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AN EIGENVALUE PROBLEM FOR LINEAR HAMILTONIAN DYNAMIC SYSTEMS

Abstract: In this paper we consider eigenvalue problems on time scales involving linear Hamiltonian dynamic systems. We give conditions that ensure that the eigenvalues of the problem are isolated and bounded below. The presented results are applicable also to Sturm-Liouville dynamic equations of higher order, and further special cases of our systems are linear Hamiltonian differential systems as well as linear Hamiltonian difference systems.

Key words: time scale, linear Hamiltonian system, eigenvalue, eigenfunction, quadratic functional, focal point, normality.

1. Introduction

A *time scale* is any nonempty closed subset of $\mathbb{R}$. For an introduction to the time scales calculus we refer the reader to [6, 7], see also [8, 12]. If f is a function on $\mathbb{T}$, we abbreviate $f_t := f(t)$ and $f^\sigma := f \circ \sigma$, where σ is the forward jump operator. The time scale derivative f_t^Δ reduces to the usual derivative $f'(t)$ if $\mathbb{T} = \mathbb{R}$ and to the forward difference $\Delta f_t = f_{t+1} - f_t$ if $\mathbb{T} = \mathbb{Z}$. The graininess function of $\mathbb{T}$ is $\mu_t := \sigma_t - t$. The set of rd-continuous functions is denoted by C_{rd} and the set of rd-continuously differentiable functions by C^1_{rd}.

Let $\mathbb{T} := [a, b]$ be a time scale interval, $a < b$. The set $\mathbb{T}$ without its possible isolated (i.e., a left-scattered) maximum will be denoted by $\mathbb{T}^\kappa$; thus $\mathbb{T}^\kappa = \mathbb{T}$ if b is left-dense. Consider the *linear Hamiltonian dynamic system*

$$\text{(H)} \qquad x^\Delta = A_t x^\sigma + B_t u, \quad u^\Delta = C_t x^\sigma - A_t^T u, \quad t \in \mathbb{T}^\kappa,$$

where $A, B, C : \mathbb{T}^\kappa \to \mathbb{R}^{n \times n}$ are real rd-continuous matrices, B_t, C_t symmetric, and $I - \mu_t A_t$ nonsingular. Motivated by [5], we consider eigenvalue problems (with formally self-adjoint boundary conditions) involving the system (H), where the matrices A_t, B_t, and C_t also depend on an eigenvalue parameter $\lambda \in \mathbb{R}$. We give conditions, among them the notion of strict controllability for system (H), that imply that the eigenvalues of (H) are isolated and bounded below, i.e., they may be arranged as

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$

counting multiplicities. An eigenvalue problem on time scales of the Sturm-Liouville type has been recently studied in [1].

The setup of this paper is as follows. In the following Section 2 we recall some preliminaries on Hamiltonian systems that are needed later. Then, in Section 3, we introduce our eigenvalue problem in detail and present basic facts about this problem, e.g., how it is possible to characterize the eigenvalues. In this section we also present our main result on isolatedness and lower boundedness of eigenvalues, which we prove by using some auxiliary results that are given in detail in the last Section .

2. Preliminaries: Hamiltonian systems

By a *solution* of (H) we mean a pair (x, u) with $x, u \in C^1_{rd}(\mathbb{T})$ satisfying the system (H) on $\mathbb{T}^\kappa$. When referring to solutions of (H) we use a usual agreement that the vector-valued solutions of (H) are denoted by small letters and the $n \times n$-matrix-valued solutions by capital ones. By $\operatorname{rank} M$, $\operatorname{Ker} M$, $\operatorname{Im} M$, $\operatorname{def} M$, $\operatorname{ind} M$, M^T, M^{T-1}, $M^\dagger$, $M \geq 0$, and $M > 0$ we denote the rank, kernel, image, defect (dimension of the kernel), index (number of negative eigenvalues), transpose, inverse of the transpose, Moore-Penrose generalized inverse (see [2, Chapter 1]), positive semidefiniteness, and positive definiteness, respectively, of the matrix M.

