of Eq. (NH_1) to be oscillatory or tend to zero as $t \to \infty$. We assume the following:

(2) There exists a bounded function $h(t)$ such that $h''(t) = g(t)$ with

$$|h(t)| < M, \qquad M > 0$$

for all t and

(3)
$$\lim_{t\to\infty} p(t) = 0.$$

THEOREM 2.1. *Under the assumptions* (2) *and* (3), *if*

(4) $$\int_0^\infty tq(t)dt < \infty,$$

then there exists a nonoscillatory solution of (NH_1).

PROOF. From (3) and (4), it follows that there exists a $T\,(>0)$ such that

$$|p(t)| < \frac{1}{3M+2} \quad \text{for} \quad t \geq T$$

and

$$\int_T^\infty tq(t)dt < \frac{1}{\gamma(3M+2)^\gamma}.$$

Consider the complete metric space X consisting of continuous functions $y(t)$ defined on $[T,\infty)$ satisfying the inequalities

$$M \leq y(t) \leq 3M+2$$

endowed with the metric

$$d(x,y) = \sup_{T<t<\infty} |x(t)-y(t)|.$$

Let $\phi(t) = \min\{\tau(t),\sigma(t)\}$ with $\phi(t)$ increasing. Define an operator A by

$$(Ay)(t) = 2M+1+p(t)y(\tau(t))+h(t)-\int_t^\infty (s-t)q(s)y^\gamma(\sigma(s))ds, \quad t \geq \phi^{-1}(T),$$

$$(Ay)(t) = 2M+1+p(t)y(\tau(t))+h(t)-\int_{\phi^{-1}(T)}^\infty (s-t)q(s)y^\gamma(\sigma(s))ds,$$

$$T \leq t < \phi^{-1}(T).$$

Let $y \in X$. Then clearly $(Ay)(t)$ is continuous,

$$(Ay)(t) \leq 2M+1+\frac{1}{(3M+2)}(3M+2)+M = 3M+2$$

and

$$(Ay)(t) \geq 2M+1-M-\int_{\phi^{-1}(T)}^\infty (s-t)q(s)y^\gamma(\sigma(s))ds \geq$$

$$\geq M+1-M-\int_{\phi^{-1}(T)}^\infty sq(s)y^\gamma(\sigma(s))ds \geq$$

$$\geq M+1-(3M+2)^{\gamma}\int_{\phi^{-1}(T)}^{\infty} sq(s)ds \geq$$

$$\geq M+1-(3M+2)^{\gamma}\int_{T}^{\infty} sq(s)ds \geq$$

$$\geq M+1-(3M+2)^{\gamma}\frac{1}{\gamma(3M+2)^{\gamma}} =$$

$$= M+1-\frac{1}{\gamma} > M \,(\Theta\gamma > 1)$$

imply that $Ay \in X$. Further, for $x, y \in X$,

$$|(Ax)(t)-(Ay)(t)| \leq |p(t)||x(\tau(t))-y(\tau(t))| +$$

$$+\int_{\phi^{-1}(T)}^{\infty} (s-t)q(s)|x^{\gamma}(\sigma(s))-y^{\gamma}(\sigma(s))|\,ds.$$

Using mean-value theorem applied on the function $f(\lambda)=\lambda^{\gamma}$, we see that

$$|(Ax)(t)-(Ay)(t)| \leq |p(t)||x(\tau(t))-y(\tau(t))| +$$

$$+\gamma\int_{\phi^{-1}(T)}^{\infty} sq(s)z^{\gamma-1}(\sigma(s))|x(\sigma(s))-y^{\gamma}(\sigma(s))|\,ds,$$

where $z(\sigma(s))$ lies between $x(\sigma(s))$ and $y(\sigma(s))$, $s > \phi^{-1}(T) \geq T$, that is, it satisfies the inequality $M \leq z(\sigma(s)) \leq 3M+2$. So, we have

$$|(Ax)(t)-(Ay)(t)| \leq |p(t)||x(\tau(t))-y(\tau(t))| +$$

$$+\gamma(3M+2)^{\gamma-1}\int_{\phi^{-1}(T)}^{\infty} sq(s)|x(\sigma(s))-y^{\gamma}(\sigma(s))|\,ds.$$

