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ON THE OSCILLATION OF CERTAIN ADVANCED FUNCTIONAL DIFFERENTIAL EQUATIONS USING COMPARISON METHODS

Abstract: Some new criteria for the oscillation of advanced functional differential equations of the form

$$\frac{d}{dt}\left(\left[\frac{1}{a_{n-1}(t)}\frac{d}{dt}\frac{1}{a_{n-2}(t)}\frac{d}{dt}\cdots\frac{1}{a_1(t)}\frac{d}{dt}x(t)\right]^{\alpha}\right) + \delta\, q(t) f(x[g(t)]) = 0$$

are presented in this paper. A discussion of neutral equations will also be included.

Key words: oscillation, nonoscillation, advanced, nonlinear, comparison.

1. Introduction

In this paper we shall deal with the oscillatory behavior of solutions of the advanced functional differential equation

$$L_n x(t) + \delta q(t) f(x[g(t)]) = 0, \tag{1.1;δ}$$

where $n \geq 3$, $\delta = \pm 1$, and

$$\begin{cases} L_0\, x(t) = x(t) \\ L_k\, x(t) = \dfrac{1}{a_k(t)}\dfrac{d}{dt}\left(L_{k-1}x(t)\right), \quad k = 1, 2, \cdots, n-1 \\ L_n\, x(t) = \dfrac{d}{dt}\left([L_{n-1}x(t)]^{\alpha}\right). \end{cases} \tag{1.2}$$

In what follows we shall assume that

(i) $a_i(t) \in C([t_0, \infty), \mathbb{R}^+ = (0, \infty))$, $t_0 \geq 0$,

$$\int^{\infty} a_i(s)ds = \infty, \quad i = 1, 2, \cdots, n-1, \tag{1.3}$$

(ii) $q(t) \in C([t_0, \infty), \mathbb{R}^+)$,

(iii) $g(t) \in C([t_0, \infty), \mathbb{R} = (-\infty, \infty))$, $g'(t) \geq 0$ and $g(t) > t$ for $t \geq t_0$,

(iv) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ and $f'(x) \geq 0$ for $x \neq 0$, and

(v) α is the quotient of positive odd integers.

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $x : [T_x, \infty) \to \mathbb{R}$ such that $L_j x(t)$, $j = 0, 1, \cdots, n$ exist and are continuous on $[T_x, \infty)$, $T_x \geq t_0$. Our attention is restricted to those solutions $x \in \mathcal{D}(L_n)$ of equation $(1.1; \delta)$ which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq T_x$. We make the standing hypothesis that equation $(1.1; \delta)$ does possess such solutions. A solution of equation $(1.1; \delta)$ is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation $(1.1; \delta)$ is called oscillatory if all its solutions are oscillatory.

Recently, the present authors [1–7] have established some results for the oscillation of equation $(1.1; \delta)$ and other related equations with general deviating arguments as well as advanced arguments. The main goal of this paper is to obtain some new criteria for the oscillation of equation $(1.1; \delta)$ with advanced arguments.

2. Preliminaries

To formulate our results we shall use the following notation: For $p_i(t) \in C([t_0, \infty), \mathbb{R})$, $i = 1, 2, \cdots$, we define $I_0 = 1$,

$$I_i(t, s; p_i, p_{i-1}, \cdots, p_1) = \int_s^t p_i(u) I_{i-1}(u, s; p_{i-1}, \cdots, p_1) du, \quad i = 1, 2, \cdots .$$

It is easy to verify from the definition of I_i that

$$I_i(t, s; p_1, \cdots, p_i) = (-1)^i I_i(s, t; p_i, \cdots, p_1)$$

and

$$I_i(t, s; p_1, \cdots, p_i) = \int_s^t p_i(u) I_{i-1}(t, u; p_1, \cdots, p_{i-1}) du.$$

We shall need the following three lemmas.

Lemma 2.1. *If $x \in \mathcal{D}(\overline{L}_n)$, where $\overline{L}_n$ is L_n defined by (1.2) with $\alpha = 1$, then the following formulas hold for $0 \leq i \leq k \leq n-1$ and $t, s \in [t_0, \infty)$*

$$L_i x(t) = \sum_{j=i}^{k-1} I_{j-i}(t, s; a_{i+1}, \cdots, a_{k-1}) L_j x(s) \tag{2.1}$$

$$+ \int_s^t I_{k-i-1}(t,u;a_{i+1},\cdots,a_{k-1})a_k(u)L_k x(u)du$$

and

$$L_i x(t) = \sum_{j=i}^{k-1}(-1)^{j-i} I_{j-i}(s,t;a_j,\cdots,a_{i+1})L_j x(s) + (-1)^{k-i}\int_t^s I_{k-i-1}(u,t;a_{k-1},\cdots,a_{i+1})a_k(u)L_k x(u)du. \tag{2.2}$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

Lemma 2.2. *Suppose condition (1.3) holds. If $x \in \mathcal{D}(\overline{L}_n)$ where $\overline{L}_n$ is as in Lemma 2.1 is eventually of one sign, then there exist a $t_x \geq t_0 \geq 0$ and an integer ℓ, $0 \leq \ell \leq n$ with $n+\ell$ even for $x(t)\overline{L}_n x(t)$ nonnegative eventually, or $n+\ell$ odd for $x(t)\overline{L}_n x(t)$ nonpositive eventually and such that for every $t \geq t_x$,*

$$\begin{cases} \ell > 0 \text{ implies } x(t)\overline{L}_k x(t) > 0, & k = 0,1,\cdots,\ell \\ \ell \leq n-1 \text{ implies } (-1)^{\ell-k}x(t)\overline{L}_k x(t) > 0, & k = \ell,\ell+1,\cdots,n. \end{cases} \tag{2.3}$$

This lemma generalizes a well–known lemma of Kiguradze and can be proved similarly.

