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BARBARA TAL-FIGIEL

SOME NONLINEAR DIFFUSION EQUATION WITH THREE NONLINEARITIES

ABSTRACT: The subject of the paper is a construction of the classical solution to the nonlinear diffusion equation $Pu(x,t) = F(x,t,u(x,t))$, $P = D_x^2 - D_t$, in the domain $D = \{(x,t) : x \in J = (0,1), t \in (0,T]\}$, satisfying the limit conditions $D_x u(x,0) = f_1(x)$, $D_x u(0,t) = F^1(t,u(0,t))$, $D_x u(1,t) = F^2(t,s(1,t))$.

KEY WORDS: limit problem for diffusion equation, nonlinear boundary-value conditions, Green function, system of Volterra integral equations, Banach fixed point method.

1. INTRODUCTION

The subject of the paper is the construction of the classical solution to the nonlinear diffusion equation

(1) $$Pu(x,t) = f(x,t,u(x,t)), \quad (x,t) \in D, \quad P = D_x^2 - D_t,$$
$$D = \{(x,t) : x \in J = (0,1), \ t \in (0,T]\},$$

satisfying the initial condition

(2) $$D_x u(x,0) = f_1(x), \qquad x \in J,$$

and the nonlinear boundary-value conditions of the Neumann type

(3) $$D_x u(0,t) = F^1(t,u(0,t)), \qquad t \in (0,T],$$

(4) $$D_x u(1,t) = F^2(t,u(1,t)), \qquad t \in (0,T].$$

The functions f, f_1, F^1, F^2 are given functions, and $(x,t) \to u(x,t)$ is unknown function.

To the solution of the above problem we apply the suitable Green function G, Green potentials, Banach fixed point method and nonlinear system of Volterra integral equations.

In [1], the similar problem for homogeneous equation is treated.

2. GREEN FUNCTION G

Let

$$U(x,t,y,s) = A(t-s)^{-1/2}\exp(B(t,s)(x-y)^2), \qquad (x,t,y,s)\in D_1,$$

(5)
$$A=(2\sqrt{\pi})^{-1}, \quad B(t,s)=(-4(t-s))^{-1},$$
$$D_1=\{(x,t,y,s) : 0\le s<t\le T,\ (x,y)\in J^2\},$$

and

$$g(x,t,y,s) = U(x,t,y,s) + K(x,t,y,s)$$

with

(6)
$$K = \sum_{i=1}^{4} K_i,$$

where

(7)
$$K_1(x,t,y,s) := A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(-x-2n-y)^2),$$

(8)
$$K_2(x,t,y,s) := A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(-x+2n+2-y)^2),$$

(9)
$$K_3(x,t,y,s) := A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(x+2n-y)^2),$$

(10)
$$K_4(x,t,y,s) := A(t-s)^{-1/2}\sum_{n=0}^{\infty}\exp(B(t,s)(x-2n-y)^2).$$

By [2] (vol. I. p. 474), g is the Green function to equation (5), to the domain D and to the Neumann boundary conditions

(11)
$$D_x g(0,t,y,s) = D_x g(1,t,y,s) = D_y g(x,t,0,s) = D_y g(x,t,1,s) = 0.$$

3. GREEN FUNCTION G TO EQUATION (5) AND DIRICHLET BOUNDARY-VALUE CONDITIONS

In the sequel, by C and C_i we shall denote the suitable positive constants. From [2] (vol.I. p.474), the function

$$G = U - K_1 - K_3 + K_2 + K_4$$

is the Green function to the equation (1) to the domain D, satisfying the Dirichlet boundary-value conditions

$$G(0,t,y,s) = G(1,t,y,s) = G(x,t,0,s) = G(x,t,1,s) = 0.$$

By the above formulas, we can prove:

