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SEQUENCES WHICH SATISFY A LOGARITHMIC LINEAR INEQUALITY

ABSTRACT: In this note we discuss the boundedness and convergence of a sequence which satisfies the following logarithmic linear inequality

$$a_{n+m} \leq a_{n+m-1}^{k_m} \dots a_n^{k_1},$$

where $k_1 + k_2 + \dots + k_m = 1$. We focus our attention especially to the case $n = 2$. Also we describe a situation where this inequality occurs naturally.

KEY WORDS: sequence, logarithmic linear inequality, boundedness.

1. INTRODUCTION

Real sequences (a_n) that satisfy the linear inequality

$$a_{n+m} \leq \sum_{i=1}^{m} k_i a_{n+m-i}, \qquad n \in N \cup \{0\},$$

where $k_i \in R$ are studied in [2], [4], [5], [7] and [8]. Sequences satisfying nonlinear inequalities or systems of inequalities with some applications has been considered in [1], [9-13]. In [6] Istratescu posed the following question: If (a_n) is a sequence of positive numbers such that

(1) $$a_n \geq a_{n+1}^{\alpha} a_{n-1}^{\alpha}, \quad a_0 = 1, \quad \alpha, \beta \geq 0, \quad \alpha + \beta = 1,$$

is then $(a_n^{1/n})$ convergent?

For the case $\alpha = \beta = 1/2$ it follows from Mclaurin theorem that the seuence $(a_n^{1/n})$ is nonincreasing and thus convergent. This theorem inspired Istratescu to ask the above question.

It is easy to show that the answer to the question is negative. Indeed, let $a_n = e^{n^2}$ then

$$a_n - a_{n+1}^{\alpha} a_{n-1}^{\beta} = e^{n^2}(1 - e^{2n(2\alpha-1)+1}).$$

Hence for $\alpha < 1/2$ and sufficiently large n, for example $n \geq n_0$, we have $a_n - a_{n+1}^{\alpha} a_{n-1}^{\beta} > 0$. Thus, the sequence defined by $a_n = e^{(n+n_0)^2 - n_0^2}$ for $n = 0,1,2,\dots$

satisfies the inequality (1). On the other hand $\lim_{n\to\infty} a_n^{1/n} = +\infty$ i.e. the sequence $(a_n^{1/n})$ is not convergent.

Let $\alpha = 0$. Then the sequence (a_n) is nondecreasing and consequently there exists finite or infinite $\lim_{n\to\infty} a_n = a$. If a is finite then $\lim_{n\to\infty} a_n^{1/n} = 1$. Let $a = +\infty$, then we can have the following three cases.

1. $\lim_{n\to\infty} a_n^{1/n} = q$, $q \ge 1$ is finite, for example $a_n = q^n$; $a_n = n$;
2. $\lim_{n\to\infty} a_n^{1/n} = +\infty$, for example $a_n = n^n$;
3. $\lim_{n\to\infty} a_n^{1/n}$ does not exist, for example let

$$(b_n) = (\ln a_n) = \left(2,2,6,6,6,\underbrace{12,12,\ldots,12}_{6},\underbrace{24,24,\ldots,24}_{12},\ldots \right)$$

then

$$\frac{b_n}{n} = \left(2,1,2,\frac{6}{4},\frac{6}{5},2,\frac{12}{7},\frac{12}{8},\frac{12}{9},\frac{12}{10},\frac{12}{11},2,\frac{24}{13},\frac{24}{14},\ldots,\frac{24}{23},2,\ldots \right).$$

Hence $\limsup b_n/n = 2$ and $\liminf b_n/n = 1$. From this we have $\limsup a_n/n = e^2$ and $\liminf a_n/n = e$.

Let $\beta = 0$, then the sequence (a_n) is nonincreasing and consequently there exists finite $\lim_{n\to\infty} a_n = a \ge 0$. If $a > 0$, then $\lim_{n\to\infty} a_n^{1/n} = 1$. Let $a = 0$, then we can have the following two cases.

