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APPROXIMATION PROPERTIES OF CERTAIN MODIFIED SZASZ-MIRAKYAN OPERATORS OF FUNCTIONS OF TWO VARIABLES

ABSTRACT: We introduce certain modified Szasz-Mirakyan operators in polynomial and exponential weighted spaces of functions of two variables and we study the degree of approximation of functions by these operators.

The similar theorems for functions of one variable were given in [4] and [5].

KEY WORDS: Szasz-Mirakyan operator, degree of approximation, polynomial and exponential weighted spac.

I. Approximation in polynomial weighted spaces

1. Preliminaries

1.1. Let as in [1], for $p \in N_0 := \{0,1,2,\dots\}$,

(1) $$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1} \quad \text{if} \quad p \geq 1, \; x \in R_0 := [0,+\infty).$$

Next, for given $p,q \in N_0$, we define the weighted function

(2) $$w_{p,q}(x,y) := w_p(x) w_q(y), \qquad (x,y) \in R_0^2 := R_0 \times R_0,$$

and the weighted space $C_{p,q}$ of all real – valued functions f continuous on R_0^2 for which $w_{p,q} f$ is uniformly continuous and bounded on R_0^2 and the norm is defined by the formula

(3) $$\| f \|_{p,q} \equiv \| f(\cdot,\cdot) \|_{p,q} := \sup_{(x,y)\in R_0^2} w_{p,q}(x,y) | f(x,y) |.$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual by the formula

(4) $$\omega(f, C_{p,q}; t,s) := \sup_{\substack{0 \leq h \leq t \\ 0 \leq \delta \leq s}} \| \Delta_{h,\delta} f(\cdot,\cdot) \|_{p,q}, \quad t,s \geq 0,$$

where $\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$ and $(x+h, y+\delta) \in R_0^2$. Moreover let $C_{p,q}^1$ be the set of all functions $f \in C_{p,q}$ which first partial derivatives belong also to $C_{p,q}$.

From (4) it follows that

(5) $$\lim_{t,s\to 0+} \omega(f, C_{p,q}; t, s) = 0,$$

for every $f \in C_{p,q}$, $p,q \in N_0$.

1.2. We introduce the following

DEFINITION 1. *Let* $R_2 := [2,+\infty)$ *and let* $r,s \in R_2$ *are fixed numbers. For functions* $f \in C_{p,q}$, $p,q \in N_0$, *we define the operators* $A_{m,n}(f; r, s; x, y) \equiv A_{m,n}(f; x, y)$

(6) $$A_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \varphi_j((mx+1)^r)\, \varphi_k((ny+1)^s) \times$$
$$\times f\left(\frac{j}{m(mx+1)^{r-1}}, \frac{k}{n(ny+1)^{s-1}}\right),$$

for $(x,y) \in R_0^2$, $m,n \in N := \{1,2,\ldots\}$, *where*

(7) $$\varphi_i(t) := e^{-t} \frac{t^i}{i!} \qquad \text{for} \quad t \in R_0, \ i \in N_0.$$

In the paper [4] were considered modified Szasz-Mirakyan operators $A_n(f; r; x)$

(8) $$A_n(f; r; x) := \sum_{k=0}^{\infty} \varphi_k\left((nx+1)^r\right) f\left(\frac{k}{n(nx+1)^{r-1}}\right), \quad x \in R_0, \ n \in N, \ r \in R_2,$$

for functions of one variable.

From (6) – (8) we deduce that $A_{m,n}(f; r, s)$ are well-defined in every space $C_{p,q}$, $p,q \in N_0$. Moreover for fixed $r,s \in R_2$ we have

(9) $$A_{m,n}(1; r, s; x, y) = 1 \quad \text{for} \quad (x,y) \in R_0^2, \ m,n \in N,$$

and if $f \in C_{p,q}$ and $f(x,y) = f_1(x) f_2(y)$ for all $(x,y) \in R_0^2$, then

(10) $$A_{m,n}(f; r, s; x, y) = A_m(f_1; r; x) A_n(f_2; s; y)$$

for all $(x,y) \in R_0^2$, $m,n \in N$.