Lemma 2.3. [11, 12]. *Consider the integro–differential inequality with advanced argument*

$$y'(t) \geq \int_t^\infty Q(t,s)y[g(s)]ds, \tag{2.4}$$

where $Q \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ and $g(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $g(t) \geq t$ for $t \geq t_0 \geq 0$. If

$$\liminf_{t\to\infty} \int_t^{g(t)} \int_s^\infty Q(s,u)duds > \frac{1}{e}, \tag{2.5}$$

then inequality (2.4) has no eventually positive solutions.

3. Main results

The equation $(1.1;\delta)$ is said to be almost oscillatory if:

(i_1). for $\delta = 1$ and n even, every solution of (1.1;1) is oscillatory,

(i_2). for $\delta = 1$ and n odd, every unbounded solution of (1.1;1) is oscillatory,

(i_3). for $\delta = -1$ and n odd, every solution of $(1.1;-1)$ is oscillatory, (i$_4$). for $\delta = -1$ and n even, every unbounded solution of $(1.1;-1)$ is oscillatory.

Now, we present the following result.

Theorem 3.1. *Let* $1 \le \ell \le n-1$, $(-1)^{n-\ell}\delta = -1$ *and*

$$f(x) \ \ge\ x^\alpha \quad \text{for} \quad x \neq 0. \tag{3.1}$$

If for $1 \le \ell \le n-2$ *and all large* $T \ge t_0$ *and* $t \ge T$,

$$\begin{aligned}(3.2;\delta)\qquad \liminf_{t\to\infty} \int_t^{g(t)} & a_1(s) I_{\ell-1}(s,T;a_2,\cdots,a_\ell) \\ & \times \left(\int_s^\infty I_{n-\ell-2}(u,s;a_{n-2},\cdots,a_{\ell+1}) a_{n-1}(u) \right. \\ & \times \left. \left(\int_u^\infty q(\tau)d\tau \right)^{1/\alpha} du \right) ds \ >\ \frac{1}{e}\end{aligned}$$

and for $\ell = n-1$ *there exists* $\eta(t) \in C([t_0,\infty), I\!R)$ *such that* $g(t) \ge \eta(t) > t$ *for all large* t *and the equation*

$$\begin{aligned}(3.2;n-1)\quad \left(\left(\frac{1}{a_{n-1}(t)} y'(t) \right)^\alpha \right)' + q(t) I_{n-2}^\alpha(g(t),\eta(t);a_1,\cdots,a_{n-2}) & \\ \times\, y^\alpha[\eta(t)] \ = \ 0 &\end{aligned}$$

is oscillatory, then $\mathcal{N}_\ell = \emptyset$, *where* $\mathcal{N}_\ell$ *is the set of all nonoscillatory solutions of equation* $(1.1;\delta)$ *satisfying (2.3).*

Proof. Let $x \in \mathcal{N}_\ell$ and assume that $x(t) > 0$ for $t \ge t_0 \ge 0$. Since $L_n x(t)$ is of one sign for $t \ge t_0$, then there exists a $t_1 \ge t_0$ such that $L_j x(t)$ $(0 \le j \le n-1)$ are also of one sign for $t \ge t_1$. Moreover,

$$L_n x(t) \ =\ \frac{d}{dt}(L_{n-1}^\alpha x(t)) \ =\ \alpha L_{n-1}^{\alpha-1} x(t) \overline{L}_n x(t),$$

where $\overline{L}_n$ is defined as in Lemma 2.1, we see that the sign of $\overline{L}_n$ and L_n are the same for $t \ge t_1$. First, we let $1 \le \ell \le n-2$. Replacing i and k by ℓ and $n-1$, respectively in (2.2), we get

$$\begin{aligned}(3.3)\qquad L_\ell\, x(t) = \sum_{j=\ell}^{n-2} (-1)^{j-\ell} I_{j-\ell}(s,t;a_j,\cdots,a_{\ell+1}) L_j x(s) + {} & (-1)^{n-\ell-1} \\ \times \int_t^s I_{n-\ell-2}(u,t;a_{n-2},\cdots,a_{\ell+1}) a_{n-1}(u) L_{n-1} x(u) du & \\ \text{for} \quad s \ge t \ge t_1. &\end{aligned}$$

Using (2.3) in (3.3), we have

$$(3.4)\quad L_\ell x(t) \;\geq\; (-1)^{n-\ell-1} \times \int_t^\infty I_{n-\ell-2}(u,t;a_{n-2},\cdots,a_{\ell+1})a_{n-1}(u)L_{n-1}x(u)du \;\text{ for }\; t\geq t_1.$$

Next, integrating equation $(1.1;\delta)$ from $u\geq t\geq t_1$ to s and letting $s\to\infty$, one can easily find

$$(3.5)\qquad \delta L_{n-1}x(u) \;\geq\; \left(\int_u^\infty q(\tau)f(x[g(\tau)])d\tau\right)^{1/\alpha} \geq \left(\int_u^\infty q(\tau)d\tau\right)^{1/\alpha} f^{1/\alpha}(x[g(u)]) \quad\text{for}\quad u\geq t\geq t_1.$$

Substituting (3.5) in (3.4), we have

$$(3.6)\quad L_\ell x(t) \;\geq\; \int_t^\infty I_{n-\ell-2}(u,t;a_{n-2},\cdots,a_{\ell+1})a_{n-1}(u) \times \left(\int_u^\infty q(\tau)d\tau\right)^{1/\alpha} f^{1/\alpha}(x[g(u)])du \quad\text{for}\quad t\geq t_1.$$