1. $\lim_{n\to\infty} a_n^{1/n} = q$, $q \in [0,1]$ is finite, for example $a_n = q^n$; $a_n = 1/n$; $a_n = 1/n^n$;
2. $\lim_{n\to\infty} a_n^{1/n}$ does not exist, for example, let

$$(b_n) = (\ln a_n) = \left(-2,-2,-6,-6,-6,\underbrace{-12,\ldots,-12}_{6},\underbrace{-24,-24,\ldots,-24}_{12},\ldots \right)$$

then

$$\frac{b_n}{n} = \left(-2,-1,-2,-\frac{6}{4},-\frac{6}{5},-2,-\frac{12}{7},-\frac{12}{8},-\frac{12}{9},-\frac{12}{10},-\frac{12}{11},-2,-\frac{24}{13},\ldots \right).$$

Hence $\liminf b_n/n = -2$ and $\limsup b_n/n = -1$. From this we have $\liminf a_n/n = e^{-2}$ and $\limsup a_n/n = e^{-1}$.

One of our goals in this paper is to show that the case $\alpha = \beta = 1/2$ is unique i.e. that in the other cases the situation is similar to the one in the case $\alpha = 0$ or

$\beta=0$. We shall show it in section 2. Also in section 2 we prove several sufficient criteria for boundedness from above for sequences which satisfy a logarithimic linear inequality.

In what follows we consider the sequences which satisfy the following difference inequality

(2) $$a_{n+1} \le a_n^{\gamma} a_{n-1}^{1-\gamma}, \qquad \gamma>0.$$

Note that the inequality (2) follows from (1) by the change $a=1/\gamma$. Also, note that we may assume $a_0=1$, since in the contrary we can consider the sequence $b_n=a_n/a_0$.

In the last section we describe an example of sequences which satisfy in a natural way a logarithmic linear inequality.

2. MAIN RESULTS

THEOREM 1. *Let* $k_i>0$ $(i=1,\dots,m)$, $k_1+\dots+k_m=1$, *and let* (a_n) *be a positive sequence of real numbers which satisfies the inequality*

(3) $$a_{n+m} \le a_{n+m-1}^{k_m} \dots a_n^{k_1}.$$

Then (a_n) *is bounded from above.*

PROOF. Appling the change $y_n=\ln a_n$ we have that

(4) $$y_{n+m} \le k_m y_{n+m-1}+\dots+k_1 y_n, \qquad n\in N_0.$$

Let (z_n) be a sequence defined by

$$z_n = \max\{y_n,\dots,y_{n+m-1}\} \quad \text{for} \quad n\in N_0.$$

By (4) we $y_{n+m} \le z_n$, $n\in N_0$ and consequently $z_{n+1}\le z_n$. Hence the result follows.

THEOREM 2. *Let* (a_n) *be a positive sequence of real numbers which satisfies the inequality* (2) *for* $\gamma\in(0,2)$. *Then* (a_n) *is bounded from above.*

PROOF. If $\gamma\in(0,1)$, the theorem is a direct consequence of Theorem 1.

Let $\gamma\in(0,2)$. Applying the change $y_n=\ln a_n$ we have that

(5) $$y_{n+m} \le \gamma y_n+(1-\gamma)y_{n-1}, \qquad n\in N_0.$$

Let us write (5) in the from

(6) $$y_{n+1}-(\gamma-1)y_n \le y_n-(\gamma-1)y_{n-1}.$$

Set

(7) $$c_n = y_{n+1}-(\gamma-1)y_n.$$

From (6) we have that (c_n) is nonincreasing. Therefore the limit $\lim_{n\to\infty} c_n$ is finite or $-\infty$.

From (7), we obtain

(8) $$y_n = c_{n-1}+c_{n-2}q+\ldots+c_0 q^{n-1}+y_0 q^n,$$

where $q=\gamma-1$.

Since (c_n) is bounded from above with c_0, we have

$$y_n \le c_0(1+q+q^2+\ldots+q^{n-1})+|y_0|q^n<\frac{c_0}{1-q}+|y_0|<+\infty,$$

as desired.

For $\gamma=1$ the result is trivial.

