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Nr 34 2004

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# GREEN FUNCTION AND SOME INHOMOGENEOUS PROBLEM FOR THE EQUATION $P^2u(x,t) = f(x,t)$

ABSTRACT: The subject of the paper is a construction of a classical solution to a biparabolic problem with initial conditions and boundary-value one.

KEY WORDS: biparabolic problem, inhomogeneous problem, Green function.

## 1. INTRODUCTION

The subject of the paper is a construction of a clssical solution to the equation

(1) $$P^2u(x,t) = f(x,t), \quad P = D_x^2 - D_t, \quad P^2 = P(P),$$

in the domain

$$D = \{(x,t) : x \in J = (0,1),\ t \in (0,T]\},$$

satisfying the initial conditions

(2) $$u(x,0) = f_0(x), \quad x \in J,$$

(3) $$Pu(x,0) = f_1(x), \quad x \in J,$$

and the boundary-value condition of Riquier type

(4) $$u(0,t) = F_1(t), \quad t \in [0,T],$$

(5) $$Pu(0,t) = F_3(t), \quad t \in [0,T],$$

(6) $$u(1,t) = F_2(t), \quad t \in [0,T],$$

(7) $$Pu(1,t) = F_4(t), \quad t \in [0,T],$$

the functions $f$, $f_0$, $f_1$, $F_i$ $(i = 1,2,3,4)$ are given and $u$ is the unknown one.

We shall solve the last problems by the suitable Green functions, and Green potentials.

## 2. GREEN FUNCTION

Consider the parabolic equation

(8) $$Pu(x,t) = 0, \qquad (x,t) \in D.$$

Let us consider the function $U$ given by the formula

$$U(x,t,y,s) = A(t-s)^{-1/2}\exp(B(t,s)(x-y)^2), \qquad (x,y)\in \bar{J}\times\bar{J},$$

$$A = (2\sqrt{\pi})^{-1}, \qquad B(t,s) = (-4(t-s))^{-1},$$

in the domain

$$D_1 = \{(x,t,y,s): 0\le s<t\le T, (x,y)\in J\times J\}.$$

Consider the function $G$ defined by the formulas

$$G(x,t,y,s) = U(x,t,y,s) - K_1(x,t,y,s) - \\ - K_3(x,t,y,s) + K_2(x,t,y,s) + K_4(x,t,y,s)$$

where

(9) $$K_1(x,t,y,s) = A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(-x-2n-y)^2),$$

(10) $$K_2(x,t,y,s) = A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(-x+2n+2-y)^2),$$

(11) $$K_3(x,t,y,s) = A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(x+2n-y)^2),$$

(12) $$K_4(x,t,y,s) = A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(x-2n-y)^2).$$

## 3. GREEN FUNCTION $G$ TO EQUATION (8) AND TO DIRICHLET BOUNDARY-VALUE CONDITIONS

In the sequel, by $C$, $C_i$ we shall denote suitable positive constants. By [2], the function $G$ is the Green function to the equation

$$P_{x,t}G(x,t,y,s) = 0,$$

in the domain $D_1$, satisfying Dirichlet boundary-value conditions

$$G(0,t,y,s) = G(1,t,y,s) = 0.$$

From [2], we obtain the formulas

$$D_yG(x,t,0,s) = \sum_{i=1}^{3} Q_i(x,t,s),$$

with

$$Q_1(x,t,s) = A(t-s)^{-3/2}x\exp(B(t,s)x^2),$$

$$(I)_1 \quad Q_2(x,t,s) = -A(t-s)^{-3/2}\sum_{n=0}^{\infty}(x-2n-2)\exp(B(t,s)(x-2n-2)^2),$$

$$Q_3(x,t,s) = -A(t-s)^{-3/2}\sum_{n=0}^{\infty}(-x+2n+2)\exp(B(t,s)(-x+2n+2)^2).$$

Similarly, we obtain the formula

$$D_yG(x,t,1,s) = \sum_{i=1}^{3} R_i(x,t,s),$$

with

$$R_1(x,t,s) = -A(t-s)^{-3/2}(x-1)\exp(B(t,s)(x-1)^2),$$

$$(II)_1 \quad R_2(x,t,s) = -A(t-s)^{-3/2}\sum_{n=0}^{\infty}(x+2n+1)\exp(B(t,s)(x+2n+1)^2),$$

$$R_3(x,t,s) = -A(t-s)^{-3/2}\sum_{n=0}^{\infty}(x-2n-1)\exp(B(t,s)(x-2n-1)^2).$$

## 4. GREEN POTENTIALS $J_1(x,t)$ AND $J_2(x,t)$

Let

$$J_1(x,t) = \int_0^t D_yG(x,t,0,s)F_1(s)ds = \sum_{i=1}^{3} J_1^i(x,t),$$

with

$$J_1^i(x,t) = \int_0^t Q_i(x,t,s)F_1(s)ds \qquad (i=1,2,3)$$

and let

$$J_2^1(x,t) = \int_0^1 (D_yG(x,t,1,s)F_1(s)ds = \sum_{i=1}^{3} J_2^i(x,t),$$

with

$$J_2^1(x,t) = \sum_0^t R_i(x,t,s)F_2(s)ds \qquad (i=1,2,3)\,.$$

Let $(k)$ denote the class of the functions $F \in C([0,T])$, such that $F(0)=F(1)=0$.