Replacing i,k and s by $1,\ell$ and t_1 respectively in (2.1), we get

$$(3.7)\qquad x'(t) = a_1(t)\sum_{j=1}^{\ell-1} I_{j-1}(t,t_1;a_2,\cdots,a_j)L_jx(t_1) + a_1(t)\int_{t_1}^t I_{\ell-2}(t,u;a_2,\cdots,a_{\ell-1})a_\ell(u)L_\ell x(u)du \geq a_1(u)I_{\ell-1}(t,t_1;a_2,\cdots,a_\ell)L_\ell x(t) \quad\text{for}\quad t\geq t_1.$$

Combining (3.6) and (3.7) and using (3.1), we obtain

$$(3.8)\quad x'(t)\geq \int_t^\infty a_1(t)I_{\ell-1}(t,t_1;a_2,\cdots,a_\ell)I_{n-\ell-2}(u,t;a_{n-2},\cdots,a_{\ell+1}) \times a_{n-1}(u)\left(\int_u^\infty q(\tau)d\tau\right)^{1/\alpha} x[g(u)]du.$$

Inequality (3.8), in view of condition $(3.2;\ell)$ and Lemma 2.3 has no eventually positive solutions, a contradiction.

Next, let $\ell = n-1$. This is the case when $\delta = 1$. Replacing i,k by 0 and $n-2$ in (2.1), we can easily obtain

$$(3.9)\qquad x(t) \;\geq\; I_{n-2}(t,s;a_1,\cdots,a_{n-2})L_{n-2}x(s) \quad\text{for}\quad t\geq s\geq t_1.$$

Replacing t and s by $g(t)$ and $\eta(t)$ respectively in (3.9), we have

$$x[g(t)] \geq I_{n-2}(g(t),\eta(t);a_1,\cdots,a_{n-2})L_{n-2}x[\eta(t)] \quad \text{for} \quad g(t) > \eta(t) \geq t_1. \tag{3.10}$$

Using (3.1) and (3.10) in equation $(1.1;\delta)$, we get

$$\begin{aligned} -L_n x(t) &= -\frac{d}{dt}\left(\frac{1}{a_{n-1}(t)}\frac{d}{dt}L_{n-2}x(t)\right)^{\alpha} = q(t)f(x[g(t)]) \\ &\geq q(t)x^{\alpha}[g(t)] \\ &\geq q(t)I^{\alpha}_{n-2}(g(t),\eta(t);a_1,\cdots,a_{n-2})\left(L_{n-2}x[\eta(t)]\right)^{\alpha}, \quad t \geq t_1. \end{aligned}$$

Set $y(t) = L_{n-1}x(t) > 0$ for $t \geq t_1$. Then, $y(t)$ satisfies

$$\left(\left(\frac{1}{a_{n-1}(t)}y'(t)\right)^{\alpha}\right)' + q(t)I^{\alpha}_{n-2}(g(t),\eta(t);a_1,\cdots,a_{n-2})y^{\alpha}[\eta(t)] \leq 0 \quad \text{for} \quad t \geq t_1.$$

Now, by applying a result in [5, Chapter 2], we see that the equation

$$\left(\left(\frac{1}{a_{n-1}(t)}z'(t)\right)^{\alpha}\right)' + q(t)I^{\alpha}_{n-2}(g(t),\eta(t);a_1,\cdots,a_{n-2})z^{\alpha}[\eta(t)] = 0$$

has an eventually positive solution, which contradicts our assumption. This completes the proof. ■

Next, we shall provide the sufficient conditions which ensure that $\mathcal{N}_n = \emptyset$, where $\mathcal{N}_n$ is the set of all nonoscillatory solutions of equation $(1.1;\delta)$ satisfying $x(t)L_jx(t) > 0,\ 0 \leq j \leq n$.

Theorem 3.2. *Let $\delta = -1$ and conditions (3.1) hold. If, either*

$$\limsup_{t\to\infty}\int_t^{g(t)} q(s)I^{\alpha}_{n-1}(g(s),g(t);a_1,\cdots,a_{n-1})ds > 1, \tag{3.11}$$

or

$$\limsup_{t\to\infty}\int_t^{g(t)} I_{n-2}(g(t),u;a_1,\cdots,a_{n-2})a_{n-1}(u) \times\left(\int_t^u q(s)ds\right)^{1/\alpha} du > 1, \tag{3.12}$$

then $\mathcal{N}_n = \emptyset$.

Proof. Let $x \in \mathcal{N}_n$ and assume that $x(t) > 0$ for $t \geq t_0 \geq 0$. Then there exists a $t_1 \geq t_0$ such that

$$L_i x(t) \ > \ 0 \quad (0 \leq i \leq n) \quad \text{on} \quad [t_1, \infty). \tag{3.13}$$

From (2.1) with i, k, t and s replaced by $0, n-1, g(s)$ and $g(t)$, respectively,

$$x[g(s)] = \sum_{j=0}^{n-2} I_j(g(s), g(t); a_1, \cdots, a_j) L_j x[g(t)] \\ + \int_{g(t)}^{g(s)} I_{n-2}(g(s), u; a_1, \cdots, a_{n-2}) a_{n-1}(u) L_{n-1} x(u) du. \tag{3.14}$$

Using (3.13) and noting that $L_{n-1}x$ is increasing, we easily get

$$x[g(s)] \ \geq \ I_{n-1}(g(s), g(t); a_1, \cdots, a_{n-1}) L_{n-1} x[g(t)] \quad \text{for} \quad t < s < g(t). \tag{3.15}$$