By setting

$$\mathcal{H} := \begin{pmatrix} -C & A^T \\ A & B \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad z := \begin{pmatrix} x \\ u \end{pmatrix}, \quad \tilde{z} := \begin{pmatrix} x^\sigma \\ u \end{pmatrix}, \tag{1}$$

the linear Hamiltonian system (H) has the form

$$\mathcal{L}[z]_t \equiv \mathcal{J} z^\Delta + \mathcal{H}_t \tilde{z} = 0, \quad t \in \mathbb{T}^\kappa. \tag{H}$$

A solution (X, U) of (H) is called a *conjoined basis* if $\operatorname{rank}(X^T \; U^T) = n$ at some (and hence at any) $t \in \mathbb{T}$, and $X^T U - U^T X \equiv 0$ on $\mathbb{T}$. The Wronski matrix $W = X^T \tilde{U} - U^T \tilde{X}$ is constant on $\mathbb{T}$ for any two solutions (X, U), $(\tilde{X}, \tilde{U})$ of (H). These two solutions are *normalized* if $W = I$. The (unique) solution (X, U), resp. $(\tilde{X}, \tilde{U})$, of (H) satisfying the initial conditions $X_a = 0$, $U_a = I$, resp. $\tilde{X}_a = -I$, $\tilde{U}_a = 0$, is called the *principal*, resp. *associated*, solution of (H) at a. Together they are called the *special normalized conjoined bases* of (H) at a.

Lemma 1. *For any* $s \in \mathbb{T}$ *and any conjoined basis* (X, U) *of* (H) *there exists another conjoined basis* $(\tilde{X}, \tilde{U})$ *such that they are normalized, and* $\tilde{X}_s$ *is invertible.*

Proof. See [13, Corollary 3.3.9], [9, Remark 5]. ■

A conjoined basis (X, U) of (H) is said to have *no focal points* in the interval $(a, b]$, provided X_t is invertible at all dense points $t \in \mathbb{T} \setminus \{a\}$, and

$$\operatorname{Ker} X^\sigma \subseteq \operatorname{Ker} X \quad \text{and} \quad D := X(X^\sigma)^\dagger \tilde{A} B \geq 0 \quad \text{on } \mathbb{T}^\kappa.$$

Recall that a point $t \in \mathbb{T}$ is *dense* if it is right-dense or left-dense. System (H) is called *disconjugate on* $\mathbb{T}$ if the principal solution of (H) at a has no focal points in $(a, b]$.

A pair (x, u) is called *admissible* if x is piecewise rd-continuously differentiable, denoted by $x \in C_p^1(\mathbb{T})$, u is piecewise rd-continuous, denoted by $u \in C_p(\mathbb{T})$, and (x, u) satisfies $x^\Delta = Ax^\sigma + Bu$ on $\mathbb{T}^\kappa$ (at points $t \in \mathbb{T}$, where x^Δ is not continuous, this is to be read as the corresponding right/left-sided limit). Let $R, S \in \mathbb{R}^{2n \times 2n}$ with S symmetric. The quadratic functional

$$\mathcal{F}(x, u) \equiv \int_a^b \left\{(x^\sigma)^T C x^\sigma + u^T B u\right\}_t \Delta t + \begin{pmatrix} -x_a \\ x_b \end{pmatrix}^T S \begin{pmatrix} -x_a \\ x_b \end{pmatrix}.$$

is called *positive definite* ($\mathcal{F} > 0$), if $\mathcal{F}(x, u) > 0$ for all admissible pairs (x, u) with $\left(\begin{smallmatrix} -x_a \\ x_b \end{smallmatrix}\right) \in \operatorname{Im} R^T$, $x \not\equiv 0$.

Following [11], system (H) is called *dense-normal on* $[a, s]$ whenever $s \in (a, b]$ is a dense point and the only solution of the system

$$u^\Delta = -A_t^T u, \quad B_t u = 0, \quad t \in [a, s]^\kappa, \tag{2}$$

is the zero solution $u_t \equiv 0$ on $[a, s]$. The hypothesis of dense-normality will be denoted by

(D) System (H) is dense-normal on any interval of the form $[a, s] \subseteq \mathbb{T}$.