Hence

$$d(Ax,Ay) \leq |p(t)|d(x,y)+\gamma(3M+2)^{\gamma-1}d(x,y)\int_{\phi^{-1}(T)}^{\infty} sq(s)\,ds \leq$$

$$\leq d(x,y)\left[|p(t)|+\gamma(3M+2)^{\gamma-1}\int_{T}^{\infty} sq(s)ds\right] \leq$$

$$\leq d(x,y)\left[\frac{1}{3M+2}+\gamma(3M+2)^{\gamma-1}\frac{1}{\gamma(3M+2)^{\gamma}}\right] =$$

$$= \frac{2}{3M+2}d(x,y) = \alpha\, d(x,y), \qquad \alpha < 1$$

that is, A is a contraction mapping. Thus, by Banach contraction mapping theorem, A has a unique fixed point $y \in X$, that is

$$y(t) = 2M + 1 + p(t)y(\tau(t)) + h(t) - \int_t^\infty (s-t)q(s)y^\gamma(\sigma(s))ds, \qquad t \geq \phi^{-1}(T).$$

Twice differentiating the above integral equation, we see that $y(t)$ satisfies Eq. (NH_1) in $[\phi^{-1}(t), \infty)$ and $0 < M < y(t) < 3M + 2$ (because $y \in X$) imply that $y(t)$ is a nonoscillatory solution of (NH_1). This complete the proof of the theorem.

REMARK. Theorem 2.1. remains valid if we assume $\gamma = 1$.

For our next theorem, we assume the following conditions:

(5) $h(t)$ is oscillatory and there exist two sequences $\{s_n\}$ and $\{s'_n\}$ tending to infinity

that

$$h\{s_n\} = \inf\{h(t);\ t \geq s_n\}$$

and

$$h\{s'_n\} = \sup\{h(t);\ t \geq s'_n\}.$$

(6) $$0 \leq p(t) \leq p < 1$$

and there exists a positive constant K such that

(7) $\sigma'(t) \geq K$, $K > 0$ and $K \in R$ for every $t \geq 0$.

THEOREM 2.2. *Suppose that* (5), (6) *and* (7) *are satisfied. If*

(8) $$\int_0^\infty tq(t)dt = \infty,$$

then all solutions of Eq. (NH_1) *are either oscillatory or tend to zero as* $t \to \infty$.

PROOF. Let $y(t)$ be a solution of (NH_1). If $y(t)$ is oscillatory, then there is nothing to prove. Let $y(t)$ be nonoscillatory. Suppose that $y(t) > 0$, $y(\delta(t)) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_0 > 0$, set $x(t) = y(t) - p(t)y(\tau(t)) - h(t)$. Then ($NH_1$) can be written as

(9) $$x''(t)+q(t)y^{\gamma}(\sigma(t)) = 0.$$

Since $q(t)>0$, then $x''(t)<0$ for $t\geq t_0$. Thus, $x'(t)>0$ or <0 for $t\geq t_1\geq t_0$. If $x'(t)<0$ for $t\geq t_1$, then $x(t)<0$ for large t, say for $t\geq T_1>t_0$. From the assumptions on $h(t)$, it is clear that $h(t)$ is bounded and $h(t)\to 0$ as $t\to\infty$. Since $x(t)<0$ for $t\geq T_1$, then

$$y(t) < h(t)+p(t)y(\tau(t)) < h(t)+p\,y(\tau(t)).$$

Hence

$$\overline{\lim_{t\to\infty}}\, y(t) \leq \overline{\lim_{t\to\infty}}\, h(t)+p\,\overline{\lim_{t\to\infty}}\, y(\tau(t)) \leq p\,\overline{\lim_{t\to\infty}}\, y(t)$$

implies that $\lim_{t\to\infty} y(t)=0$. Suppose that $x'(t)>0$ for $t\geq t_1$. Then, there exists a $t_2\geq t_1$ such that $x(t)>0$ or <0 for $t\geq t_2$. Let $x(t)>0$ for $t\geq t_2$. From the assumption on $h(t)$ and the increasingness of $x(t)$, if follows that there exists a $t_3\geq t_2$ and a real $\beta_0>0$ such that $x(t)+h(t)\geq\beta_0$, that is, $y(t)-$ $-p(t)y(\tau(t))>\beta_0$ for $t\geq t_3$. This in turn implies that there exists a positive number, β such that