Using (3.1) and (3.15) in equation $(1, 1; -1)$, we have

$$\frac{d}{ds}(L_{n-1}^{\alpha} x(s)) = q(s) f(x[g(s)]) \ \geq \ q(s) x^{\alpha}[g(s)] \\ \geq q(s) I_{n-1}^{\alpha}(g(s), g(t); a_1, \cdots, a_{n-1}) L_{n-1}^{\alpha} x[g(t)] \quad \text{for} \quad t_1 < t < s < g(t). \tag{3.16}$$

Integrating both sides of (3.16) from $t \geq t_1$ to $g(t)$, one can easily obtain

$$L_{n-1}^{\alpha} x[g(t)] \left[\int_t^{g(t)} I_{n-1}^{\alpha}(g(s), g(t); a_1, \cdots, a_{n-1}) ds - 1 \right] \ \leq \ 0.$$

This is inconsistent with (3.11).

Next, it follows from (3.14) with $g(s)$ and $g(t)$ replaced by $g(t)$ and t, respectively that

$$x[g(t)] \geq \int_t^{g(t)} I_{n-2}(g(t), u; a_1, \cdots, a_{n-2}) a_{n-1}(u) L_{n-1} x(u) du \quad \text{for} \quad t < u < g(t). \tag{3.17}$$

Integrating equation $(1, 1; -1)$ from t to u, we get

$$L_{n-1} x(u) \ \geq \ \left(\int_t^u q(s) x^{\alpha}[g(s)] ds \right)^{1/\alpha} \quad \text{for} \quad u \geq t \geq t_1. \tag{3.18}$$

Substituting (3.18) in (3.17), we have

$$x[g(t)] \geq \int_t^{g(t)} I_{n-2}(g(t), u; a_1, \cdots, a_{n-2}) a_{n-1}(u) \left(\int_t^u q(s)ds\right)^{1/\alpha} x[g(t)]du,$$

or

$$1 \; \geq \; \int_t^{g(t)} I_{n-2}(g(t), u; a_1, \cdots, a_{n-2}) a_{n-1}(u) \left(\int_t^u q(s)ds\right)^{1/\alpha} ds,$$

which contradicts condition (3.12). This completes the proof. ■

From Theorems 3.1 and 3.2 the following result follows:

Theorem 3.3. *Suppose (i) – (v) and condition (3.1) hold. Equation (1.1;δ) is almost oscillatory if*

(I$_1$). *for $\delta = 1$ and n even, condition (3.2;ℓ) ($\ell = 1, 3, \cdots, n-3$) hold and there exists $\eta(t) \in C([t_0, \infty), I\!R)$ such that $g(t) \geq \eta(t) \geq t$ for all large t and equation (3.2;n-1) is oscillatory,*

(I$_2$). *for $\delta = 1$ and n odd, condition (3.2;ℓ) ($\ell = 2, 4, \cdots, n-3$) hold and there exists $\eta(t) \in C([t_0, \infty), I\!R)$ such that $g(t) \geq \eta(t) \geq t$ for all large t and equation (3.2;n-1) is oscillatory,*

(I$_3$). *for $\delta = -1$ and n odd, condition (3.2;ℓ) ($\ell = 1, 3, \cdots, n-2$) and either (3.11) or (3.12) holds,*

(I$_4$). *for $\delta = -1$ and n even, condition (3.2;ℓ) ($\ell = 2, 4, \cdots, n-2$) and either (3.11) or (3.12) holds.*

Example 3.1. Consider the advanced differential equation

$$\left(\left(\left(e^{-t}\left(e^{-t}\left(e^{-t}x'(t)\right)'\right)'\right)\right)^{\alpha}\right)' + 4\alpha(24)^{\alpha}x^{\alpha}[4t] \;=\; 0, \quad t \geq 0 \tag{3.19}$$

where α is as in equation $(1.1;\delta)$. All conditions of Theorem 3.3 (I$_2$) are satisfied and hence all unbounded solutions of equation (3.19) are oscillatory.

We note that equation (3.19) has a bounded nonoscillatory solution $x(t) = e^{-t}$.

In the case when $\alpha = 1$, we present the following result.

Theorem 3.4. *Let $n \geq 2,\ 1 \leq \ell \leq n-1,\ (-1)^{n-\ell}\delta = -1$, conditions (i) – (iv) and (3.1) hold with $\alpha = 1$. If for all large $T \geq t_0 \geq 0$ and $t \geq T$,*

$$\liminf_{t\to\infty} \int_t^{g(t)} a_1(s) I_{\ell-1}(s, T; a_2, \cdots, a_\ell) \tag{3.20;ℓ}$$

$$\times \int_s^\infty I_{n-\ell-1}(u,s;a_{n-1},\cdots,a_{\ell+1})q(u)duds \;>\; \frac{1}{e}\,,$$

then $\mathcal{N}_\ell = \emptyset$.

