F A S C I C U L I   M A T H E M A T I C I

Nr 35 2005

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ON A GENERALIZED λ-GAUSS WEIERSTRASS SINGULAR INTEGRAL

Abstract: In the present note we consider the integral

$$W_\lambda^s(f;\, x,\, \alpha) := \frac{c(n,\, \lambda,\, s)}{\alpha^{|\lambda|}} \int_{\mathbb{R}^n} f(x+t) \exp\left(-\|t\|_\lambda^s / 4\alpha^{\frac{|\lambda|}{n}s}\right) dt,$$

where $x \in \mathbb{R}^n$, $s > 0$, $\alpha > 0$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. The integrals $W_\lambda^s(f;\, x,\, \alpha)$ will be called a generalized $\lambda-$Gauss Weierstrass singular integral. We study some approximation properties of this integral in the nonisotropic exponential weighted space.

Key words: $\lambda-$Gauss Weierstrass integral, exponential weighted space.

1. Introduction

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $\|x\|_\lambda = (|x_1|^{\frac{1}{\lambda_1}} + \ldots + |x_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}}$, $x \in R^n$. The expression $\|x-y\|_\lambda$, x, $y \in R^n$ is called the nonisotropic distance between the x and y.

It can be seen that nonisotropic distance become ordinary Euclidean distance $|x-y|$ for $\lambda_j = \frac{1}{2}$, $j = 1, 2, \ldots n$. Nonisotropic distance has the following property.

Using the inequality $(a+b)^m \le 2^m (a^m + b^m)$, $m > 1$ we obtain

$$\|x-y\|_\lambda \le M_\lambda (\|x\|_\lambda + \|y\|_\lambda), \tag{1}$$

where $M_\lambda = 2^{(1+\frac{1}{\lambda_{\min}})\frac{|\lambda|}{n}}$ and $\lambda_{\min} = \min(\lambda_1, \lambda_2, \ldots \lambda_n)$. This integral operators with the kernels depending on nonisotropic distance have important application in theory of partial differential equations and imbedding theorems. ([4], [6]).

Let us denote the nonisotropic exponential weighted space for $1 \leq p < \infty$, $q \in \overline{\mathbb{R}}_0 := [0, +\infty)$, s and $\lambda_i \in \mathbb{R}_0 := (0, \infty)$ $(i = 1, 2, \ldots n)$ with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

$$L^s_{p,q}(f) = \left\{ f : \|f\|_{p,q,s} = \left(\int_{\mathbb{R}^n} |f(x)|^p \exp\left(-q \|x\|^s_\lambda\right) dx \right)^{\frac{1}{p}} < \infty \right\}.$$

The generalized $\lambda-$Gauss Weierstrass singular integral of function f belonging to a fixed space $L^s_{p,q}$ we define by

$$W^s_\lambda(f; x, \alpha) := \frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \int_{\mathbb{R}^n} f(x+t) \exp\left(-\|t\|^s_\lambda / 4\alpha^{\frac{|\lambda|}{n}s}\right) dt, \tag{2}$$

$x \in \mathbb{R}^n$, where α, s and $\lambda_i \in \mathbb{R}_0$ $(i = 1, 2, \ldots n)$ with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $c(n, \lambda, s)$ defined in (3).

If $s = \frac{n}{|\lambda|}$, we obtain the $\lambda-$Gauss Weierstrass singular integral defined in [1], in the case of $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{2}$, generalized Gauss Weierstrass singular integral defined in [5] are obtained.

Approximation properties of classical Gauss Weierstrass singular integral have been studied systematically in the past. We refer the reader [2], [3] and the references therein. Futhermore, A. Khan and S. Umar gave a generalization of Gauss Weierstrass singular integral and obtained the order of approximation to function. In [2], it was examined that approximation properties of Picard and Gauss Weierstrass singular integral of functions of two variable in exponential weighted space. Recently, Aral [1] define the $\lambda-$Gauss Weierstrass singular integral in which the kernel depends on nonisotropic distance.

In this paper, we define a generalization $\lambda-$Gauss Weierstrass singular integral similar to one given in [5]. First, we investigate the pointwise order of convergence for the functions belonging to nonisotropic-exponential weighted space $L^s_{p,q}$ and secondly the order of convergence in $L^s_{p,q}$ norm for the same class of functions.