(10) $$y(t)-p(t)y(\tau(t)) \geq \beta x(t).$$

If this is not true, then there exists a sequence $<t_n>_{n=1}^{\infty}$, $t_n\to\infty$ as $n\to\infty$ such that

$$y(t_n)-p(t_n)y(\tau(t_n)) \geq \frac{1}{n}x(t_n).$$

So, $(1-\frac{1}{n})x(t_n)+h(t_n)\leq 0$. If $x(t_n)\to\infty$ as $n\to\infty$, then $h(t_n)$ will tend to $-\infty$, which contradicts the fact that $h(t)$ is bounded. If $x(t_n)$ tends to a constant as $n\to\infty$, then $y(t_n)-p(t_n)y(\tau(t_n))\to 0$ as $n\to\infty$, which is a contradiction to the fact that $y(t)-p(t)y(\tau(t))>\beta_0>0$. Hence from (10), we obtain

(11) $$y(t) \geq \beta x(t).$$

Set $r(t)=-tx'(t)/x^{\gamma}(\sigma(t))$. Than $r(t)<0$ for $t\geq t_2$ and

$$r'(t) = \frac{tq(t)y^{\gamma}(\sigma(t))}{x^{\gamma}(\sigma(t))}-\frac{x'(t)}{x^{\gamma}(\sigma(t))}+\frac{\gamma t x(t)x'(\sigma(t))\sigma'(t)}{x^{\gamma+1}(\sigma(t))}.$$

Using (11), the above inequality gives

$$r'(t) \geq \beta^{\gamma} t q(t) - \frac{1}{K} \frac{x'(\sigma(t))\sigma'(t)}{x^{\gamma}(\sigma(t))} + \frac{c}{t} r^{2(t)},$$

where $c = \gamma K x^{\gamma-1}(\sigma(t_3))$. Integrating the above inequality from t_3 to t, we obtain

$$r(t) \geq r(t_3) + \beta^{\gamma} \int_{t_3}^{t} s q(t) ds - \frac{1}{K} \frac{1}{-\gamma+1} \left[x^{-\gamma+1}(\sigma(t)) - x^{-\gamma+1}(\sigma(t_3)) \right] +$$

$$+ c \int_{t_3}^{t} \frac{1}{s} r^2(s) ds \geq r(t_3) + \beta^{\gamma} \int_{t_3}^{t} s q(s) ds - \frac{1}{K} \frac{1}{-\gamma+1} x^{-\gamma+1}(\sigma(t)).$$

Taking limit as $t \to \infty$, in view of the assumption (8), we obtain $r(t) \to \infty$ as $t \to \infty$, a contradiction. Hence $x(t) < 0$ for $t \geq t_2$. Then proceeding as in the first part of the theorem, we obtain $\lim_{t \to \infty} y(t) = 0$. The case $y(t) < 0$ for $t \geq t_0 \geq 0$ may be treated similarly. Thus, the theorem is proved.

COROLLARY 2.3. *Suppose that* (3), (5) *and* (7) *are satisfied. Then a necessary and sufficient condition for every solution of* (NH_1) *to be oscillatory or tend to zero is that* (8) *holds.*

REMARK. Our Corollary 2.3 extends the Corollary following Theorem 2 due to Nasr [6].

3. In this section, we consider Eq. (NH_2). We have the following theorems.

THEOREM 3.1. *Suppose that* (2) *and* (3) *are satisfied. If* (4) *holds, then there exists a nonoscillatory solution of* (NH_2).

The proof of Theorem 3.1 is quite similar to the proof of Theorem 2.1. However, for completeness, we retain the proof of the theorem.

PROOF OF THEOREM 3.1. It is possible to choose a T (>0) large enough such that

$$p(t) < \frac{1}{2(3M+2)} \quad \text{for} \quad t \geq T$$

and

$$\int_{T}^{\infty} t q(t) dt < \frac{1}{2\gamma(3M+2)^{\gamma}}.$$

Let $\phi(t)=\min\{\tau(t),\sigma(t)\}$ with $\phi(t)\to\infty$ as $t\to\infty$. Consider the complete metric space X consisting of continuous functions $y(t)$ defined on $[T,\infty)$ satisfying the inequalities

$$M \leq y(t) \leq 3M+2$$

endowed with the metric

$$d(x,y) = \sup_{T<t<\infty} |x(t)-y(t).$$

Let A be an operator defined by

$$(Ay)(t)=\begin{cases} 2M+2-p(t)y(\tau(t))+h(t)-\int_t^\infty (s-t)q(s)y^\gamma(\sigma(s))ds, & t\geq\phi^{-1}(T), \\ 2M+2-p(t)y(\tau(t))+h(t)-\int_{\phi^{-1}(T)}^\infty (s-t)q(s)y^\gamma(\sigma(s))ds, & T\leq t\leq\phi^{-1}(T), \end{cases}$$