In this paper by $M_k(\alpha, \beta)$ we shall denote suitable positive constants depending only on indicated parameters α, β.

2. LEMMAS AND THEOREMS

2.1. From (8) and (7) we get for $x \in R_0$, $n \in N$ and $r \in R_2$

(11) $$A_n(1;r;x) = 1,$$

(12) $$A_n(t-x;r;x) = \frac{1}{n}, \quad A_n((t-x)^2;r;x) = \frac{1}{n^2}\left[1+\frac{1}{(nx+1)^{r-2}}\right].$$

In the paper [4] was proved the following lemma for $A_n(f;r)$ defined by (8).

LEMMA 1. *For every fixed* $p \in N_0$ *and* $r \in R_2$ *there exist positive constants* $M_i \equiv M_i(p,r)$, $i = 1,2$, *such that for all* $x \in R_0$, $n \in N$

(13) $$w_p(x)A_n(1/w_p(t);r;x) \le M_1,$$

(14) $$w_p(x)A_n((t-x)^2/w_p(t);r;x) \le \frac{M_2}{n^2}.$$

Applying Lemma 1 we shall prove the main lemma on $A_{m,n}$ defined by (6).

LEMMA 2. *For fixed* $p,q \in N_0$ *and* $r,s \in R_2$ *there exists a positive constant* $M_3 \equiv M_3(p,q,r,s)$ *such that*

(15) $$\left\|A_{m,n}(1/w_{p,q}(t,z);r,s;\cdot,\cdot)\right\|_{p,q} \le M_3 \quad \text{for} \quad m,n \in N.$$

Moreover for every $f \in C_{p,q}$ *we have*

(16) $$\left\|A_{m,n}(f;r,s;\cdot,\cdot)\right\|_{p,q} \le M_3\|f\|_{p,q} \quad \text{for} \quad m,n \in N, \quad r,s \in N_2.$$

The formulas (6) – (7) *and the inequality* (16) *show that* $A_{m,n}$, $m,n \in N$, *defined by* (6) *are linear positive operators from the space* $C_{p,q}$ *into* $C_{p,q}$.

PROOF. The inequality (15) follows immediately from (2), (10) and (13). From (6) and (3) we get for $f \in C_{p,q}$ and $r,s \in R_2$

$$\left\|A_{m,n}(f;r,s)\right\|_{p,q} \le \|f\|_{p,q}\left\|A_{m,n}(1/w_{p,q};r,s)\right\|_{p,q}, \quad m,n \in N,$$

which by (15) implies (16).

2.2. Now we shall give two theorems on the degree of approximation of functions by $A_{m,n}$ defined by (6).

THEOREM 1. *Suppose that* $f \in C^1_{p,q}$ *with fixed* $p, q \in N_0$. *Then there exists a positive constant* $M_4 \equiv M_4(p,q,r,s)$ *such that for all* $m, n \in N$

(17) $$\left\| A_{m,n}(f;r,s;\cdot,\cdot) - f(\cdot,\cdot) \right\|_{p,q} \le M_4 \left\{ \frac{1}{m} \left\| f'_x \right\|_{p,q} + \frac{1}{n} \left\| f'_y \right\|_{p,q} \right\}.$$

PROOF. Let $(x,y) \in R^2_0$ be a fixed point. Then for $f \in C^1_{p,q}$

$$f(t,z) - f(x,y) = \int_x^t f'_u(u,z)du + \int_y^z f'_v(x,v)dv, \qquad (t,z) \in R^2_0.$$

Thus by (9)

(18) $$A_{m,n}(f(t,z);r,s;x,y) - f(x,y) =$$

$$= A_{m,n}\left(\int_x^t f'_u(u,z)du; r,s;x,y \right) + A_{m,n}\left(\int_y^z f'_v(x,v)dv; r,s;x,y \right).$$