2. Auxiliary results

In this part we shall give some auxiliary results.

Lemma 1. *For all α, s and $\lambda_i \in \mathbb{R}_0$ $(i = 1, 2, \ldots n)$ with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$*

$$\frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \int_{\mathbb{R}^n} \exp\left(-\|x\|^s_\lambda / 4\alpha^{\frac{|\lambda|}{n}s}\right) dx = 1.$$

Proof. $t=\alpha^{\lambda}x$ change of variable gives that

$$\frac{c(n,\,\lambda,\,s)}{\alpha^{|\lambda|}}\int\limits_{\mathbb{R}^n}\exp\left(-\left\|t\right\|_{\lambda}^{s}/4\alpha^{\frac{|\lambda|}{n}s}\right)dt=c(n,\,\lambda,\,s)\int\limits_{\mathbb{R}^n}\exp\left(-\left\|x\right\|_{\lambda}^{s}/4\right)dx.$$

We use generalized spherical coordinates and consider the transformation

$$\begin{aligned}
x_1 &= (u\cos\theta_1)^{2\lambda_1}\\
x_2 &= (u\sin\theta_1\cos\theta_2)^{2\lambda_2}\\
&\cdots\cdots\cdots\cdots\\
x_{n-1} &= (u\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1})^{2\lambda_{n-1}}\\
x_n &= (u\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1})^{2\lambda_n},
\end{aligned}$$

where $0\le\theta_1,\theta_2,\cdots,\theta_{n-2}\le\pi$, $0\le\theta_{n-1}\le 2\pi$. ■

Denoting the Jacobian of this transformation by $J_{\lambda}\left(u,\,\theta_1,\ldots,\,\theta_{n-1}\right)$ we obtain

$$J_{\lambda}\left(u,\,\theta_1,\ldots,\,\theta_{n-1}\right)=u^{2|\lambda|-1}\Omega_{\lambda}\left(\theta\right),$$

where $\Omega_{\lambda}\left(\theta\right)=2^{n}\lambda_1\ldots\lambda_n\prod\limits_{j=1}^{n-1}\left(\cos\theta_j\right)^{2\lambda_j}\left(\sin\theta_j\right)^{\sum\limits_{k=j+1}^{j}2\lambda_k-1}$. We can easily see that the integral $\omega_{\lambda,\,n-1}=\int\limits_{S^{n-1}}\Omega_{\lambda}\left(\theta\right)d\theta$ is finite, where S^{n-1} is the unit sphere in R^n.

Thus we have

$$\begin{aligned}
c(n,\,\lambda,\,s)\int\limits_{\mathbb{R}^n}\exp\left(-\left\|x\right\|_{\lambda}^{s}/4\right)dx &= c(n,\,\lambda,\,s)\\
&\times\int\limits_{0}^{\infty}\int\limits_{S^{n-1}}\exp\left(-u^{\frac{2|\lambda|}{n}s}/4\right)u^{2|\lambda|-1}\Omega_{\lambda}\left(\theta\right)d\theta du\\
&= c(n,\,\lambda,\,s)\frac{n}{2\left|\lambda\right|s}4^{\frac{n}{s}}\omega_{\lambda,\,n-1}\int\limits_{0}^{\infty}\exp\left(-u\right)u^{\frac{n}{s}-1}du.
\end{aligned}$$

If we choose

$$c(n,\,\lambda,\,s)^{-1}=\frac{n}{2\left|\lambda\right|s}4^{\frac{n}{s}}\omega_{\lambda,\,n-1}\Gamma\left(\frac{n}{s}\right)\tag{3}$$

then we have desired result.