LEMMA 1. *The functions g, G satisfy the conditions*

$$D_x g(x,t,y,s) = D_x G(x,t,y,s),$$

$$D_y g(x,t,y,s) = D_y G(x,t,y,s), \quad (x,t,y,s) \in D_1.$$

Next, we shall calculate

$$D_y g(x,t,0,s) = D_y G(x,t,0,s)$$

and

$$D_y g(x,t,1,s) = D_y G(x,t,1,s).$$

We obtain the formulas

$$D_y G(x,t,0,s) = \sum_{i=1}^{3} Q_i(x,t,s)$$

with

$$Q_1(x,t,s) = A(t-s)^{-3/2} x \exp(B(t,s)x^2),$$

$$Q_2(x,t,s) = -A(t-s)^{-3/2} \sum_{n=0}^{\infty} (x-2n-2)\exp(B(t,s)(x-2n-2)^2),$$

$$Q_3(x,t,s) = -A(t-s)^{-3/2} \sum_{n=0}^{\infty} (-x+2n+2)\exp(B(t,s)(-x+2n+2)^2).$$

Similarly, we obtain the formula

$$D_y G(x,t,1,s) = \sum_{i=1}^{3} R_i(x,t,y,s)$$

with

$$R_1(x,t,s) = -A(t-s)^{-3/2}(x-1)\exp(B(t,s)(x-1)^2),$$

$$R_2(x,t,s) = -A(t-s)^{-3/2} \sum_{n=0}^{\infty} (x+2n-1)\exp(B(t,s)(x+2n-1)^2),$$

$$R_3(x,t,s) = -A(t-s)^{-3/2} \sum_{n=0}^{\infty} (x-2n-1)\exp(B(t,s)(x-2n-1)^2).$$

4. GREEN POTENTIALS $J_1(x,t)$ AND $J_2(x,t)$

Let

$$J_1(x,t) = \int_0^t (D_y G(x,t,0,s)) F^1(s,u(0,s))ds = \sum_{i=1}^{3} J_1^i(x,t),$$

with

$$J_1^i(x,t) = \int_0^t Q_i(x,t,s)F^1(s,u(0,s))ds \quad (i=1,2,3),$$

and

$$J_2(x,t) = \sum_{i=1}^{3} J_2^i(x,t),$$

with

$$J_2^i(x,t) = \int_0^t R_i(x,t,s)F^2(s,u(1,s))ds \quad (i=1,2,3).$$

Let k denote the class of the functions F_1, F_2 that such $F_1(s)=F^1(s,u(1,s))$, $F_2(s)=F^2(s,u(1,s))$, $s\in[0,T]$, $F_i(s)\in C([0,T])$ $(i=1,2)$, $F_i(0)=0$ $(i=1,2)$.

LEMMA 2. *If* $F_i \in k$ $(i=1,2)$ *then*

1^0 $PJ_j^i(x,t)=0$, $(x,t)\in D$ $(j=1,2,\ i=1,2,3)$,

2^0 $J_1^1(x,t)\to F_1(t)$ *as* $(x,t)\to(0,t)$, $t\in[0,T]$,

3^0 $(J_1^2(x,t)+J_1^3(x,t))\to 0$ *as* $(x,t)\to(0,t)$, $t\in[0,T]$,

4^0 $J_1(x,t)\to 0$ *as* $(x,t)\to(1,t)$, $t\in[0,T]$,

5^0 $J_2^1(x,t)\to F_2(t)$ *as* $(x,t)\to(1,t)$, $t\in[0,T]$,

6^0 $(J_2^2(x,t)+J_2^2(x,t))\to 0$ *as* $(x,t)\to(1,t)$, $t\in[0,T]$,

7^0 $J_2(x,t)\to 0$ *as* $(x,t)\to(0,t)$, $t\in[0,T]$.

PROOF. By [2] and by properties of the potentials of the double layer, we obtain 1^0 - 7^0.

5. GREEN POTENTIALS u_0, u_1, u_2.

By [2], the solution u of the equation

(12) $$Pu(x,t) = 0, \qquad (x,t)\in D,$$

with the conditions (2)-(4) is of the form

(13) $$u(x,t) = \int_0^1 u(y,0)g(x,t,y,0)dy + \int_0^t (D_y u(0,s)g(x,t,0,s)ds +$$

$$+\int_0^t (D_y u(1,s))g(x,t,1,s)ds.$$

Let us consider the Green potentials

(14) $$u_0(x,t) = \int_0^1 f_1(y)g(x,t,y,0)dy, \quad D_x u_0(x,t) = \int_0^1 f_1(y)D_x g(x,t,y,0)dy,$$