By (1) – (3) we have

$$\left| \int_x^t f'_u(u,z)du \right| \le \left\| f'_x \right\|_{p,q} \left| \int_x^t \frac{du}{w_{p,q}(u,z)} \right| \le \left\| f'_x \right\|_{p,q} \left(\frac{1}{w_{p,q}(t,z)} + \frac{1}{w_{p,q}(x,z)} \right) |t-x|,$$

which by (1), (2), (6) and (8) – (11) implies

$$w_{p,q}(x,y) \left| A_{m,n}\left(\int_x^t f'_u(u,z)du; r,s;x,y \right) \right| \le w_{p,q}(x,y) A_{m,n}\left(\left| \int_x^t f'_u(u,z)du \right|; r,s;x,y \right) \le$$

$$\le \left\| f'_x \right\|_{p,q} w_{p,q}(x,y) \left\{ A_{m,n}\left(\frac{|t-x|}{w_{p,q}(t,z)}; r,s;x,y \right) + A_{m,n}\left(\frac{|t-x|}{w_{p,q}(x,z)}; r,s;x,y \right) \right\} \le$$

$$\le \left\| f'_x \right\|_{p,q} w_q(y) A_n\left(\frac{1}{w_q(z)}; s;y \right) \left\{ w_p(x) A_m\left(\frac{|t-x|}{w_p(t)}; r;x \right) + A_m(|t-x|; r;x) \right\}.$$

Applying the Hölder inequality and (11) – (14), we get

$$A_m(|t-x|;r;x) \le \left\{ A_m((t-x)^2;r;x) A_m(1;r;x) \right\}^{1/2} \le \frac{\sqrt{2}}{m},$$

$$w_p(x)A_m\left(\frac{|t-x|}{w_p(t)};r;x\right)\le\left\{w_p(x)A_m\left(\frac{(t-x)^2}{w_p(t)};r;x\right)\right\}^{1/2}\times$$

$$\times\left\{w_p(x)A_m\left(\frac{1}{w_p(t)};r;x\right)\right\}^{1/2}\le\frac{M_5(p,r)}{m},$$

for $x\in R_0$ and $m\in N$. Consequenty

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int_x^t f_u'(u,z)du;r,s;x,y\right)\right|\le\frac{M_6(p,r)}{m}\|f_x'\|_{p,q},\quad m\in N.$$

Analogously we obtain

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int_y^z f_v'(x,v)dv;r,s;x,y\right)\right|\le\frac{M_7(q,s)}{n}\|f_y'\|_{p,q},\quad n\in N.$$

Combining these, we derive from (18)

$$w_{p,q}(x,y)\left|A_{m,n}(f;r,s;x,y)-f(x,y)\right|\le M_8\left\{\frac{1}{m}\|f_x'\|_{p,q}+\frac{1}{n}\|f_y'\|_{p,q}\right\},$$

for all $m,n\in N$, where $M_8=M_8(p,q,r,s)=const.>0$.
Thus the proof of (17) is completed.

THEOREM 2. *Suppose that $f\in C_{p,q}$, $p,q\in N_0$. Then there exists a positive constant $M_9\equiv M_9(p,q,r,s)$ such that*

$$\|A_{m,n}(f;r,s;\cdot,\cdot)-f(\cdot,\cdot)\|_{p,q}\le M_9\,\omega\left(f;C_{p,q};\frac{1}{m},\frac{1}{n}\right),\tag{19}$$

for all $m,n\in N$.