Proof. Let $x \in \mathcal{N}_\ell$ and assume that $x(t) > 0$ for $t \geq t_0 \geq 0$. Proceeding as in the proof of Theorem 3.1 and replacing i and k by ℓ and n, respectively, in (2.2), we have

$$\begin{aligned} L_\ell\, x(t) &= \sum_{j=\ell}^{n-1}(-1)^{j-\ell} I_{j-\ell}(t,s;a_j,\cdots,a_{\ell+1})L_j x(s) \\ &\quad +(-1)^{n-\ell}\int_t^s I_{n-\ell-1}(u,t;a_{n-1},\cdots,a_{\ell+1})L_n x(u)du \\ &\geq \int_t^\infty I_{n-\ell-1}(u,t;a_{n-1},\cdots,a_{\ell+1})q(u)x[g(u)]du \quad \text{for} \quad t \geq t_1. \end{aligned} \tag{3.21}$$

Also, as in the proof of Theorem 3.1, we see (3.7) holds for $t \geq t_1$. Combining (3.7) with (3.21), we get

$$\begin{aligned} x'(t) \geq \int_t^\infty & a_1(t) I_{\ell-1}(t,t_1;a_2,\cdots,a_\ell) I_{n-\ell-1}(u,t;a_{n-1},\cdots,a_{\ell+1}) \\ & \times q(u)\, x[g(u)]du. \end{aligned} \tag{3.22}$$

Inequality (3.22), in view of condition $(3.20;\ell)$ and Lemma 2.3 has no eventually positive solution, a contradiction. This completes the proof. ■

Theorem 3.5. *Let $n \geq 2$, conditions (i) – (iv) and (3.1) hold with $\alpha = 1$. Equation $(1.1;\delta)$ is almost oscillatory if*

(i_1). *for $\delta = 1$ and n even, condition* (3.20;ℓ) ($\ell = 1,3,\cdots,n-1$),

(i_2). *for $\delta = 1$ and n odd, condition (3.20;ℓ)* ($\ell = 2,4,\cdots,n-1$),

(i_3). *for $\delta = -1$ and n odd, condition (3.20;ℓ)* ($\ell = 1,3,\cdots,n-2$) *and either condition (3.11) or (3.12),*

(i_4). *for $\delta = -1$ and n even, condition (3.20;ℓ)* ($\ell = 2,4,\cdots,n-2$) *and either condition (3.11) or (3.12).*

Note advanced differential equations can differ from ordinary differential equations with respect to oscillation. For example

$$\left(\frac{1}{t}x'(t)\right)' + \frac{3}{4t^3}x[ct] \;=\; 0, \quad t \geq 1$$

is oscillatory by Theorem 3.5 (i_1) for all $c > \exp(8/3e)$, while the corresponding ordinary differential equation

$$\left(\frac{1}{t}x'(t)\right)' + \frac{3}{4t^3}x(t) \;=\; 0, \quad t \geq 1$$

has a nonoscillatory solution $x(t) = \sqrt{t}$.

Next, we obtain the following results.

Theorem 3.6. *Let* $1 \le \ell \le n-1$, $(-1)^{n-\ell}\delta = -1$ *and*

$$\int^{\pm\infty} \frac{du}{f^{1/\alpha}(u)} < \infty. \tag{3.23}$$

If for $1 \le \ell \le n-1$ *and all large* $T \ge t_0$, $t \ge T$,

$$\int^{\infty} a_1(s) I_{\ell-1}(s,T;a_2,\cdots,a_\ell)\Big(\int_s^{\infty} I_{n-\ell-2}(u,s;a_{n-2},\cdots,a_{\ell+1}) \times a_{n-1}(u)\Big(\int_u^{\infty} q(\tau)d\tau\Big)^{1/\alpha} du\Big)\, ds = \infty, \tag{3.24;ℓ}$$

and for $\ell = n-1$,

$$\int^{\infty} a_1[g(s)]g'(s)\left(\int_s^{g(s)} I_{n-2}(g(s),u;a_2,\cdots,a_{n-2}) \times a_{n-1}(u)\left(\int_u^{\infty} q(\tau)d\tau\right)^{1/\alpha} du\right) ds = \infty, \tag{3.24;$n-1$}$$

then $\mathcal{N}_\ell = \emptyset$.

Proof. Let $x \in \mathcal{N}_\ell$ and assume that $x(t) > 0$ for $t \ge t_0 \ge 0$. As in the proof of Theorem 3.1, we obtain (3.6) and (3.7), $t \ge t_1$, $1 \le \ell \le n-2$. Combining (3.6) and (3.7), we obtain

$$\frac{x'(t)}{f^{1/\alpha}(x(t))} \ge a_1(t) I_{\ell-1}(t,t_1;a_2,\cdots,a_\ell) \times \int_t^{\infty} I_{n-\ell-2}(u,t;a_{n-2},\cdots,a_{\ell+1}) a_{n-2}(u)\left(\int_u^{\infty} q(\tau)d\tau\right)^{1/\alpha} du. \tag{3.25}$$

Integrating (3.25) from t_1 to $T \ge t_1$, we have

$$\int_{t_1}^{T} a_1(t) I_{\ell-1}(t,t_1;a_2,\cdots,a_\ell)\left(\int_t^{\infty} I_{n-\ell-2}(u,t;a_{n-2},\cdots,a_{\ell+1}) \times a_{n-2}(u)\left(\int_u^{\infty} q(\tau)d\tau\right)^{1/\alpha} du\right) dt \le \int_{x(t_1)}^{x(T)} \frac{du}{f^{1/\alpha}(u)}.$$

Letting $T \to \infty$ in the above inequality and using (3.23) we arrive at a contradiction to (3.24;ℓ), $1 \le \ell \le n-2$.