Lemma 2. *Suppose that* $f \in L^s_{p,\,q/(2M_\lambda)^s}$ *with fixed* $q \in \overline{\mathbb{R}}_0$, $1 \le p < \infty$, s *and* $\lambda_i \in \mathbb{R}_0$ $(i = 1, 2, \dots n)$ *with* $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. *Then for* $0 < \alpha < \left(\frac{p}{4q}\right)^{\frac{n}{|\lambda| s}}$

$$||W^s_\lambda (f;\, x,\, \alpha)||_{p,\,q,\,s} \le K(\alpha,\, p,\, q,\, \lambda,\, s)\, ||f||_{p,\,q/(2M_\lambda)^s,\,s}$$

where

$$K(\alpha,\, p,\, q,\, \lambda,\, s) := \frac{c(n,\, \lambda,\, s)}{\alpha^{|\lambda|}} \int\limits_{\mathbb{R}^n} \exp\left(-\, ||t||^s_\lambda \left(\frac{1}{4\alpha^{\frac{|\lambda|}{n} s}} - \frac{q}{p} \right) \right) dt.$$

The integral W^s_λ *is a linear positive operator from the space* $L^s_{p,\,q/(2M_\lambda)^s}$ *into* $L^s_{p,\,q}$ *provided* $0 < \alpha < \left(\frac{p}{4q}\right)^{\frac{n}{|\lambda| s}}$.

Remark. Since $2M_\lambda > 1$, $q \in \overline{\mathbb{R}}_0$ and $s \in \mathbb{R}_0$ then we have $\exp\left(-\, ||t||^s_\lambda\right) \le \exp\left(-q\, ||t||^s_\lambda \,/\, (2M_\lambda)^s\right)$. Thus $L^s_{p,\,q/(2M_\lambda)^s} \subset L^s_{p,\,q}$ holds, for $1 \le p < \infty$.

3. Pointwise order of convergence

In this section we shall consider pointwise order of convergence problem in the space $L^s_{p,\,q}$.

Theorem 1. *Suppose that* $f \in L^s_{p,\,q/(2M_\lambda)^s}$ *with same* $q \in \overline{\mathbb{R}}_0$, $1 \le p < \infty$, s *and* $\lambda_i \in \mathbb{R}_0$ $(i = 1, 2, \dots n)$ *with* $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. *If the point* $x \in \mathbb{R}^n$ *satisfy the condition for some* $\delta > 0$ *and* $\beta > 0$

(4)

$$\sup_{0<h<\delta} \left(\frac{1}{h^{2|\lambda|+\beta}} \int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}} < h} |f(x+t) - f(x)| \exp\left(-q\, ||t||^s_\lambda \,/\, (2M_\lambda)^s\right) dt \right) < \infty$$

then we have

$$|W^s_\lambda (f;\, x,\, \alpha) - f(x)| = 0\left(\alpha^{\frac{\beta}{2}}\right) \quad as\ \alpha \to 0^+ \quad .$$

Proof. Let $x \in \mathbb{R}^n$ satisfy the condition (4). This mean that, for any $\delta > 0$, there exists constant C such that

$$\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}} < h} |f(x+t) - f(x)| \exp\left(-q\, ||t||^s_\lambda \,/\, (2M_\lambda)^s\right) dt < Ch^{2|\lambda|+\beta},$$

provided $h \leq \delta$. Changing to generalized spherical coordinates we can write

$$G_{\lambda}(r) := \int_0^r g_{\lambda}(u)\, u^{2|\lambda|-1} du < C r^{2|\lambda|+\beta}, \tag{5}$$

where

$$g_{\lambda}(u) = \exp\left(-q u^{\frac{2|\lambda|}{n}s} / (2M_{\lambda})^s\right) \int_{S^{n-1}} \left| f\left(x + (u\theta)^{2|\lambda|}\right) - f(x) \right| \Omega_{\lambda}(\theta)\, d\theta.$$

From Lemma 1, we have

$$\begin{aligned}
\left|W_{\lambda}^{s}(f; x, \alpha) - f(x)\right| &\leq \frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \\
&\quad \times \int_{\mathbb{R}^n} |f(x+t) - f(x)| \exp\left(-\|t\|_{\lambda}^{s} / 4\alpha^{\frac{|\lambda|}{n}s}\right) dt \\
&\leq \frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \int_{\|t\|_{\lambda}^{\frac{n}{2|\lambda|}} < \delta} |f(x+t) - f(x)| \exp\left(-\|t\|_{\lambda}^{s} / 4\alpha^{\frac{|\lambda|}{n}s}\right) dt \\
&\quad + \frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \int_{\|t\|_{\lambda}^{\frac{n}{2|\lambda|}} \geq \delta} |f(x+t) - f(x)| \exp\left(-\|t\|_{\lambda}^{s} / 4\alpha^{\frac{|\lambda|}{n}s}\right) dt \\
&:= I_1(\alpha) + I_2(\alpha).
\end{aligned}$$