(15) $$u_1^1(x,t) = \int_0^t g(x,t,0,s)F^1(s,u(0,s))ds,$$

$$D_x u_1^1(x,t) = \int_0^t (D_x g(x,t,0,s))F^1(s,u(0,s))ds,$$

(16) $$u_1^2(x,t) = \int_0^t g(x,t,1,s)F^2(s,u(1,s))ds,$$

$$D_x u_1^2(x,t) = \int_0^1 (D_x g(x,t,1,s))F^1(s,u(1,s))ds.$$

6. PROPERTIES OF THE FUNCTIONS u_0

Denote by K_1 the class of all functions $f_1 \in C^2(\bar{J})$ such that

$$D_y^i f_1(0) = D_y^i f_1(1) = 0 \quad (i=0,1), \qquad f_1(y)=0 \text{ for } y\in R\setminus J$$

and by K_2 the class of all functions

$$t \to \bar{h}(t) = (h_1(t), h_2(t)), \quad t\in[0,T],$$

such that

$$h_i(t)\in C([0,T]), \quad h_i(0)=0 \quad (i=1,2).$$

LEMMA 3. *If* $F_1 \in K_2$ *then*

1^0 $Pu_0(x,t)=0, \ (x,t)\in D,$

2^0 $D_x u_0(x,0) = f_1(x), \ x\in \bar{J},$

3^0 $D_x u_0(0,t) = h_1(t), \ h_1(t) = \int_0^1 f_1(y)g(0,t,y,0)dy, \ t\in[0,T], \ h_1(0)=0,$

4^0 $D_x u_0(1,t) = h_2(t),\ h_2(t) = \int_0^1 f_1(y) g(1,t,y,0)dy,\ t \in [0,T],\ h_2(0) = 0,$

5^0 $\bar{h}(t) \in K_2$.

PROOF. Ad 1^0. By [2], we obtain 1^0.

Ad 2^0. We have

$$D_x u_0(x,t) = \int_0^1 f_1(y) D_x g(x,t,y,0)dy = \int_0^1 f_1(y)(-D_y G(x,t,y,0))dy,$$

and, integrating by parts, we obtain

$$D_x u_0(x,t) = \int_0^1 f_1(y) G(x,t,y,0)dy.$$

Moreover, we have

$$\int_0^1 f_1(y) G(x,t,y,0)dy = \int_0^1 f_1(y) U(x,t,y,0)dy + \\ + \int_0^1 f_1(y)(G(x,t,y,0) - U(x,t,y,0))dy$$

and

$$\lim_{t\to 0}\int_0^1 f_1(y) G(x,t,y,0)dy = \lim_{t\to 0}\int_0^1 f_1(y) U(x,t,y,0)dy + \\ + \lim_{t\to 0}\int_0^1 f_1(y)(G(x,t,y,0) - U(x,t,y,0))dy = f_1(x),\quad x \in J.$$

Ad 3^0, 4^0. By the last formulas and boundary properties of function G, we obtain 3^0, 4^0.

Ad 5^0. From 3^0, 4^0, we get 5^0.

7. THE CLASSES K_4^1 AND K_4^2

Denote by K_4^1 the class of all functions $s \to F(s, u(0,s))$, $s \in [0,T]$, such that:

1^0 $u(0,\cdot) \in C([0,T])$, 2^0 $F(\cdot, u(0,\cdot)) \in C([0,T])$, 3^0 $F(0, u(0,0)) = 0$,

4^0 $|F(s, u(0,s))| < M$, $M > 0$ is a constant,

5^0 F satisfies the Lipschitz condition with respect to the argument $u(0,s)$ and together with a constant q.

Denote by K_4^2 the class of all functions $s \to F(s,u(1,s))$ satisfying the conditions $1^0 - 5^0$ with respect to the argument $u(1,s)$.