Moreover, we say that (H) is *normal* on $\mathbb{T}$ if whenever $x_t \equiv 0$ on $\mathbb{T}$, then $u_t \equiv 0$ on $\mathbb{T}$, i.e., system (H) is normal on $\mathbb{T}$ if whenever u_t solves (2) with $s = b$ (not necessarily dense), then $u_t \equiv 0$ on $\mathbb{T}$.

The differentiation with respect to λ will be denoted by $\frac{d}{d\lambda} z = \dot{z}$. We require throughout that

$$\frac{d}{d\lambda}\left\{\begin{pmatrix} X \\ U \end{pmatrix}^\Delta (\lambda)\right\} = \left\{\frac{d}{d\lambda}\begin{pmatrix} X \\ U \end{pmatrix}(\lambda)\right\}^\Delta \tag{3}$$

for every conjoined basis (X, U) of (H). This assumption is rather restrictive, but it certainly holds for any time scale which has constant graininess, in particular for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

3. The eigenvalue problem

Let be given constant matrices $R, R^{\#} \in \mathbb{R}^{2n\times 2n}$ such that $\operatorname{rank}(R^{\#}\ R) = 2n$ and $R^{\#}R^T$ is symmetric. In this paper, the superscript $^{\#}$ does not mean a generalized inverse, but it is just an ordinary upper index. For $\lambda \in \mathbb{R}$, we consider the linear Hamiltonian system

$$(\mathrm{H}_\lambda) \qquad x^\Delta = A_t(\lambda)x^\sigma + B_t(\lambda)u, \quad u^\Delta = C_t(\lambda)x^\sigma - A_t^T(\lambda)u, \quad t \in \mathbb{T}^\kappa,$$

subject to the (formally self-adjoint) boundary conditions

$$R^{\#}\begin{pmatrix} -x_a \\ x_b \end{pmatrix} + R\begin{pmatrix} u_a \\ u_b \end{pmatrix} = 0. \tag{4}$$

We employ the following general assumption

$$\left\{\begin{array}{l} \text{For all } \lambda \in \mathbb{R},\ A(\lambda), B(\lambda), C(\lambda) \in C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n}),\ B(\lambda), C(\lambda) are \\ \text{symmetric, and } I - \mu A(\lambda) \text{ is nonsingular on } \mathbb{T}^\kappa. \text{ For all } t \in \mathbb{T}^\kappa,\ A_t(\cdot), \\ B_t(\cdot) \text{ and } C_t(\cdot) \text{ are continuously differentiable with respect to } \lambda. \end{array}\right.$$

We denote $\tilde{A}_t(\lambda) := [I - \mu_t A_t(\lambda)]^{-1}$.

First we derive the Lagrange identity for (H) on any time scale $\mathbb{T}$.

Lemma 2 (Lagrange identity). *For any $z, w \in C^1_{rd}(\mathbb{T}, \mathbb{R}^{2n})$, where $z = \binom{x}{u}$ and $w = \binom{y}{v}$, and with notation* (1)*, we have*

$$\int_a^b \left\{\tilde{w}^T\mathcal{L}[z] - \mathcal{L}^T[w]\tilde{z}\right\}_t \Delta t = w_t^T\mathcal{J}z_t\Big|_a^b.$$

Proof. Let $z = \binom{x}{u}$, $w = \binom{y}{v}$, and $\tilde{z} = \binom{x^\sigma}{u}$, $\tilde{w} = \binom{y^\sigma}{v}$. For brevity, we omit the argument t in the following computation. The integration by parts in the third equality sign and the symmetry of $\mathcal{H}_t$ yield