**LEMMA 1.** *If* $F_i \in (k)$ $(i=1,2)$, *then:*

$$(III)_1 \qquad PJ_i^j(x,t) = 0 \quad (i=1,2, \quad j=1,2,3)$$

$1^0$ $J_1^1(x,t) \to F_1(t)$, as $(x,t) \to (0,t)$, $t \in (0,T]$,

$2^0$ $J_1^2(x,t) + J_1^3(x,t) \to 0$, as $(x,t) \to (0,t)$, $t \in (0,T)$,

$3^0$ $J_1(x,t) \to 0$, as $(x,t) \to (1,t)$, $t \in (0,T]$,

$4^0$ $J_2^1(x,t) \to F_2(t)$, as $(x,t) \to (1,t)$, $t \in (0,T]$,

$5^0$ $J_2^1(x,t) + J_2^3(x,t) \to 0$, as $(x,t) \to (1,t)$, $t \in (0,T]$,

$6^0$ $J_2(x,t) \to 0$, as $(x,t) \to (0,t)$, $t \in (0,T]$.

**PROOF.** By [2], and by properties of the potentials of the double layer we obtain $1^0, 4^0$. By formulas $(I)_1$, $(II)_1$, we get $2^0, 3^0$, $5^0, 6^0$ and $(III)_1$.

## 5. CONSTRUCTION OF THE SOLUTION TO PROBLEM (1)-(7)

To solve problem (1)-(7) we change the biparabolic problem to the problems with new unknown functions $U$, $V$, $W$ satisfying the system of differential limit problems:

$$Pu(x,t) = 0, \qquad (x,t) \in D,$$

$$(I) \qquad u(x,0) = f_0(x),\ x \in J,\ u(0,t) = F_1(t),\ u(1,t) = F_2(t),\ t \in [0,T],$$

and

$$P^2V(x,t) = 0, \qquad (x,t) \in D,$$

$$(II) \qquad V(0,t) = V(1,t) = 0,\ V(x,0) = 0,\ x \in J,$$

$$PV(x,0) = f_2(x),\ x \in J,\ PV(0,t) = F_3(t),\ PV(1,t) = F_4(t),\ t \in [0,T].$$

From the fundamental formula, the solution $U$ of the equation

$$(13) \qquad Pu(x,t) = 0, \quad (x,t) \in D,$$

with the conditions from $(I)$ is of the form

$$(14) \qquad U(x,t) = \int_0^1 U(y,0)G(x,t,y,0)dy + \int_0^t U(0,s)D_yG(x,t,0,s)ds + \int_0^1 U(1,s)g(x,t,1,s)ds.$$

According to the fundamental formula consider the potentials

$$(15) \qquad U_0(x,t) = \int_0^1 f_0(y)G(x,t,y,0)dy,$$

(16) $$U_1^1(x,t) = \int_0^t F_1(s) D_y G(x,t,0,s) ds,$$

(17) $$U_1^2(x,t) = \int_0^t F_2(s) D_y G(x,t,1,s) ds.$$

## 6. PROPERTIES OF THE FUNCTION $U_0$

Denote by $(K_1)$ the class of all functions $f_0 \in C^2(\bar{J})$ such that

$$D_y^i f_0(0) = D_y^i f_0(1) \qquad (i=0,1,2),$$

$$f_0(y) = 0 \ \text{ for } \ y \in (R \setminus \bar{J}).$$

**LEMMA 2.** *If* $f_0 \in (K_1)$ *then*

$1^0$ $PU_0(x,t) = 0, \quad (x,t) \in D,$

$2^0$ $U_0(x,t) = f_0(x), \qquad x \in \bar{J}.$

**PROOF.** By [2], we obtain $1^0$ and $2^0$.

**LEMMA 3.** *If* $F_i \in (k) \ \ (i=1,2)$ *then*

$1^0$ $PU_1^i(x,t) = 0, \quad (x,t) \in D, \quad (i=1,2),$

$2^0$ $U_1^1(x,t) \to F_1(t),$ *as* $(x,t) \to (0,t), \ t \in [0,T],$

$3^0$ $U_1^2(x,t) \to F_2(t),$ *as* $(x,t) \to (1,t), \ t \in [0,T].$

**PROOF.** By Lemma 1, we obtain assertions $1^0 - 3^0$.

## 7. SOLUTIONS OF PROBLEMS (*I*), (*II*), (*III*)

By foregoing lemmas, we obtain the following results: The solution *U* of the problem (*I*) is of the form