One can easily prove the following (see, [8])

LEMMA 1. *Let α_i $(i=0,\ldots,n-1)$ be real, $\sum_{i=0}^{n-1}\alpha_i=1$, $P_n(z)=z^n-\alpha_{n-1}\times z^{n-1}-\Lambda-\alpha_0$ and $P_n(q)=0$ for some $q\in C\setminus\{1\}$. Then $P_n(z)=(z-q)P_{n-1}(z)$ where $P_{n-1}(z)=z^{n-1}-\beta_{n-2}z^{n-2}-\Lambda-\beta_0$ is such that $\sum_{i=0}^{n-2}\beta_i=1$.*

The following Theorem is a natural generalization of Theorem 2 in the case $\gamma\in(1,2)$.

THEOREM 3. *Let (a_n) be a positive sequence of real numbers which satisfies the inequality (3), such that all zeros of the polynomial $P_m(z)=z^m-k_m z^{m-1}-\Lambda-k_1$, which are different from 1 belong to the interval $(0,1)$ and if $z=1$ is a zero of $P_m(z)$ it is simple. Then (a_n) is bounded from above.*

PROOF. We will prove this theorem by indunction. If $m=2$ and $z=1$ is a zero of $P_2(z)$, our statement is a consequence of Theorem 2. Let $p,q\in(0,1)$ are zeros of $P_2(z)$. Thus, applying the chang $y_n=\ln a_n$ we obtain

$$y_{n+1}-qy_n \le p(y_n-qy_{n-1}), \qquad n\in N.$$

Let $d_n=y_n-qy_{n-1}$ and $d_{n+1}-pd_n=c_n$. Then we have

$$d_n = c_{n-1} + c_{n-2}p + \Lambda + c_0 p^{n-1} + y_0 p^n, \qquad n \in N.$$

Since $c_n \leq 0$ and $p \in (0,1)$ we obtain $d_n \leq | y_0 |$ and consequently

$$y_n - q y_{n-1} \leq | y_0 |, \qquad n \in N.$$

Now we can finich the proof in this case as in Theorem 2.

Let $m > 2$ and $z = 1$ be a zero of $P_m(z)$. Applying the change $y_n = \ln a_n$ we have that

$$y_{n+m} \leq k_m y_{n+m-1} + \Lambda\ k_1 y_n, \qquad n \in N_0.$$

Let $q,\ q \neq 1$ be a zero of $P_m(z)$ and $z_n = y_{n+1} - q y_n$. Then

$$z_{n+m-1} \leq \sum_{i=0}^{m-2} \beta_i z_{n+i},$$

where $\beta_i = 0, \ldots, m-2$, are coefficients of the polynomial $P_{m-1}(z)$ from the Lemma 1. Since $\sum_{i=0}^{m-2} \beta_i = 1$, by the inductive hypothesis we have that the sequence (z_n) is bounded from above i.e. there exists $M \in R$ such that

$$y_{n+1} - q y_n \leq M, \qquad n \in N.$$

Now we finish the proof as in Theorem 2.

When $z = 1$ is not zero of $P_m(z)$ the proof is similar and we omit it.

The following interesting theorem which is closely related to Theorem 3 was established in [3].

THEOREM A. *Consider the difference equation* $a_{n+1} = g(a_n) f(a_{n-1})$, $n \in N \cup \{0\}$ *with nonnegative initial conditions where the functions f and g satisfy the following hypotheses:*

a) $f \in C([0,\infty),(0,\infty))$, $g \in C[[0,\infty),[0,\infty)]$;

b) g is increasing and f is nonincreasing;

c) The function $G(x) = f(x) g(x)$, *has the property that there exist points* $\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_m$ *such that* $0 \leq \overline{x}_0 < \overline{x}_1 < \Lambda < \overline{x}_m < \infty$ *and such that for each* $i = 0, \ldots, m$, $G(\overline{x}_i) = \overline{x}_i$ *and* $G(x) \neq x$ *for* $x \in (\overline{x}_i, \overline{x}_{i+1})$;

d) There exist numbers $L, p, q \in [0,\infty)$ and $A, B \in (0,\infty)$ such that

$$g(x) \leq A x^p \quad and \quad f(x) \leq \frac{B}{x^q} \quad for \quad x \geq L,$$

where either $p = 0$ *or* $0 < p^2 < 4q$.