$$\begin{aligned}
\int_a^b \tilde{w}^T\mathcal{L}[z]\,\Delta t &= \int_a^b \left\{\begin{pmatrix} y^\sigma \\ v\end{pmatrix}^T \mathcal{J}\begin{pmatrix} x \\ u\end{pmatrix}^\Delta + \tilde{w}^T\mathcal{H}\tilde{z}\right\}\Delta t \\
&= \int_a^b \left\{(y^\sigma)^T u^\Delta - v^T x^\Delta + \tilde{w}^T\mathcal{H}\tilde{z}\right\}\Delta t \\
&= y^T u\Big|_a^b - v^T x\Big|_a^b + \int_a^b \left\{-(y^\Delta)^T u + (v^\Delta)^T x^\sigma + \tilde{w}^T\mathcal{H}\tilde{z}\right\}\Delta t \\
&= \begin{pmatrix} y_b \\ v_b\end{pmatrix}^T \mathcal{J}\begin{pmatrix} x_b \\ u_b\end{pmatrix} - \begin{pmatrix} y_a \\ v_a\end{pmatrix}^T \mathcal{J}\begin{pmatrix} x_a \\ u_a\end{pmatrix} + \int_a^b \left\{\begin{pmatrix} y^\Delta \\ v^\Delta\end{pmatrix}^T \mathcal{J}^T\begin{pmatrix} x^\sigma \\ u\end{pmatrix} + \tilde{w}^T\mathcal{H}\tilde{z}\right\}\Delta t \\
&= w^T\mathcal{J}z\Big|_a^b + \int_a^b \left\{\mathcal{J}w^\Delta + \mathcal{H}\tilde{w}\right\}^T\tilde{z}\,\Delta t \\
&= w^T\mathcal{J}z\Big|_a^b + \int_a^b \mathcal{L}^T[w]\tilde{z}\,\Delta t.
\end{aligned}$$

Therefore, the required identity follows. ■

The boundary conditions (4) are called *formally self-adjoint* if $w_t^T \mathcal{J} z_t \big|_a^b = 0$ for all $z, w \in C_{rd}^1(\mathbb{T}, \mathbb{R}^{2n})$ satisfying the given boundary conditions, i.e., $z = \binom{x}{u}$ and $w = \binom{y}{v}$ satisfy (4) and

$$R^{\#} \begin{pmatrix} -y_a \\ y_b \end{pmatrix} + R \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0,$$

respectively. Let us now remark that, in view of the next result, the symmetry of $R^{\#} R^T$ is a natural assumption when considering formally self-adjoint eigenvalue problems with the system (H).

Lemma 3 (Formally self-adjoint boundary conditions)**.** *Let $R^{\#}$ and R be real $2n \times 2n$-matrices such that* $\operatorname{rank}(R^{\#}\ R) = 2n$. *Then the boundary conditions* (4) *are formally self-adjoint iff $R^{\#} R^T$ is symmetric.*

Proof. The proof is the same as the proof of [13, Proposition 2.1.1]. ■

Remark 1. By [13, Remark 2.2.1], there exist matrices $S, S^{\#} \in \mathbb{R}^{2n \times 2n}$, such that S is symmetric, $\operatorname{rank}(S^{\#}\ R) = 2n$, $\operatorname{Im}(S^{\#})^T = \operatorname{Ker} R$, and $R^{\#} = RS + S^{\#}$.

Definition 1 (Eigenvalue problem)**.** *The* eigenvalue problem

$$\text{(E)} \qquad (\mathrm{H}_\lambda), \ \lambda \in \mathbb{R}, \quad (4),$$

consists of the linear Hamiltonian dynamic system (H_λ) and the boundary conditions (4)*. A number $\lambda \in \mathbb{R}$ is called an* eigenvalue *of (E) if there exists a nontrivial solution (x, u) of (H_λ) satisfying* (4)*. Such a solution is then called an* eigenfunction *corresponding to the eigenvalue λ. The set of all eigenfunctions corresponding to λ together with the zero function is called an* eigenspace*, and its dimension is referred to as the* multiplicity *of the eigenvalue λ.*

Theorem 1 (Characterization of eigenvalues)**.** *Let $\lambda \in \mathbb{R}$ and let (X, U), $(\tilde{X}, \tilde{U})$ be any normalized conjoined bases of* (H_λ)*. Then λ is an eigenvalue of* (E) *iff the matrix $\Lambda \in \mathbb{R}^{2n \times 2n}$ defined by*

$$\Lambda := R^{\#} \begin{pmatrix} -X_a & -\tilde{X}_a \\ X_b & \tilde{X}_b \end{pmatrix} + R \begin{pmatrix} U_a & \tilde{U}_a \\ U_b & \tilde{U}_b \end{pmatrix}$$

is singular, and then $\operatorname{def} \Lambda$ *is the multiplicity of the eigenvalue λ.*