PROOF. We apply the Stieklov function $f_{h,\delta}$ for $f\in C_{p,q}$

$$f_{h,\delta}(x,y):=\frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v)dv,\qquad (x,y)\in R_0^2,\ h,\delta>0.\tag{20}$$

From (20) it follows that

$$f_{h,\delta}(x,y)-f(x,y)=\frac{1}{h\delta}\int_0^h du\int_0^\delta \Delta_{u,v}f(x,y)dv,$$

$$(f_{h,\delta})'_x(x,y) = \frac{1}{h\delta}\int_0^{\delta}(\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y))\,dv,$$

$$(f_{h,\delta})'_y(x,y) = \frac{1}{h\delta}\int_0^{h}(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y))\,du.$$

Thus

(21) $$\left\|f_{h,\delta}-f\right\|_{p,q} \le \omega(f;C_{p,q};h,\delta),$$

(22) $$\left\|(f_{h,\delta})'_x\right\|_{p,q} \le 2h^{-1}\omega(f;C_{p,q};h,\delta),$$

(23) $$\left\|(f_{h,\delta})'_y\right\|_{p,q} \le 2\delta^{-1}\omega(f;C_{p,q};h,\delta),$$

for all $h,\delta>0$, which show that $f_{h,\delta}\in C^1_{p,q}$ if $f\in C_{p,q}$ and $h,\delta>0$.

Now, for $A_{m,n}$ defined by (6), we can write

$$\begin{aligned} w_{p,q}(x,y)\left|A_{m,n}(f;r,s;x,y)-f(x,y)\right| &\le \\ \le w_{p,q}(x,y)\Big\{&\left|A_{m,n}(f(t,z)-f_{h,\delta}(t,z);r,s;x,y)\right| + \\ &+\left|A_{m,n}(f_{h,\delta}(t,z);r,s;x,y)-f_{h,\delta}(x,y)\right| + \\ &+\left|f_{h,\delta}(x,y)-f(x,y)\right|\Big\} := T_1+T_2+T_3. \end{aligned}$$

By (3), (16) and (21),

$$T_1 \le \left\|A_{m,n}(f-f_{h,\delta};r,s;\cdot,\cdot)\right\|_{p,q} \le M_3\left\|f-f_{h,\delta}\right\|_{p,q} \le M_3\,\omega(f;C_{p,q};h,\delta),$$

$$T_3 \le \omega(f;C_{p,q};h,\delta).$$

Applying Theorem 1 and (22) and (23), we get

$$T_2 \le M_5\left\{\frac{1}{m}\left\|(f_{h,\delta})'_x\right\|_{p,q}+\frac{1}{n}\left\|(f_{h,\delta})'_y\right\|_{p,q}\right\} \le 2M_4\,\omega(f;C_{p,q};h,\delta)\left\{h^{-1}\frac{1}{m}+\delta^{-1}\frac{1}{n}\right\}.$$

Consequently there exists $M_{10}\equiv M_{10}(p,q,r,s)$ such that

(24) $$\left\|A_{m,n}(f;r,s;\cdot,\cdot)-f(\cdot,\cdot)\right\|_{p,q} \le M_{10}\,\omega(f;C_{p,q};h,\delta)\left\{1+h^{-1}\frac{1}{m}+\delta^{-1}\frac{1}{n}\right\},$$

for $m,n\in N$ and $h,\delta>0$. Now, for $m,n\in N$ setting $h=\frac{1}{m}$ and $\delta=\frac{1}{n}$ to (24), we obtain (19).

From Theorem 2 and the property (5) follows

COROLLARY 1. *Let* $f \in C_{p,q}$, $p,q \in N_0$. *Then*

(25) $$\lim_{m,n\to\infty} \left\| A_{m,n}(f;r,s;\cdot,\cdot) - f(\cdot,\cdot) \right\|_{p,q} = 0.$$

Theorem 2 and Corollary 1 in our paper show that operators $A_{m,n}$, $m,n \in N$, give better degree of approximation of functions $f \in C_{p,q}$ than classical Szasz–Mirakyan operator $S_{m,n}$, considered in [3] for continuous and bounded functions.