LEMMA 4. *If* $F^1 \in K_4^1$ *and* $F^2 \in K_4^2$ *then the functions* u_1^i $(i=1,2)$, *satisfy the conditions:*

1^0 $Pu_1^i(x,t)=0$, $(i=1,2)$, $(x,t)\in D$, 2^0 $u_1^1(0,t)=\int_0^t g(0,t,0,s)F^1(s,u(0,s))ds$,

3^0 $u_1^1(0,0)=0$, 4^0 $u_1^2=(0,t)=\int_0^t g(0,t,1,s)F^2(s,u(1,s))ds$, 5^0 $u_1^2(0,t)=0$,

6^0 $u_1^1=(1,t)=\int_0^t g(1,t,0,s)F^1(s,u(0,s))ds$, 7^0 $D_x u_1^1(0,t)=F^1(t,u(0,t))$,

8^0 $D_x u_1^1(1,t)=0$, 9^0 $D_x u_1^2(0,t)=0$, 10^0 $D_x u_1^2(1,t)=F^2(t,u(1,t))$,

11^0 $u_1^1(x,0)=u_1^2(x,0)=0$, $x\in J$, *for* $t\in[0,T]$.

PROOF. Ad 1^0. By [2], we obtain 1^0.

Ad 2^0-6^0. Consider the integrals $I_j(x,t)=\int_0^t g(x,t,y,s)\tilde{F}^i(s)ds$ $(j=1,2)$ with the majorants $Ct^{1/2}$ (uniformly for $x\in \bar{J}$). Consequently, they are locally uniformly convergent at the points $x=0$ and $x=1$. Hence, we obtain 2^0-6^0.

By Lemma 2, we get 7^0-10^0.

8. PROBLEM (I)

Applying the Green potentials u_0, u_1^i $(i=1,2)$, we shall solve problem (I) concerning to the construction to of the solution u to the equation $Pu(x,t)=0$, $(x,t)\in D$, where

(17) $$u(x,t)=u_0(x,t)+u_1^1(x,t)+u_1^2(x,t), \quad (x,t)\in D,$$

satisfying the initial condition

$$D_x u(x,0)=f_1(x), \quad x\in \bar{J},$$

and the boundary-value conditions (3), (4).

To the solution of the problem (I) we apply the formulas

$$D_x u(x,t) = D_x u_0(x,t) + D_x u_1^1(x,t) + D_x u_1^2(x,t), \quad (x,t) \in D,$$
$$D_x u(0,t) = D_x u_0(0,t) + D_x u_1^1(0,t) + D_x u_1^2(0,t), \quad t \in [0,T],$$
$$D_x u(1,t) = D_x u_0(1,t) + D_x u_1^1(1,t) + D_x u_1^2(1,t), \quad t \in [0,T].$$

Let us introduce the unknown functions

$$t \to H_1(t) = u(0,t), \quad t \in [0,T],$$
$$t \to H_2(t) = u(1,t), \quad t \in [0,T].$$

9. THE INTEGRAL EQUATION TO THE FUNCTIONS H_i $(i = 1,2)$

By foregoing formulas we obtain the Volterra system of the integral equations

$$(18) \quad \begin{aligned} H_1(t) &= h_1(t) + \int_0^t g(0,t,0,s) F^1(s, H_1(s)) ds + \int_0^t g(0,t,1,s) F^2(s, H_2(s)) ds, \\ H_2(t) &= h_2(t) + \int_0^t g(0,t,1,s) F^1(s, H_1(s)) ds + \int_0^t g(1,t,1,s) F^2(s, H_2(s)) ds. \end{aligned}$$

By Lemma 4, the last integrals have the majorants of the form

$$M(t) = C_2 t^{1/2} \le M(T) = C_2 T^{1/2}.$$

Consequently, we obtain:

LEMMA 5. *The compatibility conditions*

$$u_0(0,0) = u_1^1(0,0) = u_1^2(0,0) = u_0(1,0) = u_1^1(1,0) = u_1^2(1,0) = 0$$

hold.