■

We consider I_1. Using above observation, we get

$$\begin{aligned}
I_1(\alpha) &= \frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \int_0^{\delta} g_{\lambda}(u)\, u^{2|\lambda|-1} \exp\left(-A u^{\frac{2|\lambda|}{n}s}\right) du \\
&= \frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \int_0^{\delta} \exp\left(-A u^{\frac{2|\lambda|}{n}s}\right) dG_{\lambda}(u),
\end{aligned}$$

where $A = \left(1/4\alpha^{\frac{|\lambda|}{n}s} - q/(2M_{\lambda})^s\right) > 0$. Thus, using integration by parts and (5) we obtain

$$\begin{aligned}
I_1(\alpha) &= \frac{c(n, \lambda, s)}{\alpha^{|\lambda|}} \\
&\quad \times \left\{ G_{\lambda}(\delta) \exp\left(-A\delta^{\frac{2|\lambda|}{n}s}\right) + \int_0^{\delta} G_{\lambda}(u)\, d\left(-\exp\left(-A u^{\frac{2|\lambda|}{n}s}\right)\right) \right\}
\end{aligned}$$

$$\leq \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}$$

$$\times\, C\left\{\delta^{2|\lambda|+\beta}\exp\left(-A\delta^{\frac{2|\lambda|}{n}s}\right)+\int_0^{\delta}u^{2|\lambda|+\beta}d\left(-\exp\left(-Au^{\frac{2|\lambda|}{n}s}\right)\right)\right\}$$

$$= \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}C\left\{(2|\lambda|+\beta)\int_0^{\delta}\exp\left(-Au^{\frac{2|\lambda|}{n}s}\right)u^{2|\lambda|+\beta-1}du\right\}$$

$$\leq C\,(2|\lambda|+\beta)\frac{nc(n,\lambda,s)}{2|\lambda|s}\alpha^{\frac{\beta}{2}}\left(\frac{4(2M_\lambda)^s}{(2M_\lambda)^s-4q\alpha^{\frac{|\lambda|}{n}s}}\right)^{\frac{n}{s}+\frac{n}{2|\lambda|s}\beta}$$

$$\times\,\Gamma\left(\frac{n}{s}+\frac{n}{2|\lambda|s}\beta\right) = 0(\alpha^{\frac{\beta}{2}}) \quad \text{as} \quad \alpha\to 0^+.$$

To estimate I_2, if $\frac{1}{p}+\frac{1}{p'}=1$ then by Hölder's inequality and (1)

$$I_2(\alpha)=\frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\delta}|f(x+t)-f(x)|\exp\left(-||t||_\lambda^s/4\alpha^{\frac{|\lambda|}{n}s}\right)dt$$

$$\leq \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}\left\{|f(x)|\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\delta}\exp\left(-||t||_\lambda^s/4\alpha^{\frac{|\lambda|}{n}s}\right)dt\right.$$

$$+\left(\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\delta}|f(x+t)|^p\exp\left(-q\,||t||_\lambda^s\right)dt\right)$$

$$\left.\times\left(\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\delta}\exp\left(-||t||_\lambda^s\left(\frac{p'}{4\alpha^{\frac{|\lambda|}{n}s}}-\frac{p'}{p}q\right)\right)dt\right)^{\frac{1}{p'}}\right\}$$

$$\leq \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}\left\{|f(x)|A_1(\alpha)+A_2(\alpha)\exp\left(||x||_\lambda^s\right)||f||_{p,(2M_\lambda)^s q}\right\},$$

where

$$A_1(\alpha) = \int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\delta}\exp\left(-||t||_\lambda^s/4\alpha^{\frac{|\lambda|}{n}s}\right)dt$$

and

$$A_2(\alpha) = \left(\int\limits_{\|t\|_\lambda^{\frac{n}{2|\lambda|}} \geq \delta} \exp\left(-\|t\|_\lambda^s \left(\frac{p'}{4\alpha^{\frac{|\lambda|}{n}s}} - \frac{p'}{p} q \right) \right) dt \right)^{\frac{1}{p'}}.$$

Now we consider A_1.