Next, let $\ell = n-1$. Replacing i, k, s and t by $1, n-1, t$ and $g(t)$, respectively, we have

$$x'[g(t)] \;\geq\; a_1[g(t)] \int_t^{g(t)} I_{n-3}(g(t), u; a_2, \cdots, a_{n-2}) \times a_{n-1}(u) L_{n-1}x(u)du. \tag{3.26}$$

As in the proof of Theorem 3.1, we obtain (3.5). Combining (3.5) and (3.26) we have

$$\begin{aligned} x'[g(t)]g'(t) \;&\geq\; a_1[g(t)]g'(t) \int_t^{g(t)} I_{n-3}(g(t), u; a_2, \cdots, a_{n-2}) \\ &\times\, a_{n-1}(u) \left(\int_u^{\infty} q(\tau) d\tau \right)^{1/\alpha} f^{1/\alpha}(x[g(u)])du, \quad t \geq t_1 \end{aligned}$$

or

$$\begin{aligned} \frac{x'[g(t)]g'(t)}{f^{1/\alpha}(x[g(t)])} \;&\geq\; a_1[g(t)]g'(t) \int_t^{g(t)} I_{n-3}(g(t), u; a_2, \cdots, a_{n-2}) \\ &\times a_{n-1}(u) \left(\int_u^{\infty} q(\tau) d\tau \right)^{1/\alpha} du. \end{aligned}$$

The rest of the proof is similar to the above case and hence omitted. This completes the proof. ■

Theorem 3.7. *Let $\delta = -1$. If, either*

$$-f(-xy) \;\geq\; f(xy) \;\geq\; f(x)f(y) \quad \text{for} \quad xy > 0, \tag{3.27}$$

$$\frac{u^{\alpha}}{f(u)} \;\to\; 0 \quad \text{as} \quad u \to \infty \tag{3.28}$$

and

$$\limsup_{t\to\infty} \int_t^{g(t)} q(s) f(I_{n-1}(g(s), g(t); a_1, \cdots, a_{n-1}))ds \;>\; 0 \tag{3.29}$$

or

$$\limsup_{t\to\infty} \int_t^{g(t)} I_{n-2}(g(t), u; a_1, \cdots, a_{n-2}) \times a_{n-1}(u) \left(\int_t^{u} q(s) ds \right)^{1/\alpha} du \;>\; 0, \tag{3.30}$$

then $\mathcal{N}_n = \emptyset$.

Proof. The proof can be modelled on that of Theorem 3.2 and hence omitted. ■

Theorem 3.8. *Let $\delta = -1$, condition (3.27) hold and*

$$\int^{\pm\infty} \frac{du}{f(u^{1/\alpha})} < \infty. \tag{3.31}$$

If

$$\int^{\infty} q(s) f(I_{n-1}(g(s), s; a_1, \cdots, a_{n-1}) ds = \infty, \tag{3.32}$$

then $\mathcal{N}_n = \emptyset$.

Proof. Let $x \in \mathcal{N}_n$ and assume that $x(t) > 0$ for $t \geq t_0 \geq 0$. Then there exists a $t_1 \geq t_0$ such that (3.13) holds on $[t_1, \infty)$. Replacing i, k, t and s in (2.1) by $0, n-1, g(t)$ and t, respectively, we get

$$\begin{aligned} x[g(t)] &= \sum_{j=0}^{n-2} I_j(g(t), t; a_1, \cdots, a_j) L_j x(t) \\ &\quad + \int_t^{g(t)} I_{n-2}(g(t), u; a_1, \cdots, a_{n-2}) a_{n-1}(u) L_{n-1} x(u) du \\ &\geq \int_t^{g(t)} I_{n-2}(g(t), u; a_1, \cdots, a_{n-2}) a_{n-1}(u) L_{n-1} x(u) du \\ &\geq I_{n-1}(g(t), t; a_1, \cdots, a_{n-1}) L_{n-1} x(t), \quad t \geq t_1. \end{aligned}$$

Set $u(t) = L_{n-1}^{\alpha} x(t)$. Then, $u(t)$ satisfies

$$\begin{aligned} u'(t) &= L_n x(t) = -\delta L_n x(t) = q(t) f(x[g(t)]) \\ &\geq q(t) f(I_{n-1}(g(t), t; a_1, \cdots, a_{n-1})) f(u^{1/\alpha}(t)) \quad \text{for} \quad t \geq t_1. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{t_1}^{T} q(t) f(I_{n-1}(g(t), t; a_1, \cdots, a_{n-1})) dt &\leq \int_{t_1}^{T} \frac{u'(t)}{f(u^{1/\alpha})} \\ &= \int_{u(t_1)}^{u(T)} \frac{dw}{f(w^{1/\alpha})}. \end{aligned}$$

Letting $T \to \infty$, we find

$$\int_{t_1}^{\infty} q(t) f(I_{n-1}(g(t), t; a_1, \cdots, a_{n-1})) dt \leq \int_{u(t_1)}^{\infty} \frac{dw}{f(w^{1/\alpha})} < \infty.$$

This contradicts (3.32) and completes the proof. ■

Combining Theorems 3.6 – 3.8, we have the following result.

Theorem 3.9. *Suppose that (i) – (v) and condition (3.23) hold. A sufficient condition for equation (1.1;δ) to be almost oscillatory is that*

(I$_1$). *when $\delta = 1$ and n even, condition (3.24;ℓ) ($\ell = 1, 3, \cdots, n-3$) and (3.24;n-1) hold,*

(I$_2$). *when $\delta = 1$ and n odd, condition (3.24;ℓ) ($\ell = 2, 4, \cdots, n-3$) and (3.24;n-1) hold,*

(I$_3$). *when $\delta = -1$ and n odd, condition (3.24;ℓ) ($\ell = 1, 3, \cdots, n-2$) and either (3.27) and (3.29), (3.30) or (3.27) and (3.32) hold,*

(i$_4$). *when $\delta = -1$ and n even, condition (3.24;ℓ) ($\ell = 2, 4, \cdots, n-2$) and either (3.27) and (3.29), (3.30) or (3.27) and (3.32) hold.*

When $\alpha = 1$, we can easily obtain the following immediate results.