10. SOLUTION OF THE NONLINEAR PROBLEM (I)

To solve system (18) we apply the Banach fixed point method.
Introduce the denotations

$$H_1(t) = h_1(t) + F_1(t, H_1, H_2),$$
$$H_2(t) = h_2(t) + F_2(t, H_1, H_2),$$

with

$$F_1(t,H_1,H_2)=\int_0^t g(0,t,0,s)F^1(s,H_1(s))ds + \int_0^t g(0,t,1,s)F^2(s,H_2(s))ds$$

and

$$\text{(19)}\quad F_2(t,H_1,H_2)=\int_0^t g(0,t,1,s)F^1(s,H_1(s))ds + \int_0^t g(1,t,1,s)F^2(s,H_2(s))ds.$$

Let

$$\overline{h}(t) = (h_1(t),h_2(t)), \qquad t\in[0,T],$$

$$\overline{Z}(t) = (H_1(t),H_2(t)), \quad t\in[0,T].$$

We can write system (19) in the form

$$\overline{Z}(t) = \overline{H}(t) + \overline{F}(t,\overline{Z}(t)),$$

with

$$\overline{F}(t) = (F_1(t,\overline{Z}(t)),F_2(t,\overline{Z}(t)).$$

11. Some Banach spaces of continuous functions

Let K_5 denote the class of all vector fucntions $\overline{F}$ which are defined and continuous in the set

$$D_2 = \{(t,\overline{Z}): t\in[0,T],\ H_1,H_2\in R\}$$

bounded in D_2 i.e.

$$\|\overline{F}\| \le M, \quad M>0 \text{ is a constant,}$$

satisfying the Lipschitz condition

$$\|\overline{F}(t,\overline{Z}^1)-\overline{F}(t,\overline{Z}^2)\| \le q\|\overline{Z}^1-\overline{Z}^2\|, \quad q=C_1\,T^{1/2},$$

with

$$\overline{Z}^i = (Z_1^i,Z_2^i) \quad (i=1,2),$$

uniformly for $t\in[0,T]$.

Introduce the Banach space of the continuous functions

$$B^1 = \{t\to\overline{h}(t),\ t\in\{0,T]\},$$

with the norm

$$\|\overline{h}\| = \sup_{t\in[0,T]}|h_1(t)| + \sup_{t\in[0,T]}|h_2(t)| = N_1$$

and the Banach space

$$B^2 = \{\overline{F} : \overline{F} = (F_1, F_2)\}$$

with the norm

$$\| \overline{F} \| = \| F_1 + F_2 \| \leq \| F_1 \| + \| F_2 \| .$$

Let

$$B^3 = B^1 \times B^2$$

denote the cartesian product of the spaces B^1, B^2 being the set of the functions

$$B^3 = \{t \to \overline{w}(t) : \overline{w}(t) = (\overline{h}(t), \overline{F}(t, \overline{Z}(t)))\}$$

with the norm

$$\| \overline{w} \| = \| \overline{h} \| + \| \overline{F} \| = N_1 + \| \overline{F} \|.$$

12. THE BALLS IN THE SPACE B^3

Let $\overline{O}$ denote the vector function with the coordinates equal identically zero. Moreover, let $R^1 = N_1 + \| \overline{F} \|$,

$K(\overline{O}, R^1)$ be the ball with the center $\overline{O}$ and radius R^1, being the set of the functions $\overline{w} = (\overline{h}, \overline{F})$ such that

$$\| \overline{w} \| < R^1,$$

$K(\overline{O}, qR^1)$ the ball with the center $\overline{O}$ and radius qR^1, being the set of the functions $\overline{h} \in B^1$ such that

$$\| \overline{h} \| < qR^1,$$

$K(\overline{O}, (1-q)R^1)$ be the ball with the center $\overline{O}$ and radius $(1-q)R^1$, being the set of the functions $\overline{F} \in B^2$ such that

$$\| \overline{F} \| < (1-q)R^1.$$

13. TRANSFORMATION TO THE BANACH FIXED POINT METHOD

Let us consider the transformation

$$(\overline{S}) \qquad S : t \to \overline{S}(t, \overline{h}(t), \overline{F}(t, \overline{Z}(t))) = \overline{h}(t) + \overline{F}(t, \overline{Z}(t)), \quad t \in [0, T].$$

LEMMA 6. *If* $\overline{h} \in K(\overline{O}, qR^1)$, $\overline{F} \in K(\overline{O}, (1-q)R^1)$, *and* $q \in (0,1)$ *then the transformation* $\overline{S}$ *satisfies the conditions:*

1^0 $\overline{S}$ *transforms the ball* $K(\overline{O}, R^1)$ *into itself,*

2^0 $\overline{S}$ *is a contraction with the coefficient* q.