Then every solution of given equation is bounded from above by a positive constant.

It is an interesting question whether the condition $f \in C([0,\infty),(0,\infty))$, can be replaced by $f \in C((0,\infty),(0,\infty))$?

Let us consider the following difference inequality

$$a_{n+2} \le a_n^{-r^2}, \qquad n \in N.$$

Let $a_{2n} = c_n$, then we have $c_{n+1} \le c_n^{-r^2}$, $n \in N$. It is easy to see that the sequence

$$(c_n) = \left(e, e^{-1/r^4}, e^{1/r^2}, e^{-1/r^6}, e^{1/r^4}, e^{-1/r^8}, e^{1/r^6}, e^{-1/r^{10}}, \dots \right),$$

satisfies the last inequality and is unbounded.

Hence this example shows that the condition $f \in C([0,\infty),(0,\infty))$, can not be replaced by $f \in C((0,\infty),(0,\infty))$.

The following theorem is the main result in this paper.

THEOREM 4. *Let* $\gamma \in (0,1) \cup (1,2)$. *Then*

(*a*) *there are positive sequence* (a_n) *which satisfy the inequality* (2) *and such that* $\lim_{n\to\infty} a_n^{1/n}$ *exists and its finite*;

(*b*) *there are positive sequence* (a_n) *which satisfy the inequality* (2) *such that the sequence* $(a_n^{1/n})$ *is bounded and divergent.*

PROOF. Applying the change $y_n = \ln a_n$ we obtain (5) and consequently (6). Set $c_n = y_{n+1} - (\gamma - 1) y_n$. From (6) we have that (c_n) is nonincreasing, therefore $\lim_{n\to\infty} c_n$ is finite or $-\infty$.

Let $\lim_{n\to\infty} c_n = c > -\infty$. From (6), we obtain

$$(9) \qquad y_n = c_{n-1} + c_{n-2} q + \Lambda + c_0 q^{n-1} + y_0 q^n =$$
$$= c(1 + q + q^2 + \Lambda + q^{n-1}) + d_{n-1} + d_{n-2} q + \Lambda + d_0 q^{n-1} + y_0 q^n,$$

where $q = \gamma - 1$.

Since $\lim_{n\to\infty} d_n = 0$ and $0 < |q| < 1$, clearly

$$\lim_{n\to\infty} (d_{n-1} + d_{n-2} q + \Lambda + d_0 q^{n-1} + y_0 q^n) = 0.$$

Thus we have

$$\lim_{n\to\infty} y_n = \frac{c}{1-q}$$

and consequently

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} e^{y_n/n} = 1.$$

The sequence $a_n = \exp(c\frac{(\gamma-1)^n-1}{\gamma})$, $n \in N$, is an example of such a situation.

Let $\lim_{n\to\infty} c_n = -\infty$, $|q| \in (0,1)$. Set $q = 1/r$. It is clear that $|r| > 1$. We can write (8) in the from

$$y_n = \frac{y_0 + c_0 r + \cdots + c_{n-2} r^{n-1} + c_{n-1} r^n}{r^n}.$$

We may suppose that $c_n < 0$, $n \in N_0$. Let

$$(c_n) = \left(-3,-3,-6,-6-6,\underbrace{-12,-12,\ldots,-12}_{6},\underbrace{-24,-24,\ldots,-24}_{12},\ldots \right).$$

Let us show that $\lim_{n\to\infty} y_n/n$ does not exist. From this we immediately obtain that $\lim_{n\to\infty} a_n^{1/n}$ does not exist, as desired. For the sake of simplicity we shall consider $\lim_{n\to\infty} -y_n/n$. Also we may assume $y_0 = 0$.