$$U(x,t) = U_0(x,t) + U_1^1(x,t) + U_1^2(x,t).$$

**LEMMA 4.** *If* $f_1 \in (K_1), \ F_i \in (k) \ \ (i=3,4)$ *then the function V given by the formula*

$$V(x,t) = V_0(x,t) + V_1^1(x,t) + V_1^2(x,t),$$

*where*

$$V_0(x,t) = \int_0^1 f_1(y)tG(x,t,y,0)dy,$$

$$V_1^1(x,t) = \int_0^t F_3(s)(t-s)D_yG(x,t,0,s)ds,$$

$$V_1^2(x,t) = \int_0^t F_4(s)(t-s)D_yG(x,t,1,s)ds,$$

*is a solution to problem (II).*

**PROOF.** By properties of the function $G$, we have, by [2], $P^2V_0(x,t)=0$, $(x,t)\in D$, $\sqrt{V_0(x,0)}=0$, $PV_0(x,t)=\int_0^1 f_1(y)G(x,t,y,0)dy$, and $PV_0(x,0)=$ $=f_1(x)$, $x\in \bar{J}$, $V_0(0,t)=0$, $t\in[0,T]$, $V_0(1,t)=0$, $t\in[0,T]$, and

$$PV_0(0,t) = PV_0(1,t) = 0, \qquad t\in[0,T].$$

By [2], we have $P^2V_1^1(x,t)=0$, $(x,t)\in D$.

From the inequality $|V_1^1(x,t)| \le Ct\int_0^t| F_3(s)D_yG(x,t,0,s)ds$, we obtain $V_1^1(0,t)=0$, $t\in[0,T]$. Since $PV_1^1(x,t) = \int_0^t| F_3(s)D_yG(x,t,0,s)ds$, thus $PV_1^1(0,t)=F_3(t)$, $t\in[0,T]$ and $PV_1^1(1,t)=0$.

Similarly, we obtain the to properties of the function $V_1^2$.

Problem (*III*) concerns the construction of a solution $W$ to the equation

$$P^2W(x,t) = f(x,t), \qquad (x,t)\in D,$$

satisfying homogeneous limit conditions

$$W(x,0) = 0, \qquad x\in J,$$

$$W(0,t) = 0 = W(1,t), \qquad t\in[0,T],$$

$$PW(x,0) = 0, \qquad PW(0,t) = 0 = PW(1,t), \quad t\in[0,T].$$

Let $G_1(x,t,y,s) = (t-s)G(x,t,y,s)$.

By [2], we obtain following theorem:

**THEOREM 1.** *If the function f is of the class* $C^1(D)$ *then the function W, given by*

$$W(x,t) = \int_0^t \int_0^1 G_1(x,t,y,s) f(y,s) dy ds,$$

*satisfies the conditions:*

$1^0$ $P^2 W(x,t) = f(x,t), \quad (x,t) \in D,$

$2^0$ $W(x,0) = 0, \quad x \in J,$

$3^0$ $W(0,t) = W(1,t) = 0, \quad t \in (0,T],$

$4^0$ $PW(x,0) = 0, \quad x \in J,$

$5^0$ $PW(0,t) = PW(1,t) = 0.$

**PROOF.** We have $P_{x,t} G_1(x,t,y,s) = G(x,t,y,s)$ and, by the Poisson theorem, we obtain

$$PW(x,t) = \int_0^t \int_0^1 G(x,t,y,s) f(y,s) dy ds$$

and

$$P^2 W(x,t) = f(x,t), \qquad (x,t) \in D.$$

By properties of the functions $G_1$, $G$, we have $3^0$.

Since $PW(x,t) = \int_0^t \int_0^1 G(x,t,y,s) f(y,s) dy ds$ then $4^0$, $5^0$ hold.

## 8. FUNDAMENTAL THEOREM

Let

$$Z(x,t) = u(x,t) + V(x,t) + W(x,t), \quad (x,t) \in D,$$

By foregoing lemmas and by Theorem 1 we obtain the fundamental theorem:

**THEOREM 2.** *If the assumptions of the foregoing lemmas and Theorem* 1 *are satisfied then function Z is a solution of problem* (1)-(7).

## REFERENCES

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Received on 20.11.2001 and, in revised form, on 12.09.2002.