Theorem 3.10. *Let $n \geq 2$, $\alpha = 1$, $1 \leq \ell \leq n-1$, $(-1)^{n-\ell}\delta = -1$, conditions (i) – (iv) hold and*

$$\int^{\pm\infty} \frac{du}{f(u)} < \infty. \tag{3.33}$$

If for all large $T \geq t_0$, $t \geq T$,

$$\int^{\infty} a_1(s) I_{\ell-1}(s, T; a_2, \cdots, a_\ell) \int_s^{\infty} I_{n-\ell-1}(u, s; a_{n-1}, \cdots, a_{\ell+1}) \times q(u)\, du ds = \infty, \tag{3.34; ℓ}$$

then $\mathcal{N}_\ell = \emptyset$.

Theorem 3.11. *Let $n \geq 2$, conditions (i) – (iv) and (3.33) hold. A sufficient condition for equation (1.1;δ) with $\alpha = 1$ to be almost oscillatory is that*

(i$_1$). *when $\delta = 1$ and n even, condition (3.34;ℓ) ($\ell = 1, 3, \cdots, n-1$) hold,*

(i$_2$). *when $\delta = 1$ and n odd, condition (3.34;ℓ) ($\ell = 2, 4, \cdots, n-1$) hold,*

(i$_3$). *when $\delta = -1$ and n odd, condition (3.34;ℓ) ($\ell = 1, 3, \cdots, n-2$) and either (3.27) and (3.29), (3.30), or (3.27) and (3.32) with $\alpha = 1$ hold,*

(i$_4$). *when $\delta = -1$ and n even, condition (3.34;ℓ) ($\ell = 2, 4, \cdots, n-2$) and either (3.27) and (3.29), (3.30), or (3.27) and (3.32) with $\alpha = 1$ hold.*

4. Oscillation of neutral equations

In this section, we shall extend the results of Section 3 to neutral equations of the type

$$(4.1;\delta) \qquad \frac{d}{dt}\left(L_{n-1}(x(t)+p(t)x[\sigma(t)])\right)^{\alpha}+\delta q(t)f(x[g(t)]) \;=\; 0,$$

where conditions (i) – (v) hold, and

(vi). $p(t)\in C([t_0,\infty),[0,\infty))$,

(vii). $\sigma(t)\in C([t_0,\infty),\mathbb{R})$ and $\lim_{t\to\infty}\sigma(t)=\infty$.

If we define

$$(4.2) \qquad z(t) \;=\; x(t)+p(t)x[\sigma(t)],$$

then equation (4.1) becomes

$$(4.3;\delta) \qquad \frac{d}{dt}(L_{n-1}z(t))^{\alpha}+\delta q(t)f(x[g(t)]) \;=\; 0.$$

If $x(t)$ is a nonoscillatory solution of equation $(4.1;\delta)$, say, $x(t)>0$ and $x[\sigma(t)]>0$ for $t\ge t_0\ge 0$, then $z(t)>0$ for $t\ge t_0$ and there exists a $t_1\ge t_0$ and an integer ℓ, $1\le \ell\le n$ such that

$$(4.4) \qquad z'(t) \;>\; 0 \quad \text{for} \quad t\ge t_1.$$

Now, we shall examine the following two cases:

$$(I).\ \{0\le p(t)\le 1,\ \ \sigma(t)<t\} \quad \text{and} \quad (II).\ \{p(t)\ge 1,\ \ \sigma(t)>t\}.$$

For the case (I), we assume that

$$(4.5) \qquad 0\le p(t)\le 1,\ \ \sigma(t)<t \ \text{ and } \ \sigma(t) \ \text{ is strictly increasing for } \ t\ge t_0 \\ \text{and} \ \ p(t)\not\equiv 1 \ \ \text{eventually.}$$

Now, we have for $t\ge t_1$,

$$(4.6) \qquad \begin{aligned} x(t) &= z(t)-p(t)x[\sigma(t)] \\ &= z(t)-p(t)[z[\sigma(t)]-p[\sigma(t)]x[\sigma\circ\sigma(t)]] \\ &\ge z(t)-p(t)z[\sigma(t)] \ \ge\ (1-p(t))z(t). \end{aligned}$$

Using (4.6) in equation $(4.3;\delta)$, we have

$$(4.7;\delta) \qquad -\delta\frac{d}{dt}(L_{n-1}z(t))^{\alpha}=q(t)f(x[g(t)])$$

$$\geq q(t)f((1-p[g(t)])z[g(t)]) \quad \text{for} \quad t \geq t_1.$$

Next, for the case (II), we assume that

$$(4.8) \qquad p(t) \geq 1 \;\text{ and }\; p(t) \not\equiv 1 \;\text{ eventually,}\; \sigma(t) > t \\ \text{and } \sigma(t) \text{ is strictly increasing for } t \geq t_0 \geq 0.$$

We also let

$$p^*(t) \;=\; \frac{1}{p[\sigma^{-1}(t)]}\left(1 - \frac{1}{p[\sigma^{-1}\circ\sigma^{-1}(t)]}\right) \quad \text{for all large} \quad t,$$

where σ^{-1} is the inverse function of σ.