First, let us calculate $\lim_{n\to\infty} -y_{3\cdot 2^{k-1}}/3\cdot 2^{k-1} r^{3\cdot 2^{k-1}}$. Note that

$$\begin{aligned} -y_{3\cdot 2^{k-1}} r^{3\cdot 2^{k-1}} &= 3(r+r^2) + 3\cdot 2(r^3 + r^4 + r^5) + \cdots + \\ &\quad + 3\cdot 2^{k-1}(r^{3\cdot 2^{k-2}} + \cdots + r^{3\cdot 2^{k-2}-1}) + 3\cdot 2^k r^{3\cdot 2^{k-1}} = \\ &= 3r\frac{r^2-1}{r-1} + 3\cdot 2r^3\frac{r^3-1}{r-1} + \cdots + 3\cdot 2^{k-1} r^{3\cdot 2^{k-2}}\frac{r^{3\cdot 2^{k-2}}-1}{r-1} + 3\cdot 2^k r^{3\cdot 2^{k-1}} = \\ &= \frac{3\cdot 2^{k-1} r^{3\cdot 2^{k-1}}(2r-1) - (3r + 3r^3 + 3\cdot 2r^6 + \cdots + 3\cdot 2^{k-2} r^{3\cdot 2^{k-2}})}{r-1}. \end{aligned}$$

Thus we have

$$\begin{aligned} \lim_{k\to k} -\frac{y_{3\cdot 2^{k-1}}}{3\cdot 2^{k-1}} &= \frac{2r-1}{r-1} - \lim_{k\to\infty} \frac{3r + 3r^3 + 3\cdot 2r^6 + \cdots + 3\cdot 2^{k-2} r^{3\cdot 2^{k-2}}}{3\cdot 2^{k-1} r^{3\cdot 2^{k-1}}(r-1)} = \\ &= \frac{2r-1}{r-1} - \frac{1}{r-1}\lim_{k\to\infty}\frac{3\cdot 2^{k-2} r^{3\cdot 2^{k-2}}}{3\cdot 2^{k-1} r^{3\cdot 2^{k-1}} - 3\cdot 2^{k-2} r^{3\cdot 2^{k-2}}} = \frac{2r-1}{r-1}. \end{aligned}$$

On the other hand

$$\begin{aligned} -\frac{y_{3\cdot 2^{k-1}+1}}{3\cdot 2^{k-1}+1} &= -\frac{y_{3\cdot 2^{k-1}}}{2\cdot 2^{k-1} r}\cdot\frac{3\cdot 2^{k-1}}{3\cdot 2^{k-1}+1} + \frac{3\cdot 2^k}{(3\cdot 2^{k-1}+1)} \to \\ &\to \frac{2r^2-1}{r(r-1)}, \quad \text{as} \quad k\to\infty. \end{aligned}$$

Since

$$\frac{2r^2-1}{r(r-1)} \neq \frac{2r-1}{r-1}, \quad \text{for} \quad r \neq 1,$$

we have that $\lim\limits_{n\to\infty} \frac{y_n}{n}$ does not exist.

REMARK 1. Note that by Theorem 1 every sequence (a_n) in Theorem 4 must be bounded.

For readers who are interesting in this area we offer the following conjecture.

CONJECTURE 1. *Let (a_n) be a positive sequence of real numbers which satisfy the inequality* (2) *for $\gamma > 2$. Then each of the following three cases is possible:*

(a) *there exists finite* $\lim\limits_{n\to\infty} a_n^{1/n}$;

(b) $\lim\limits_{n\to\infty} a_n^{1/n} = +\infty$;

(c) the *sequence* $(a_n^{1/n})$ *is bounded and divergent.*

In fact, form all above mentioned we only need to show that there is a sequence (a_n) which satisfy (2) for $\gamma > 2$, such that the sequence $(a_n^{1/n})$ is bounded and divergent.

3. APPLICATIONS

In this section we describe a situation of sequences which satisfy inequality (2) in a natural way. Let T denote a linear continuous mapping from the Hilbert space H into itself.

DEFINICTION 1. *An operator T is said to be of class (N,l,k) if for all $x \in H$, $\|x\| = 1$ we have the following inequality*

$$\|T^l x\| \geq \|Tx\|^k.$$

For $k = l = 2$ this is called class N.