II. APPROXIMATION IN EXPONENTIAL WEIGHTED SPACES

3. PRELIMINARIES

3.1. Let as in [2] and [5], for a fixed $p > 0$ and $r \in R_2$,

(26) $$v_{pr}(x) := \exp(-prx), \qquad x \in R_0,$$

and let for fixed $p,q > 0$ and $r,s \in R_2$

(27) $$v_{pr,qs}(x,y) := v_{pr}(x) v_{qs}(y), \quad (x,y) \in R_0^2.$$

Denote by $C_{pr,qs}$ the set of all real – valued functions f continuous on R_0^2 for which $f v_{pr,qs}$ is uniformly continuous and bounded on R_0^2 and the norm is defined by

(28) $$\|f\|_{pr,qs} \equiv \|f(\cdot,\cdot)\|_{pr,qs} := \sup_{(x,y)\in R_0^2} v_{pr,qs}(x,y) \left| f(x,y) \right|.$$

The modulus of continuity of function $f \in C_{pr,qs}$ we define as in §1.1 by formula

$$\omega(f, C_{pr,qs}; t,z) := \sup_{\substack{0\le u\le t \\ 0\le v\le z}} \left\| \Delta_{u,v} f(\cdot,\cdot) \right\|_{pr,qs}, \qquad t,z \ge 0,$$

and we have

(29) $$\lim_{t,z\to 0+} \omega(f, C_{rp,qs}; t,z) = 0 \qquad \text{for} \quad f \in C_{pr,qs}.$$

Analogously as in §1.1, for fixed $p,q > 0$ and $r,s \in R_2$, we denote by $C^1_{pr,qs}$ the set of all functions $f \in C_{pr,qs}$ which first partial derivatives belong also to $C_{pr,qs}$.

3.2. Similarly as in Section I we introduce

DEFINITION 2. *Let* $r,s\in R_2$ *are fixed numbers. For functions* $f\in C_{pr,qs}$, $p,q>0$, *we define the operators*

$$(30)\quad B_{m,n}(f;p,q,r,s;x,y):=B_{m,n}(f;x,y)\equiv$$

$$\equiv\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_j\left((mx+1)^r\right)\varphi_k((ny+1)^s)\,f\left(\frac{j}{m(mx+1)^{r-1}+pr},\frac{k}{n(ny+1)^{s-1}+qs}\right),$$

for $(x,y)\in R_0^2$ *and* $m,n\in N$.

In [5] were examined the operators

$$(31)\qquad B_n(f;x)\equiv B_n(f;r;x):=\sum_{k=0}^{\infty}\varphi_k\left((nx+1)^r\right)f\left(\frac{k}{n(nx+1)^{r-1}+pr}\right),$$

$x\in R_0$, $n\in N$, for functions f of one variable, belonging to exponential weighted spaces.

In this paper we shall give similar results for operators $B_{m,n}(f)$.

4. LEMMAS AND THEOREMS

4.1. From (30) and (7) we deduce that $B_{m,n}(f)$ is well-defined in every space $C_{pr,qs}$, $p,q>0$, $r,s\in R_2$. In particular

$$(32)\qquad B_{m,n}(1;x,y)=1,\qquad (x,y)\in R_0^2,\ m,n\in N,$$

and if $f\in C_{pr,qs}$ and $f(x,y)=f_1(x)f_2(y)$ for all $(x,y)\in R_0^2$, then

$$(33)\qquad B_{m,n}(f;p,q,r,s;x,y)=B_m(f_1;p,r;x)B_n(f_2;q,s;y)$$

for all $x,y\in R_0$ and $m,n\in N$. Moreover from (31) we get

$$(34)\qquad B_n(1;p,r;x)=1,\qquad x\in R_0,\ n\in N.$$

In the paper [5] were proved the following two lemmas for $B_n(f;p,r;\cdot)$ definied by (31).