Now, since (4.4) holds, we have

$$\begin{aligned}(4.9) \qquad x(t) &= \frac{1}{p[\sigma^{-1}(t)]}\left(z[\sigma^{-1}(t)] - x[\sigma^{-1}(t)]\right)\\ &= \frac{z[\sigma^{-1}(t)]}{p[\sigma^{-1}(t)]} - \frac{1}{p[\sigma^{-1}(t)]}\left(\frac{z[\sigma^{-1}\circ\sigma^{-1}(t)]}{p[\sigma^{-1}\circ\sigma^{-1}(t)]} - \frac{x[\sigma^{-1}\circ\sigma^{-1}(t)]}{p[\sigma^{-1}\circ\sigma^{-1}(t)]}\right)\\ &\geq \frac{z[\sigma^{-1}(t)]}{p[\sigma^{-1}(t)]} - \frac{z[\sigma^{-1}\circ\sigma^{-1}(t)]}{p[\sigma^{-1}(t)]p[\sigma^{-1}\circ\sigma^{-1}(t)]}\\ &\geq \frac{1}{p[\sigma^{-1}(t)]}\left[1 - \frac{1}{p[\sigma^{-1}\circ\sigma^{-1}(t)]}\right] z[\sigma^{-1}(t)]\\ &= p^*(t)z[\sigma^{-1}(t)] \quad \text{for} \quad t \geq t_1.\end{aligned}$$

Using (4.9) in equation $(4.3;\delta)$, we get

$$\begin{aligned}(4.10;\delta) \qquad -\delta\frac{d}{dt}(L_{n-1}z(t))^\alpha &= q(t)f(x[g(t)])\\ &\geq q(t)f(p^*[g(t)]z[\sigma^{-1}\circ g(t)]) \quad \text{for} \quad t \geq t_1.\end{aligned}$$

It follows from the above discussion that Theorem 3.3 (as well as other results of Section 3) can be applied to equation $(4.1;\delta)$ if in addition we assume that conditions (vi), (vii) and (4.5) hold. In this case, $q(t)$ in Theorem 3.3 is replaced by $q(t)(1-p[g(t)])^\alpha$.

Also, we see that Theorem 3.3 (say) is applicable to equation $(4.1;\delta)$ provided that conditions (vi), (vii) and (4.8) hold. In this case, $q(t)$ in Theorem 3.3 is replaced by $q(t)(p^*[g(t)])^\alpha$ and $g(t)$ is replaced by $\sigma^{-1}\circ g(t)(\;>\;t)$.

The formulation of these results as well as others are left to the reader.

5. Further results for the oscillation of equation (1.1;1)

In this section we shall extend some of the results given in the previous sections to equation (1.1;1) when the function f need not be monotonic.

We need the following notations and a lemma due to Mahfoud [10]. Let

$$\mathbb{R}_{t_0} = \begin{cases} (-\infty, -t_0] \cup [t_0, \infty) & \text{if} \quad t_0 > 0 \\ (-\infty, 0) \cup (0, \infty) & \text{if} \quad t_0 = 0 \end{cases}$$

and

$$C_B(\mathbb{R}_{t_0}) = \{f \in C(\mathbb{R}) : f \ \text{ is of bounded variation on any interval } [a,b] \subset \mathbb{R}_{t_0}\}.$$

Lemma 5.1. [10]. *Suppose $t_0 > 0$ and $f \in C(\mathbb{R})$. Then, $f \in C_B(\mathbb{R}_{t_0})$ if and only if $f(x) = H(x)G(x)$ for all $x \in \mathbb{R}$, where $G : \mathbb{R}_{t_0} \to \mathbb{R}^+$ is nondecreasing on $(-\infty, -t_0)$ and nonincreasing on (t_0, ∞) and $H : \mathbb{R}_{t_0} \to \mathbb{R}$ is nondecreasing on $\mathbb{R}_{t_0}$.*

To obtain such extensions, we assume that $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$ and let G and H be a pair of continuous components of f and H being the nondecreasing one.

As in the proofs presented above, if $x(t)$ is a nonoscillatory solution of equation (1.1;1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$, then there exist a $t_1 \geq t_0$ and a constant $b > 0$ such that

$$L_{n-1}x(t) \leq b \quad \text{for} \quad t \geq t_1. \tag{5.1}$$

Integrating (5.1), $(n-1)$–times, there exist a $t_2 \geq t_1$ and a constant $K > 0$ such that $g(t) \geq t_1$ for $t \geq t_2$ and

$$\begin{aligned} x[g(t)] &\leq K \int_{t_1}^{g(t)} a_1(s_1) \int_{t_1}^{s_1} a_2(s_2) \int_{t_1}^{s_2} \cdots \\ &\quad \times \int_{t_1}^{s_{n-2}} a_{n-1}(s) ds ds_{n-2} \cdots ds_1 \\ &= KI(g(t), t_1) \quad \text{for} \quad t \geq t_2. \end{aligned} \tag{5.2}$$

Now, it follows from equation (1.1;1) and Lemma 5.1 that

$$\begin{aligned} -\frac{d}{dt}(L_{n-1}x(t))^\alpha &= q(t)f(x[g(t)]) = q(t)G(x[g(t)])H(x[g(t)]) \\ &\geq q(t)G(KI(g(t), t_1))H(x[g(t)]) \quad \text{for} \quad t \geq t_2. \end{aligned}$$

It follows from the above discussion that Theorem 3.3–(I_1), (I_2) (as well as other results in Sections 3 and 4) is applicable to equation (1.1;1) if f is replaced by H and $q(t)$ is replaced by $q(t)G(cI(g(t),T))$ for every constant $c > 0$ and all large $T \geq t_0$ with $g(t) \geq T$ and I is defined as in (5.2). The formulation of this result as well as others are left to the reader.

The following functions are not monotonic:

(i) $f(x) = \dfrac{|x|^{\beta-1}x}{1+|x|^{\gamma}}$, *where* β *and* γ *are positive constants,*

(ii) $f(x) = |x|^{\beta-1}x\exp(-|x|^{\gamma})$, *where* β *and* γ *are positive constants,*

(iii) $f(x) = |x|^{\beta-1}x \operatorname{sech} x$, *where* β *is a positive constant.*

We note that the results of Section 3 are not applicable to equation $(1.1;\delta)$ with any one of the above choices of f.

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