$$\begin{aligned} A_1(\alpha) &= \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}} \omega_{\lambda,n-1} \int\limits_{\delta}^{\infty} \exp\left(-u^{\frac{2|\lambda|}{n}s} / 4\alpha^{\frac{|\lambda|}{n}s} \right) u^{2|\lambda|-1} du \\ &= c(n,\lambda,s)\omega_{\lambda,n-1} \int\limits_{\frac{\delta}{\alpha^{1/2}}}^{\infty} \exp\left(-u^{\frac{2|\lambda|}{n}s} / 4 \right) u^{2|\lambda|-1} du \\ &= c(n,\lambda,s)\omega_{\lambda,n-1} 4^{\frac{n}{s}} \frac{2n}{|\lambda|\, s} \int\limits_{\delta^{\frac{|\lambda|}{n}s} / 2\alpha^{\frac{|\lambda|}{2n}s}}^{\infty} \exp\left(-t^2\right) t^{\frac{2n}{s}-1} dt \\ &= c(n,\lambda,s)\omega_{\lambda,n-1} 4^{\frac{2n}{s}} \frac{n}{|\lambda|\, s} \exp\left(-\delta^{\frac{|\lambda|}{n}s} / 2\alpha^{\frac{|\lambda|}{4n}s} \right) \Gamma\left(\frac{2n}{s}\right) \\ &= 0\left(\alpha^{\frac{\beta}{2}}\right) \text{ as } \alpha \to 0^+. \end{aligned}$$

Similarly for A_2 we have the following estimate.

$$\begin{aligned} A_2(\alpha) = {} & \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}} \\ & \times \left[\omega_{\lambda,n-1} \int\limits_{\delta}^{\infty} \exp\left(-u^{\frac{2|\lambda|}{n}s} \left[\left(\frac{p'}{4\alpha^{\frac{|\lambda|}{n}s}} - \frac{p'}{p} q \right) \right] \right) u^{2|\lambda|-1} du \right]^{\frac{1}{p'}} \\ = {} & 0\left(\alpha^{\frac{\beta}{2}}\right) \text{ as } \alpha \to 0^+. \end{aligned}$$

This proves the Theorem 1.

Theorem 2. *Suppose that* $f \in L^s_{p,\, q/(2M_\lambda)^s}$ *with same* $1 \leq p < \infty$, $q \in \overline{\mathbb{R}}_0$, s *and* $\lambda_i \in \mathbb{R}_0$ $(i = 1, 2, \ldots n)$ *with* $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. *If the point* $x \in \mathbb{R}^n$ *satisfy the condition for some* $\delta > 0$ *and* $\beta > 0$

$$(6) \quad \sup_{0<h<\delta} \frac{1}{h^{2|\lambda|+\beta}} \left(\int\limits_{\|t\|_\lambda^{\frac{n}{2|\lambda|}} < h} |f(x+t) - f(x)|^p \exp\left(\frac{-q\, \|t\|_\lambda^s}{(2M_\lambda)^s} \right) dt \right)^{\frac{1}{p}} < \infty$$

then we have

$$|W_\lambda^s(f;x,\alpha)-f(x)|=0\left(\alpha^{\frac{\beta}{2p}}\right)\quad as\quad \alpha\to 0^+.$$

Proof. Since $\frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}\int\limits_{\mathbb{R}^n}\exp\left(-\|x\|_\lambda^s/4\alpha^{\frac{|\lambda|}{n}s}\right)dx=1$, from Hölder's inequality we have

$$\begin{aligned}|W_\lambda^s(f;x,\alpha)-f(x)| &\le \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}\\ &\quad\times\int\limits_{\mathbb{R}^n}|f(x+t)-f(x)|\exp\left(-\|t\|_\lambda^s/4\alpha^{\frac{|\lambda|}{n}s}\right)dt\\ &\le \left(\frac{c(n,\lambda,s)}{\alpha^{|\lambda|}}\int\limits_{\mathbb{R}^n}|f(x+t)-f(x)|^p\exp\left(-\|t\|_\lambda^s/4\alpha^{\frac{|\lambda|}{n}s}\right)dt\right)^{\frac{1}{p}}.\end{aligned}$$

Proceeding in a similar manner as in the proof of Theorem 1, we can complete the proof of Theorem 2. ■

4. Norm convergence

In this section we will state some estimate of convergence the integral W_λ^s as $\alpha\to 0^+$ in the $L_{p,q}^s$ norm.