PROOF.

Ad 1^0. Let $(\overline{h}, \overline{F}) \in K(\overline{O}, R^1)$. Then

$$\|(\overline{h}, \overline{F})\| < \|\overline{h}\| + \|\overline{F}\| < qR^1 + (1-q)R^1 = R^1.$$

Ad 2^0. Let $\overline{h} \in K(\overline{O}, qR^1)$, $\overline{F} \in K(\overline{O}, (1-q)R^1)$. Then

$$\|\overline{S}(t, \overline{h}(t), \overline{F}(t, \overline{Z}^1(t))) - \overline{S}(t, \overline{h}(t), \overline{F}(t, \overline{Z}^2(t)))\| =$$
$$= \|\overline{F}(t, \overline{Z}^1(t)) - \overline{F}(t, \overline{Z}^2(t))\| < q\|\overline{Z}^1 - \overline{Z}^2\|.$$

By the Banach fixed point theorem, there exists a fixed point $\overline{Z}$ to the transformation $\overline{S}$,

$$\overline{Z}(t) = \overline{Z}(t, H_1, H_2)$$

such that

(20) $$\overline{Z}(t) = \overline{h}(t) + \overline{F}(t, \overline{Z}(t)), \quad t \in [0, T].$$

14. CONSTRUCTION OF THE FIXED POINT TO EQUATION (20)

Let us introduce the sequence

$$\overline{Z}_0(t) = \overline{h}(t), \quad \overline{Z}_n(t) = \overline{Z}(t, H_1^n, H_2^n) = \overline{h}(t) + F(\overline{Z}_{n-1}(t)), \quad n = 1,2...$$

Moreover, let us consider the Cauchy sequence

$$\overline{Z}_{m,n}(t) = \overline{Z}_n(t) - \overline{Z}_m(t), \quad m, n = 0,1,...$$

LEMMA 7. *The sequence* $\overline{Z}_{m,n}$ *satisfies the condition*

$$\|\overline{Z}_n - \overline{Z}_m\| \le q^m (1-q)^{-1} \|\overline{Z}_1 - \overline{Z}_0\|.$$

PROOF. We have

$$\|\overline{Z}_n - \overline{Z}_{n-1}\| \le q\|\overline{Z}_{n-1} - \overline{Z}_{n-2}\| \le ... \le q^{n-1}\|\overline{Z}_1 - \overline{Z}_0\|.$$

We can write

$$\overline{Z}_n = \overline{Z}_0 + \sum_{j=1}^{n} (\overline{Z}_j - \overline{Z}_{j-1}).$$

For the sequence $\overline{Z}_{m,n}$ we have the formula

$$\overline{Z}_n - \overline{Z}_m = \sum_{j=m+1}^{n} (\overline{Z}_j - \overline{Z}_{j-1}).$$

By the last formulas, we obtain the inequality

$$\| \overline{Z}_{m,n} \| \leq \sum_{j=m+1}^{n} q^{j+1} \| \overline{Z}_1 - \overline{Z}_0 \| \leq \| \overline{Z}_1 - \overline{Z}_0 \| q^m \sum q^j \leq \| \overline{Z}_1 - \overline{Z}_0 \| q^m (1-q)^{-1}$$

for arbitrary positive numbers $m,n > N$ and the sequence $\overline{Z}_{m,n}$ of the functions of the class $C([0,T])$ satisfies the Cauchy condition.

Remark 1. From the above considerations, there exists

$$\lim_{n\to\infty} \overline{Z}_n(t) = \overline{Z}(t)$$

and

$$\overline{Z} \in C([0,T])$$

because the space $C([0,T])$ is the complete space.

By the above studies we obtain:

Theorem 1. *If* $\overline{h} \in K(\overline{O}, qR^1)$, $F_1 \in K_4^1$, $F^2 \in K_4^2$, $\overline{F} \in K(\overline{O},(1-q)R^1$, $q \in (0,1)$, *then the function u, given by formula* (17), *is a solution to problem* (I) *if and only if function* $\overline{Z}$ *is a solution to equation* (20).