LEMMA 3. *Let* $p>0$ *be given number and let* $r\in R_2$. *Then for all* $x\in R_0$ *and* $n\in N$ *we have*

$$B_n(t-x;p,r;x) = \frac{(nx+1)^r}{n(nx+1)^{r-1}+pr} - x,$$

$$B_n((t-x)^2;p,r;x) = \left(\frac{(nx+1)^r}{n(nx+1)^{r-1}+pr} - x\right)^2 + \frac{(nx+1)^r}{[n(nx+1)^{r-1}+pr]^2},$$

$$B_n(e^{prt};p,r;x) = e^{p_n(nx+1)^r},$$

$$B_n((t-x)^2e^{prt};p,r;x) = \left\{\left(\frac{(nx+1)^r}{n(nx+1)^{r-1}+pr}e^{\frac{pr}{n(nx+1)^{r-1}+pr}} - x\right)^2 + \right.$$

$$\left. + \frac{(nx+1)^r}{[n(nx+1)^{r-1}+pr]^2}e^{\frac{pr}{n(nx+1)^{r-1}+pr}}\right\}e^{p_n(nx+1)^r}.$$

where

$$p_n := e^{\frac{pr}{n(nx+1)^{r-1}+pr}} - 1 \quad \textit{for} \quad n\in N.$$

LEMMA 4. *For every fixed* $p>0$ *and* $r\in R_2$ *there exist positive constants* $M_i \equiv M_i(p,r)$, $i=11,12$, *such that for all* $x\in R_0$, $n\in N$

$$v_{pr}(x)\,B_n\big(1/v_{pr}(t);p,r;x\big) \le M_{11},$$

$$v_{pr}(x)\,B_n\left(\frac{(t-x)^2}{v_{pr}(t)};p,r;x\right) \le \frac{M_{12}}{n^2}.$$

Applying (26) – (28) and (32) – (34) and Lemma 4 and arguing as in the proof of Lemma 2, we can prove the basic property of $B_{m,n}(f)$.

LEMMA 5. *For fixed* $p,q>0$ *and* $r,s\in R_2$ *there exists a positive constant* $M_{13} \equiv M_{13}(p,q,r,s)$ *such that*

(35) $$\left\|B_{m,n}(1/v_{pr,qs}(t,z);p,r,q,s;\cdot,\cdot)\right\|_{pr,qs} \le M_{13} \quad \textit{for} \quad m,n\in N.$$

Moreover for every $f\in C_{pr,qs}$ *we have*

(36) $$\left\|B_{m,n}(f;p,q,r,s;\cdot,\cdot)\right\|_{pr,qs} \le M_{13}\left\|f\right\|_{pr,qs} \quad \textit{for} \quad m,n\in N,\ r,s\in R_2.$$

The formulas (30) *and* (7) *and the inequality* (36) *show that* $B_{m,n}$, $m,n\in N$, *defined by* (30) *are linear positive operators from the space* $C_{pr,qs}$ *into* $C_{pr,qs}$.

4.2. Applying Lemma 3 – Lemma 5 and (26) – (28) and (32) – (34) and reasoning as in the proof of Theorem 1, we can prove the following

THEOREM 3. *Suppose that* $f\in C^1_{pr,qs}$ *with given* $p,q>0$ *and* $r,s\in R_2$. *Then there exists a positive constant* $M_{14}\equiv M_{14}(p,q,r,s)$ *such that for all* $m,n\in N$

$$\left\|B_{m,n}(f;p,q,r,s;\cdot,\cdot)-f(\cdot,\cdot)\right\|_{pr,qs}\le M_{14}\left\{\frac{1}{m}\left\|f'_x\right\|_{pr,qs}+\frac{1}{n}\left\|f'_y\right\|_{pr,qs}\right\}.$$

THEOREM 4. *Suppose that* $f\in C_{pr,qs}$, $p,q\in N_0$, $r,s\in R_2$. *Then there exists a positive constant* $M_{15}\equiv M_{15}(p,q,r,s)$ *such that*

$$\left\|B_{m,n}(f;p,q,r,s;\cdot,\cdot)-f(\cdot,\cdot)\right\|_{pr,qs}\le M_{15}\,\omega\left(f;C_{pr,qs};\frac{1}{m},\frac{1}{n}\right), \tag{37}$$

for all $m,n\in N$.