Proof. Let (x, u) be a nontrivial solution of (H_λ). We put

$$d := \begin{pmatrix} X_a & \tilde{X}_a \\ U_a & \tilde{U}_a \end{pmatrix}^{-1} \begin{pmatrix} x_a \\ u_a \end{pmatrix} = \begin{pmatrix} \tilde{U}_a^T & -\tilde{X}_a^T \\ -U_a^T & X_a^T \end{pmatrix} \begin{pmatrix} x_a \\ u_a \end{pmatrix} \neq 0,$$

and thus $\left(\begin{smallmatrix} x_t \\ u_t \end{smallmatrix}\right) = \left(\begin{smallmatrix} X_t & \tilde{X}_t \\ U_t & \tilde{U}_t \end{smallmatrix}\right) d$ on $\mathbb{T}$. Hence,

$$R^{\#}\begin{pmatrix} -x_a \\ x_b \end{pmatrix} + R\begin{pmatrix} u_a \\ u_b \end{pmatrix} = R^{\#}\begin{pmatrix} -X_a & -\tilde{X}_a \\ X_b & \tilde{X}_b \end{pmatrix} d + R\begin{pmatrix} U_a & \tilde{U}_a \\ U_b & \tilde{U}_b \end{pmatrix} d = \Lambda d.$$

Thus, (x,u) satisfies the boundary conditions (4) iff $\Lambda d = 0$, i.e., λ is an eigenvalue of (E) iff Λ is singular. ■

Corollary 1 (Separated boundary conditions)**.** *Assume that separated boundary conditions are given, i.e.,*

$$R = \begin{pmatrix} R_a & 0 \\ 0 & R_b \end{pmatrix}, \quad R^{\#} = \begin{pmatrix} -R_a^{\#} & 0 \\ 0 & R_b^{\#} \end{pmatrix},$$

where the $n \times n$-matrices $R_a, R_b, R_a^{\#}, R_b^{\#}$ satisfy $\operatorname{rank}(R_a^{\#}\ R_a) = \operatorname{rank}(R_b^{\#}\ R_b) = n$, $R_a(R_a^{\#})^T = R_a^{\#}R_a^T$, *and* $R_b(R_b^{\#})^T = R_b^{\#}R_b^T$. *Let (X,U) be the conjoined basis of* (H_λ), $\lambda \in \mathbb{R}$, *with $X_a = -R_a^T$, $U_a = (R_a^{\#})^T$. Then λ is an eigenvalue of* (E) *iff the matrix $\Omega \in \mathbb{R}^{n\times n}$ given by*

$$\Omega := R_b^{\#}X_b + R_bU_b$$

is singular.

Proof. Let $\lambda \in \mathbb{R}$. For (X,U) there exists a conjoined basis $(\tilde{X},\tilde{U})$ of (H_λ) such that they are normalized, by Lemma 1. Then, Theorem 1 implies that λ is an eigenvalue of (E) iff Λ is singular. Since

$$\Lambda = R^{\#}\begin{pmatrix} -X_a & -\tilde{X}_a \\ X_b & \tilde{X}_b \end{pmatrix} + R\begin{pmatrix} U_a & \tilde{U}_a \\ U_b & \tilde{U}_b \end{pmatrix} = \begin{pmatrix} 0 & -I \\ \Omega & R_b^{\#}\tilde{X}_b + R_b\tilde{U}_b \end{pmatrix},$$

we have $\Lambda\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = 0$ iff $c_2 = 0$ and $\Omega c_1 + (R_b^{\#}\tilde{X}_b + R_b\tilde{U}_b)c_2 = 0$, i.e., iff $\Omega c_1 = 0$. Hence, Λ is singular iff Ω is singular. ■

Definition 2 (Strict dense-normality)**.** *The set of systems* $(\mathrm{H}_\mathbb{R}) := \{(\mathrm{H}_\lambda), \lambda \in \mathbb{R}\}$, *is called* strictly dense-normal on $\mathbb{T}$ *if*

(i) (H_λ) *satisfies* (D) *for all* $\lambda \in \mathbb{R}$.