Theorem 3. *Suppose that* $f\in L^s_{p,q/(2M_\lambda)^s}$ *with same* $1\le p<\infty$, $q\in\overline{\mathbb{R}}_0$, s *and* $\lambda_i\in\mathbb{R}_0$ $(i=1,2,\dots n)$ *with* $|\lambda|=\lambda_1+\lambda_2+\dots+\lambda_n$. *If the point* $x\in\mathbb{R}^n$ *satisfy the condition for some* $\delta>0$ *and* $\beta>0$

$$\sup_{0<h<\delta}\frac{1}{h^{2|\lambda|+\beta}}\int\limits_{\|t\|_\lambda^{\frac{n}{2|\lambda|}}<h}\|f(x+t)-f(x)\|_{p,q,s(x)}\,dt<\infty \tag{7}$$

satisfies then we have

$$\|W_\lambda^s(f;x,\alpha)-f(x)\|_{p,q,s}=0\left(\alpha^{\frac{\beta}{2}}\right)\quad as\quad \alpha\to 0^+.$$

Proof. Using Lemma 1 and generalized Hölder's inequality we have

$$\begin{aligned}\|W_\lambda^s(f;x,\alpha)-f(x)\|_{p,q,s} &=\\ &= \left(\int\limits_{\mathbb{R}^n}|W_\lambda^s(f;x,\alpha)-f(x)|^p\exp\left(-q\|x\|_\lambda^s\right)dx\right)^{\frac{1}{p}}\end{aligned}$$

$$\leq \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}} \int_{\mathbb{R}^n} \exp\left(\frac{-\|t\|_\lambda^s}{4\alpha^{\frac{|\lambda|}{n}s}} \right) \|f(x+t) - f(x)\|_{p,q,s(x)}\, dt$$

$$= \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}} \int_{\|t\|_\lambda^{\frac{n}{2|\lambda|}} < \delta} \exp\left(\frac{-\|t\|_\lambda^s}{4\alpha^{\frac{|\lambda|}{n}s}} \right) \|f(x+t) - f(x)\|_{p,q,s(x)}\, dt$$

$$+ \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}} \int_{\|t\|_\lambda^{\frac{n}{2|\lambda|}} \geq \delta} \exp\left(\frac{-\|t\|_\lambda^s}{4\alpha^{\frac{|\lambda|}{n}s}} \right) \|f(x+t) - f(x)\|_{p,q,s(x)}\, dt$$

$$= J_1(\alpha) + J_2(\alpha),$$

for $\delta > 0$.

Similarly as in the estimate of I_1 using (7) then we get $J_1(\alpha) = 0\left(\alpha^{\frac{\beta}{2}}\right)$ as $\alpha \to 0^+$ for $\beta > 0$.

Observe that for the estimate of J_2.

$$\|f(x+t) - f(x)\|_{p,q,s(x)} \leq \left(\int_{\mathbb{R}^n} |f(x+t)|^p \exp(-q\|x\|_\lambda^s)\, dx \right)^{\frac{1}{p}}$$

$$+ \left(\int_{\mathbb{R}^n} |f(x)|^p \exp(-q\|x\|_\lambda^s)\, dx \right)^{\frac{1}{p}}$$

$$\leq \exp\left(\frac{q}{p} \|t\|_\lambda^s \right) \|f\|_{p,q/(2M_\lambda)^s,s} + \|f\|_{p,q,s}.$$

Thus

$$J_2(\alpha) \leq \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}} \|f\|_{p,q/(2M_\lambda)^s,s} \int_{\|t\|_\lambda^{\frac{n}{2|\lambda|}} \geq \delta} \exp\left(-\|t\|_\lambda^s \left(\frac{1}{4\alpha^{\frac{|\lambda|}{n}s}} - \frac{q}{p} \right) \right) dt$$

$$+ \frac{c(n,\lambda,s)}{\alpha^{|\lambda|}} \|f\|_{p,q,s} \int_{\|t\|_\lambda^{\frac{n}{2|\lambda|}} \geq \delta} \exp\left(-\|t\|_\lambda^s / 4\alpha^{\frac{|\lambda|}{n}s} \right) dt.$$