PROOF. Similarly as in the proof of Theorem 2 we shall apply the Stieklov function $f_{h,\delta}$ for $f\in C_{pr,qs}$, defined by (20).
Analogously as (21) – (23) we get

$$\left\|f_{h,\delta}-f\right\|_{pr,qs}\le\omega(f;C_{pr,qs};h,\delta), \tag{38}$$

$$\left\|(f_{h,\delta})'_x\right\|_{pr,qs}\le 2h^{-1}\,\omega(f;C_{pr,qs};h,\delta), \tag{39}$$

$$\left\|(f_{h,\delta})'_y\right\|_{pr;qs}\le 2\delta^{-1}\,\omega(f;C_{pr,qs};h,\delta), \tag{40}$$

for all $h,\delta>0$, which show that $f_{h,\delta}\in C^1_{pr,qs}$ if $f\in C_{pr,qs}$ and $h,\delta>0$. Now, for $B_{m,n}$ defined by (30), we can write

$$\begin{aligned}
&v_{pr,qs}(x,y)\left|B_{m,n}(f;p,q,r,s;x,y)-f(x,y)\right|\le\\
&\le v_{pr,qs}(x,y)\Big\{\left|B_{m,n}(f(t,z)-f_{h,\delta}(t,z);p,q,r,s;x,y)\right|+\\
&+\left|B_{m,n}(f_{h,\delta}(t,z);p,q,r,s;x,y)-f_{h,\delta}(x,y)\right|+\\
&+\left|f_{h,\delta}(x,y)-f(x,y)\right|\Big\}:=T_1+T_2+T_3.
\end{aligned}$$

By (28), (36) and (38),

$$T_1 \le \left\| B_{m,n}(f - f_{h,\delta}; p,q,r,s;\cdot,\cdot) \right\|_{pr,qs} \le M_{13} \left\| f - f_{h,\delta} \right\|_{pr,qs} \le M_{13}\omega(f;C_{pr,qs};h,\delta),$$

$$T_3 \le \omega(f;C_{pr,qs};h,\delta).$$

Applying Theorem 3 and (39) and (40), we get

$$T_2 \le M_{15}\left\{ \frac{1}{m}\left\|(f_{h,\delta})'_x\right\|_{pr,qs} + \frac{1}{n}\left\|(f_{h,\delta})'_y\right\|_{pr,qs} \right\} \le$$

$$\le 2M_{14}\,\omega(f;C_{pr,qs}h,\delta)\left\{ h^{-1}\frac{1}{m} + \delta^{-1}\frac{1}{n} \right\}.$$

From the above we deduce that there exists a positive constant $M_{16} \equiv M_{16}(p,q,r,s)$ such that

$$\text{(41)}\quad \left\| B_{m,n}(f;p,q,r,s;\cdot,\cdot) - f(\cdot,\cdot) \right\|_{pr,qs} \le M_{16}\,\omega(f;C_{pr,qs};h,\delta)\left\{ 1 + h^{-1}\frac{1}{m} + \delta^{-1}\frac{1}{n} \right\},$$

for $m,n \in N$ and $h,\delta > 0$. Now, for $m,n \in N$, setting $h = \frac{1}{m}$ and $\delta = \frac{1}{n}$ to (41), we obtain (37).

Theorem 4 and (29) imply

COROLLARY 2. *Let* $f \in C_{pr,qs}$, $p,q \in N_0$, $r,s \in R_2$. *Then*

$$\lim_{m,n\to\infty} \left\| B_{m,n}(f;p,q,r,s;\cdot,\cdot) - f(\cdot,\cdot) \right\|_{pr,qs} = 0.$$

Theorem 4 and Corollary 2 in our paper show that operators $B_{m,n}$, $m,n \in N$, give better degree of approximation of functions belonging to exponential weighted spaces than classical Szasz–Mirakyan operator $S_{m,n}$, examined for continuous and bounded functions in [3].

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