(ii) *For all $\lambda \in \mathbb{R}$, for any $s \in \mathbb{T} \setminus \{a\}$, for any solution (x,u) of* (H_λ), *if*

$$\dot{\mathcal{H}}_t(\lambda)\begin{pmatrix} x_t^\sigma \\ u_t \end{pmatrix} = 0 \quad \text{for all } t \in [a,s]^\kappa,$$

then $x_t = u_t \equiv 0$ on $\mathbb{T}$.

Remark 2. We are particularly interested in the case when $A_t(\lambda) \equiv A_t$ and $B_t(\lambda) \equiv B_t$ are independent of λ and C depends on λ linearly, i.e., it is of the form $C_t - \lambda \tilde{C}_t$. In this remark we discuss some features of this special case.

(i) First, we note that

(ii) *implies* (i) *in Definition 2.*

To show this, let $\lambda \in \mathbb{R}$ and take any solution (x, u) of (H_λ) such that $x_t = 0$ on $[a, s]$, where $s \in \mathbb{T}$ is a dense point. We have

$$\dot{\mathcal{H}}_t(\lambda) \begin{pmatrix} x_t^\sigma \\ u_t \end{pmatrix} = \begin{pmatrix} -\dot{C}_t(\lambda) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_t^\sigma \\ u_t \end{pmatrix} = -\dot{C}_t(\lambda) x_t^\sigma = \tilde{C}_t x_t^\sigma = 0$$

on $[a, s]^\kappa$, hence (ii) implies $x_t = u_t \equiv 0$ on $\mathbb{T}$, so that (H_λ) is dense-normal on $[a, s]$.

(ii) Next, we show that

eigenvectors corresponding to different eigenvalues are orthogonal.

More precisely, let $R, S^\#, S, \tilde{S} \in \mathbb{R}^{2n \times 2n}$ be such that $S, \tilde{S}$ are symmetric, $\operatorname{rank}(S^\# \; R) = 2n$, $\operatorname{Im}(S^\#)^T = \operatorname{Ker} R$, and put

$$S(\lambda) := S - \lambda \tilde{S}, \quad R^\#(\lambda) := R S(\lambda) + S^\#.$$

Consider the eigenvalue problem

$$(\tilde{\mathrm{E}}) \qquad \begin{cases} x^\Delta = A_t x^\sigma + B_t u, \quad u^\Delta = (C_t - \lambda \tilde{C}_t) x^\sigma - A_t^T u, \quad t \in \mathbb{T}^\kappa, \\ \lambda \in \mathbb{R}, \quad R^\#(\lambda) \begin{pmatrix} -x_a \\ x_b \end{pmatrix} + R \begin{pmatrix} u_a \\ u_b \end{pmatrix} = 0. \end{cases}$$

If (x, u) and (y, v) are eigenfunctions of $(\tilde{\mathrm{E}})$ belonging to eigenvalues λ and ν, respectively, $\lambda \neq \nu$, then $x \perp y$ with respect to $\tilde{C}$ and $\tilde{S}$, i.e.,

$$\langle x, y \rangle := \int_a^b (x_t^\sigma)^T \tilde{C}_t y_t^\sigma \, \Delta t + \begin{pmatrix} -x_a \\ x_b \end{pmatrix}^T \tilde{S} \begin{pmatrix} -y_a \\ y_b \end{pmatrix} = 0.$$

To show this, we follow the proof of [13, Proposition 2.2.2]. Since (x, u) solves (H_λ) and (y, v) solves (H_ν), integration by parts implies

$$(5) \qquad \int_a^b \left\{ (x^\sigma)^T (C - \lambda \tilde{C}) y^\sigma + u^T B v \right\}_t \Delta t = u_t^T y_t \Big|_a^b,$$

$$(6) \qquad \int_a^b \left\{ (y^\sigma)^T (C - \nu \tilde{C}) x^\sigma + v^T B u \right\}_t \Delta t = v_t^T x_t \Big|_a^b.$$