Let $y\in X$. Then, $(Ay)(t)$ is continuous, $(Ay)(t)\leq 3M+2$ and

$$(Ay)(t) \geq 2M+2-p(t)y(\tau(t))+h(t)-\int_{\phi^{-1}(T)}^\infty sq(s)y^\gamma(\sigma(s))ds \geq$$

$$\geq 2M+2-\frac{1}{2(3M+2)}(3M+2)-M-(3M+2)^\gamma\frac{1}{2\gamma(3M+2)^\gamma}=$$

$$= M+\frac{3}{2}-\frac{1}{2\gamma} > M(\because\gamma>1)$$

Imply that $Ay\in X$. Now, we shall that A is a contraction mapping. For $x,y\in X$,

$$|(Ax)(t)-(Ay)(t)| \leq |p(t)||x(\tau(t))-y(\tau(t))| +$$

$$+\int_{\phi^{-1}(T)}^\infty (s-t)q(s)|x^\gamma(\sigma(s))-y^\gamma(\sigma(s))|ds \leq$$

$$\leq |p(t)|x(\tau(t))-y(\tau(t))| + \int_{\phi^{-1}(T)}^\infty sq(s)|x^\gamma(\sigma(s))-y^\gamma(\sigma(s))|ds,$$

witch further yields, using mean value theorem

$$|(Ax)(t)-(Ay)(t)| \le |p(t)||x(\tau(t))-y(\tau(t))| +$$
$$+\gamma(3M+2)^{\gamma-1} \int_{\phi^{-1}(T)}^{\infty} sq(s)|x(\sigma(s))-y(\sigma(s))|\,ds.$$

Hence

$$d(Ax,Ay) \le |p(t)|d(x,y)+\gamma(3M+2)^{\gamma-1}d(x,y)\int_{\phi^{-1}(T)}^{\infty} sq(s)ds \le$$
$$\le d(x,y)\left[\frac{1}{2(3M+2)}+\gamma(3M+2)^{\gamma-1}\frac{1}{2\gamma(3M+2)^{\gamma}}\right]=$$
$$=\frac{1}{(3M+2)}d(x,y).$$

Hence A is a contraction mapping. Consequently, by Banach contraction mapping theorem, A has a unique fixed point $y\in X$, that is

$$y(t) = 2M+2-p(t)y(\tau(t))+h(t)-\int_{t}^{\infty}(s-t)q(s)y^{\gamma}(\sigma(t))ds, \qquad t\ge\phi^{-1}(T).$$

Twice differentiation of the above integral equation shows that $y(t)$ satisfies (NH_2) in $[\phi^{-1}(T),\infty)$ with $0<M<y(t)<3M+2$. Hence the theorem is proved.

REMARK. Theorem 3.1 remains valid for $\gamma=1$.

THEOREM 3.2. *Assume that* (5), (6) and (7) *hold. Then the condition* (8) *is sufficient for all solutions of Eq.* (NH_2) *to be oscillatory.*

PROOF. If possible suppose that $y(t)$ is a nonoscillatory solution of (NH_2). Let $y(t)>0$, $y(\tau(t))>0$ and $y(\sigma(t))>0$ for $t\ge t_0>0$. Set $x(t)=y(t)+$ $+p(t)y(\tau(t))-h(t)$. Then Eq. (NH_2) reduces to

$$x''(t) + q(t)y^{\gamma}(\sigma(t)) = 0. \tag{12}$$

Hence $x''(t)<0$ for $t\ge t_0$ and $x(t)$ is nonoscillatory for $t\ge t_1\ge t_0$. Since $h(t)$ is oscillatory, then $x(t)>0$ and $t\ge t_1$. This ultimately yields that $x'(t)>0$ for large t, say for $t\ge t_2\ge t_1$. We claim that $y(t)$ is bounded. If not, $y(t)$ is unbounded. Then $x(t)$ is unbounded. Consequently, $\lim_{t\to\infty}x(t)=\infty$. It is possible to find $t_3>t_2$ such that for $t\ge t_3$, we have

$$\begin{aligned}(1-p)x(t) &\le x(t)-p(t)x(\tau(t)) = \\ &= y(t)-p(t)p(\tau(t))y(\tau(\tau(t)))-h(t)+p(t)h(\tau(t)) < \\ &< y(t)+|h(t)|+|h(\tau(t))| < y(t)+\varepsilon,\end{aligned}$$

where $\varepsilon > 0$. Setting $z(t) = (1-p)x(t)-\varepsilon$, we see that $z(t) < y(t)$ for $t \ge t_3$, $\lim_{t\to\infty} z(t) = \infty$ and (12) may be written as