15. Problem (II)

Problem (II) concerns the construction of the solution to the inhomogeneous nonlinear parabolic equation

(21) $$PW(x,t) = f(x,t,W(x,t)),$$

satisfying the conditions (2), (3), (4).

To solve the last problem we assume that its solution U is of the form

(22) $$U(x,t) = u(x,t) + \int_0^t \int_0^1 G(x,t,y,s) f(y,s,W(y,s)) dyds,$$

where u is a solution of the problem (I) and the Green potential of the single layer is a solution of the inhomogeneous equation.

To solve the last problem we shall apply the suitable transformation S_1 to the Banach fixed point method and the method of the succesive approximations.

16. THE TRANSFORMATION S_1

Let us consider the transformation S_1, given by the formulas

$$(S_1) \qquad (S_1(u,W))(x,t) = u(x,t) + N(x,t,W),$$

$$N(x,t,W) = \int_0^t \int_0^1 G(x,t,y,s) f(y,s,W(y,s)) dyds.$$

In the sequel we shall construct the fixed point V to the transformation S_1 being the solution of problem (II) such that

$$(S_2) \qquad V(x,t) = u(x,t) + \int_0^t \int_0^1 G(x,t,y,s) f(y,s,V(y,s)) dyds.$$

17. THE CLASS K_4 AND SOME BANACH SPACES

Let K_4 denote the class of all functions $(y,s,W) \to f(y,s,W) = f(y,s,W(y,s))$ satisfying the conditions:

(a) *the functions f are defined and continuous in the set*

$$D_3 = \{(x,t,W) : (x,t) \in \overline{D}, W \in R\},$$

(b) *the functions f are bounded i.e.* $|f| < M$, *is a constant,*

(c) *the functions f satisfy the Lipschitz condition in the supremum norm with a Lipschitz constant* $L \in (0,1)$ *with respect to W.*

Introduce the Banach spaces of continuous functions $X_1 = \{(x,t) \to u(x,t)\}$, defined by formula (17), with the norm $\| u \| = N_1$, the space X_2 of the functions $N(x,t,W) = \int_0^t \int_0^1 G(x,t,y,s) f(y,s,W(y,s)) dyds$ with the norm $\| N(W) \| = MCt^{1/2} \leq C_1 T^{1/2}$, with the suitable constant C_1, and $X_3 = X_1 \times X_2$ – the cartesian product of the spaces X_1, X_2, with the norm $\| \cdot \|_{X_3} = N_1 + C_1 T^{1/2}$.

18. THE BALLS IN SPACE X_3

Let $\overline{O}$ denote the vector function with all coordinates equal zero. Let $R^1 = N_1 + C_1 T^{1/2} = N_2(T)$. Consider three balls:

Let $K^1(\overline{O}, R^1)$ denote the ball with center $\overline{O}$ and radius R^1, being the set of all functions $(u, N(W))$ for which $\| (u, N(W)) \| \leq R^1$, $K^2(\overline{O}, LR^1)$ the ball with the center $\overline{O}$ and radius LR^1, being the set of all functions u for which $\| u \| \leq LR^1$, $K^3(\overline{O},(1-L)R^1)$ the ball with the center $\overline{O}$ and radius $(1-L)R^1$, being the set of all functions $N(W)$ for which $\| N(W) \| \leq (1-L)R^1$.

19. PROPERTIES OF THE TRANSFORMATION S_1

LEMMA 8. *If* $L \in (0,1)$, $u \in K^2(\overline{O}, LR^1)$, $N(W) \in K^3(\overline{O},(1-L)R^1)$ *then the transformation* S_1 *satisfies the conditions:* 1^0 S_1 *transforms the ball* $K^1(\overline{O}, R^1)$ *into itself,* 2^0 S_1 *is a contraction with the coefficient L.*

PROOF.