By substracting (6) from (5) we obtain

$$(7)\qquad (\nu-\lambda)\int_a^b \left\{(x^\sigma)^T\tilde{C}y^\sigma\right\}_t \Delta t = y_t^T u_t\Big|_a^b - x_t^T v_t\Big|_a^b .$$

Observe that $S^\# R^T = 0$ and $R^\#(\lambda) + \lambda R\tilde{S} = RS + S^\#$. Moreover, from [13, Proposition 2.1.2] it follows that (x,u) and (y,v) satisfy the boundary conditions in $(\tilde{\mathrm{E}})$ iff

$$\begin{pmatrix}-x_a\\ x_b\end{pmatrix} = -R^T c,\ \begin{pmatrix}u_a\\ u_b\end{pmatrix} = \{R^\#(\lambda)\}^T c,\ \begin{pmatrix}-y_a\\ y_b\end{pmatrix} = -R^T d,\ \begin{pmatrix}v_a\\ v_b\end{pmatrix} = \{R^\#(\nu)\}^T d,$$

for some $c, d \in \mathbb{R}^{2n}$. Thus, from (7) we have

$$\begin{aligned}
(\nu-\lambda)\langle x,y\rangle &= (\nu-\lambda)\int_a^b (x_t^\sigma)^T\tilde{C}_t y_t^\sigma\,\Delta t + \begin{pmatrix}-x_a\\ x_b\end{pmatrix}^T (\nu-\lambda)\tilde{S}\begin{pmatrix}-y_a\\ y_b\end{pmatrix}\\
&= y_t^T u_t\Big|_a^b - x_t^T v_t\Big|_a^b + \begin{pmatrix}-x_a\\ x_b\end{pmatrix}^T (\nu-\lambda)\tilde{S}\begin{pmatrix}-y_a\\ y_b\end{pmatrix}\\
&= \begin{pmatrix}-y_a\\ y_b\end{pmatrix}^T \begin{pmatrix}u_a\\ u_b\end{pmatrix} - \begin{pmatrix}-x_a\\ x_b\end{pmatrix}^T \begin{pmatrix}v_a\\ v_b\end{pmatrix} + \begin{pmatrix}-x_a\\ x_b\end{pmatrix}^T (\nu-\lambda)\tilde{S}\begin{pmatrix}-y_a\\ y_b\end{pmatrix}\\
&= -d^T R\{R^\#(\lambda)\}^T c + c^T R\{R^\#(\nu)\}^T d + (\nu-\lambda)\,c^T R\tilde{S}R^T d\\
&= -d^T R(RS+S^\#)^T c + c^T R(RS+S^\#)^T d = 0.
\end{aligned}$$

Hence, $x \perp y$ and the proof is complete.

(iii) If the system is strictly dense-normal and if $\tilde{S}$ and $\tilde{C}_t$ are all positive semidefinite (which is satisfied in the present setting – note that throughout this paper, with the exception of this remark, we assume $\tilde{S} = 0$ – subject to conditions (V_1) and (V_2) given after this remark), then

all eigenvalues are real.

To see this, let (x,u) be an eigenfunction corresponding to an eigenvalue λ. Then $(\bar{x},\bar{u})$ is an eigenfunction corresponding to the eigenvalue $\bar{\lambda}$, and we may use the calculation from the second part of this remark to obtain

$$\begin{aligned}
0 &= (\lambda-\bar{\lambda})\langle\bar{x},x\rangle\\
&= (\lambda-\bar{\lambda})\left\{\int_a^b \left\{(\bar{x}^\sigma)^T\tilde{C}x^\sigma\right\}_t \Delta t + \begin{pmatrix}-\bar{x}_a\\ \bar{x}_b\end{pmatrix}^T \tilde{S}\begin{pmatrix}-x_a\\ x_b\end{pmatrix}\right\}.
\end{aligned}$$

Clearly, $\langle\bar{x},x\rangle \neq 0$, since otherwise the positive semidefiniteness of $\tilde{S}$ and $\tilde{C}_t$ implies

$$\tilde{S}\begin{pmatrix}-x_a\\ x_b\end{pmatrix} = 0 \quad\text{and}\quad \tilde{C}x^\sigma = 0 \text{ on } [a,b]^\kappa$$

and hence $x = u \equiv 0$ by strict dense-normality, which is impossible. Therefore, $\lambda - \bar{\lambda} = 0$ and our claim $\lambda \in \mathbb{R}$ follows.