DEFINITION 2. *An operator T defined on Banach space X, is called of class (c) if for each $x \in X$ the sequence $(\|T^n x\|^{1/n})_{n\in N}$ is convergent.*

The following theorem was established in [6].

THEOREM B. *Every operator of class (N) is of class (c).*

The author poses the following question: is the class (c) larger than class (N)? In order to answer this question he noyes that if T, S are operators of the class (N)

and $TS = ST$ then it is known that TS is not necessarily of class (*N*). We may suppose that $\| T \| \leq 1$ and $\| S \| \leq 1$. Further, we have

$$\| (TS)^2 x \| = \left\| T^2 \left(\frac{S^2 x}{\| S^2 x \|} \right) \right\| \| S^2 x \| \geq \left\| T \left(\frac{S^2 x}{\| S^2 x \|} \right) \right\|^2 \| S^2 x \| =$$

$$= \left\| S^2 \left(\frac{Tx}{\| Tx \|} \right) \right\|^2 \frac{\| Tx \|^2}{\| S^2 x \|} \geq \frac{\| TSx \|^4}{\| S^2 x \| \| Tx \|^2},$$

for all $x \in X$, $\| x \| = 1$. This implies that TS is of the class $(N,2,4)$.

Let $T_1 = TS$, then

$$\| T_1^{n+1} x \| = \left\| T_1^2 \left(\frac{T_1^{n-1} x}{\| T_1^{n-1} x \|} \right) \right\| \| T_1^{n-1} x \| \geq \left\| T_1 \left(\frac{T_1^{n-1} x}{\| T_1^{n-1} x \|} \right) \right\|^2 \| T_1^{n-1} x \| =$$

$$= \frac{\| T_1^n x \|^4}{\| T_1^{n-1} x \|^3}$$

i.e.

(10) $$\| T_1^{n+1} x \| \cdot \| T_1^{n-1} x \|^3 \geq \| T_1^n x \|^4 .$$

Let $a_n = 1 / \| T_1^n x \|$, then we can write (10) in the following form

(11) $$a_{n+1} \leq a_n^4 \, a_{n-1}^{-3} .$$

Istratesku hoped that for such sequences $(a_n^{1/n})$ are convergent which will mean that T_1 is of the class (*c*) and so (*c*) will be larger than the class (*N*).

Unfortunately we saw that there exist sequences which satisfy inequality (11) and diverge. Note that $a_0 = 1$ and

$$a_1^{-1} = \| T_1 x \| = \| TSx \| \leq \| T \| \| S \| \| x \|.$$

Since we may suppose that $\| T \| \leq 1$ and $\| S \| \leq 1$ we have $a_1 \geq 1$.

LEMMA 2. *Let operator T be of class (N,l,k). Then*

$$\| T^{n+1} x \| \cdot \| T^{n-l+1} x \|^{k-1} \geq \| T^{n-l+2} x \|^k ,$$

for every $x \in X$.

PROOF. The proof is similar to the previous proof concerning T_1.

If we set $a_n = 1 / \| T^n x \|$ in (11) we obtain

$$a_{n+1} \leq a_{n-l+2}^k \, a_{n-l+1}^{1-k}$$

In particular for $l = 2$ we obtain the inequality (2) where $\gamma = k$.

THEOREM 5. *Let T be operator of class* $(N,2,k)$, *such that* $\|Tx\| \geq 1$ *for all* $x \in X$, $\|x\|=1$. *Then T is of class* (c).

PROOF. By lemma 2 we have

$$\left(\frac{a_{n-1}}{a_n}\right)^{k-1} \leq \frac{a_n}{a_{n+1}},$$

where $a_n = 1/\|T^n x\|$. From conditions of the theorem we have $a_0 = 1$ and $a_1 \leq 1$, for all $x \in H$, $\|x\|=1$. Thus we have

$$\frac{a_n}{a_{n+1}} \geq \frac{a_{n-1}}{a_n} \geq 1.$$

Therefore

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and is finite.

By a well known theorem

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} a_n^{1/n}, \qquad n \in N,$$

and the result follows.

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