Ad 1^0. Let $u \in K^2(\overline{O}, LR^1)$, and $N(W) \in K^3(\overline{O},(1-L)R^1)$. Then:

$$\| u + N(W) \| \leq \| u \| + \| N(W) \| \leq LR^1 + (1-L)R^1 = R^1.$$

Ad 2^0. We have

$$\| S_1(u,W^1) - S_1(u,W^2) \| = \| N(W^1) - N(W^2) \| \leq L \| W^1 - W^2 \|.$$

By the Banach fixed point theorem we obtain:

THEOREM 2. *There exists a fixed point V to the transformation* S_1 *for which the following equation:*

$$(S_2) \quad V(x,t) = u(x,t) + \int_0^t \int_0^1 f(y,s,V(y,s)) G(x,t,y,s) dyds, \quad (x,t) \in D,$$

is satisfied.

20. CONSTRUCTION OF THE SOLUTION $(x,t) \to V(x,t)$ BY THE METHOD OF THE SUCCESIVE APPROXIMATIONS

Let us consider the sequence $V_n (n = 1,2,...)$ by the formulas:

$$V_0(x,t) = u(x,t), \quad (x,t) \in D,$$

$$V_1(x,t) = u(x,t) + \int_0^t \int_0^1 f(y,s,V_0(y,s)) G(x,t,y,s) dyds,$$

....

$$(22) \qquad V_n(x,t) = u(x,t) + \int_0^t \int_0^1 f(y,s,V_{n-1}(y,s)) G(x,t,y,s) dyds, \quad n = 1,2,....$$

LEMMA 9. *The sequence* $V_n (n = 1,2,...)$ *satisfies the conditions*:

(23) $$\| V_n - V_{n-1} \| \le L \| V_{n-1} - W_{n-2} \| < \mathrm{K} \le L^{n-1} \| V_1 - V_0 \|.$$

PROOF. By formula (22), we obtain inequality (23).
Next, let us consider the Cauchy sequence

(24) $$V_{m,n}(x,t) = V_n(x,t) - V_m(x,t), \quad (x,t) \in D, \quad n > m \quad (n, m = 1,2,...).$$

LEMMA 10. *The inequality*

(25) $$\| V_n - V_m \| \le L^m (1-L)^{-1} \| V_1 - V_0 \|$$

holds.

PROOF. We can write

(26) $$V_n(x,t) - V_m(x,t) = \sum_{j=m+1}^{n} (V_j(x,t) - V_{j-1}(x,t)).$$

By the last formula, we obtain the inequality

(27) $$\| V_{m,n} \| \le \sum_{j=m+1}^{n} L^{j+1} \| V_1 - V_0 \| \le \sum_{j=0}^{\infty} L^{j+1} \| V_1 - V_0 \| \le$$
$$\le \| V_1 - V_0 \| L^m (1-L)^{-1}$$

for arbitrary integers $m, n > N$, $(x,t) \in \overline{D}$. The sequence $V_{m,n}$ satisfies the Cauchy condition.

REMARK 2. From the completness of the space $C(\overline{D})$ and from the condition $V_{m,n} \in C(\overline{D})$, it follows that there exists $\lim_{n\to\infty} V_n(x,t) = V(x,t)$, $V \in C(\overline{D})$, because the space $C(\overline{D})$ is the complete space. Consequently, we obtain the fundamental theorem:

THEOREM 2. *If the assumptions of the foregoing lemmas and of Theorem* 1 *are satisfied then function V is a solution to problem* (II).

21. UNIQUENESS THEOREM

From the integral equation, we obtain the differential equation.

Indeed, assume that $F_1(x,t) = f(x,t,V(x,t))$. Function F_1 satisfies the Lispschitz condition. Applying the Poisson theorem, we obtain the differential equation $PV(x,t) = Pu(x,t) + f(x,t,V(x,t))$ with the suitable limit conditions.

Hence, the differential and integral problems are equivalent. Since solution u of problem (I) is unique thus we obtain:

THEOREM 3. *The function*

$$(x,t) \to V(x,t) = u(x,t) + \int_0^t \int_0^1 f(y,s,V(y,s))G(x,t,y,s)dyds$$

is the unique solution to problem (II).

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(Institute of Chemical Engineering and Physical Chemistry, Cracow University of Technology, Warszawska 24, Kraków)

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