Since $\frac{1}{\alpha^{|\lambda|}}\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\delta}\exp\left(-\left\|x\right\|_\lambda^s/4\alpha^{\frac{|\lambda|}{n}s}\right)dt=\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\frac{\delta}{\sqrt{\alpha}}}\exp\left(-\left\|t\right\|_\lambda^s/4\right)dt$ we see that the second summand tends to 0 as $\alpha\to0^+$. For the first summand, changing to generalized spherical coordinates we can write as

$$\begin{aligned}\frac{1}{\alpha^{|\lambda|}}\int\limits_{||t||_\lambda^{\frac{n}{2|\lambda|}}\geq\delta}\exp\left(-B\left\|t\right\|_\lambda^s\right)dt &= \frac{\omega_{\lambda,n-1}}{\alpha^{|\lambda|}B^{\frac{n}{s}}}\int\limits_{\delta B^{\frac{n}{2|\lambda|s}}}^{\infty}\exp\left(-u^{\frac{n}{2|\lambda|}s}\right)u^{2|\lambda|-1}du\\ &\leq \frac{\omega_{\lambda,n-1}8^{\frac{n}{s}}n\Gamma\left(\frac{n}{s}\right)}{2\left|\lambda\right|s\left(1-\frac{4q}{p}\alpha^{\frac{|\lambda|}{n}s}\right)^{\frac{n}{s}}}\exp\left(-\frac{\delta^{\frac{n}{2|\lambda|}s}}{2}B\right)\\ &= 0\left(\alpha^{\frac{\beta}{2}}\right)\quad\alpha\to0^+,\end{aligned}$$

where $B=\frac{1}{4\alpha^{\frac{|\lambda|}{n}s}}-\frac{q}{p}$.

This completes the proof of the Theorem3. ■

Examples:

We give examples satisfying the conditions of the Theorems.. If we choose $n=1$ and $\lambda=\frac{1}{2}$ then $|\lambda|=\frac{1}{2}$ and the condition (4) becomes for some $\delta>0$ and $\beta>0$

$$(4')\qquad \sup_{0<h<\delta}\left(\frac{1}{h^{1+\beta}}\int\limits_{|t|<h}\left|f\left(x+t\right)-f\left(x\right)\right|\exp\left(-q\left|t\right|^s\right)dt\right)<\infty.$$

Since $\exp\left(-q\left|t\right|^s\right)\leq1$ for $|t|<h$ then to obtain $\left(4'\right)$ we must have

$$(4'')\qquad \sup_{0<h<\delta}\left(\frac{1}{h^{1+\beta}}\int\limits_{|t|<h}\left|f\left(x+t\right)-f\left(x\right)\right|dt\right)<\infty.$$

If $f\in L^s_{p,q}$ then for any $a>0$, $f\in L_1(-a,a)=L^s_{p,0}(-a,a)$. In other words $(4'')$ (and $(4')$) are finite for $\delta_1<h<\delta$. It is left to show that $\left(4''\right)$ holds as $h\to0$. Since $f\in L_1(-a,a)$, for almost all x

$$\lim_{h\to0}\frac{1}{h}\int\limits_{|t|<h}\left|f\left(x+t\right)-f\left(x\right)\right|dt=0$$

(see [6], page 11). Then, to get $(4'')$ it is enough to show that

$$(4''') \qquad \frac{1}{h}\int\limits_{|t|<h} |f(x+t)-f(x)|\,dt = 0\left(h^{\beta}\right) \quad \text{as } h \to 0.$$

Consequently, if $f \in Lip(\beta) := \left\{f : |f(t)-f(x)| \leq |t-x|^{\beta},\ 0<\beta\leq 1\right\}$ and $f \in L^{s}_{p,q}$ at x point then it is obvious that $\left(4'''\right)$ holds. Simultaneously, $\left(4'\right)$ holds.

Similar examples satisfying the conditions (6) and (7) can be given. Under the same conditions, by using the similar method if we take $f \in Lip\left(\beta+\frac{1}{p'}\right)$, $\frac{1}{p}+\frac{1}{p'}=1$ at x point then (6) holds. Also, $f \in Lip(\beta)$ at all $x \in \mathbb{R}$ then (7) holds.

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Received on 08.07.2003 and, in revised from, on 14.01.2004