(13) $$z''(t) + q_1(z)z^{\gamma}(\sigma(t)) = 0$$

where

$$q_1(t) = \frac{(1-p)q(t)y^{\gamma}(\sigma(t))}{((1-p)x(\sigma(t))-\varepsilon)^{\gamma}}.$$

Since $z(t) < y(t)$ and $z(t) > 0$ for large t, then there exists $t_4 > t_3$ such that for $t \ge t_4$, $q_1(t) \ge (1-p)q(t)$ and from (8), we obtain

$$\int_0^{\infty} tq_1(t)dt = \infty.$$

Then by Lemma 1.1, all solutions Eq. (13) are oscillatory, which is a contradiction because $z(t) > 0$ is a solution of (13). Thus $y(t)$ is bounded and hence $x(t)$ is bounded. As $x(t) > 0$, $x'(t) > 0$ for $t \ge t_2$, then $\lim_{t\to\infty} x(t) = \alpha > 0$ exists. Choosing $0 < \varepsilon < (1-p)\alpha$, setting $w(t) = (1-p)x(t)-\varepsilon$ and proceding as above, we obtain $w(t) < y(t)$ and

(14) $$w''(t) + q_2(t)w^{\gamma}(\sigma(t)) = 0$$

for $t \ge t_3 \ge t_2$, where

$$q_2(t) = \frac{(1-p)q(t)y^{\gamma}(\sigma(t))}{((1-p)x(t)-\varepsilon)^{\gamma}}.$$

Now $w'(t) = (1-p)x'(t) > 0$, $w''(t) = (1-p)x''(t) < 0$ implies that $w(t)$ is nonoscillatory. If $w(t) < 0$ for $t \ge t_5 \ge t_2$, then $\lim_{t\to\infty} w(t) = \beta$, $\beta \le 0$ exists. Taking limit as $t \to \infty$ in $w(t) = (1-p)x(t)-\varepsilon$, we obtain $0 \ge (1-p)\alpha - \varepsilon > 0$, a contradiction. So $w(t) > 0$ for $t \ge t_5$. It is possible to find $t_6 > t_5$ such that $q_2(t) \ge (1-p)q(t)$ for $t \ge t_6$ and hence

$$\int_{t_6}^{\infty} q_2(t)dt = \infty.$$

By Lemma 1.1, all solutions of Eq. (14) are oscillatory, which is a contradiction. This case $y(t)<0$ may be treated similarly. This completes the proof of the theorem.

COROLLARY 3.3. *Suppose that* (3), (5) *and* (7) *are satisfied. Then a necessary and sufficient condition for all solutions of* (NH_2) *to be oscillatory is that* (8) *holds.*

REMARK. Corollary 3.3 is a generalization of the corollary following Theorem 2 due to Nasr [6].

REFERENCES

[1] D.D. Bainov, D.P. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, Adam Hilger, Bristol 1991.

[2] P. Das, Oscillations and asymptotic behaviour of solutions for second order neutral delay differential equations, *J. Indian Math. Soc.* 60(1994), 159-170.

[3] P. Das, N. Mishra, A necessary and sufficient condition for the solutions of a functional differential equation to be oscillatory or tend to zero, *J. Math. Anal. Appl.* 204(1997), 78-87.

[4] J. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equations*, Clarendon Press, Oxford 1991.

[5] J. Hale, *Functional Differential* Equations, Springer-Verlag, New York 1991.

[6] A.H. Nasr, Necessary and sufficient conditions for the oscillation of forced nonlinear second order differential equations with delayed argument, *J. Math. Anal. Appl.* 212(1997), 51-59.

[7] N. Parhi, P.K. Mohanty, Oscillation of solutions of forced neutral differential equations of n-th order, *Czechoslovak Math. J.* 45(1995), 413-433.

[8] N. Parhi, P.K. Mohanty, On oscillatory forced neutral differential equations of second order, *J. Indian Math. Soc.* 61(1995), 241-254.

([1]Department of Mathematics, Jagannath Institute for Technology and Management, Paralakhemundi – 761 211, Orissa, India

[2]Department of Mathematics, Berhampur University, Berhampur – 760 007, India)

Received on 22.05.2000 and, in revised form, on 13.08.2003.