Let us continue with the investigation of the general eigenvalue problem (E). Given the eigenvalue problem (E), we define the quadratic functional

$$\mathcal{F}(x,u;\lambda) := \int_a^b \left\{(x^\sigma)^T C(\lambda) x^\sigma + u^T B(\lambda) u\right\}_t \Delta t + \begin{pmatrix} -x_a \\ x_b \end{pmatrix}^T S \begin{pmatrix} -x_a \\ x_b \end{pmatrix},$$

where the matrix S is determined by Remark . We consider the following assumptions:

(V_1) ($H_\mathbb{R}$) is strictly dense-normal on $\mathbb{T}$.

(V_2) $\lambda_1 \le \lambda_2$ always implies $\mathcal{H}_t(\lambda_1) \le \mathcal{H}_t(\lambda_2)$ for all $t \in \mathbb{T}^\kappa$.

(V_3) There exists $\underline{\lambda} \in \mathbb{R}$ such that $\mathcal{F}(\cdot;\underline{\lambda}) > 0$ and $\lambda \le \underline{\lambda}$ always imply for all $t \in \mathbb{T}^\kappa$

$$\operatorname{Ker} B(\underline{\lambda}) \subseteq \operatorname{Ker} B(\lambda) \quad \text{and} \quad B(\lambda)\left\{B^\dagger(\lambda) - B^\dagger(\underline{\lambda})\right\} B(\lambda) \ge 0.$$

(V_4) (H_λ) is normal on $\mathbb{T}$ for all $\lambda \in \mathbb{R}$.

Now the main result of this paper reads as follows.

Theorem 2. *Assume* (V_1)–(V_4). *Then, if there exist eigenvalues of* (E), *they may be arranged by*

$$-\infty < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots,$$

counting multiplicities. More precisely,

(i) (V_1) *and* (V_2) *imply that the eigenvalues are isolated.*

(ii) (V_2)–(V_4) *imply that the eigenvalues are bounded below by* $\underline{\lambda}$, *provided* (H_λ) *satisfies* (D) *for all* $\lambda \in \mathbb{R}$.

Proof. Part (i) – isolatedness. Let $(X(\lambda), U(\lambda))$, $(\tilde{X}(\lambda), \tilde{U}(\lambda))$ be the special normalized conjoined bases of (H_λ) at a for each $\lambda \in \mathbb{R}$. Fix $\lambda_0 \in \mathbb{R}$. Then by Lemma 6 there exists $\varepsilon > 0$ such that $X_b(\lambda)$ is invertible and

$$\begin{pmatrix} I & 0 \\ U_b(\lambda) & \tilde{U}_b(\lambda) \end{pmatrix} \begin{pmatrix} 0 & I \\ X_b(\lambda) & \tilde{X}_b(\lambda) \end{pmatrix}^{-1}$$

is strictly decreasing for all $\lambda \in \mathcal{U}(\lambda_0,\varepsilon)$, where $\mathcal{U}(\lambda_0,\varepsilon) := [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$ is the closed ε-interval around λ_0 without λ_0. It follows from the Index Theorem (Proposition in the next section) that the singular points of

$$\begin{aligned}\Lambda(\lambda) &= R^\# \begin{pmatrix} 0 & I \\ X_b(\lambda) & \tilde{X}_b(\lambda) \end{pmatrix} + R \begin{pmatrix} I & 0 \\ U_b(\lambda) & \tilde{U}_b(\lambda) \end{pmatrix} \\ &= R^\# \begin{pmatrix} -X_a(\lambda) & -\tilde{X}_a(\lambda) \\ X_b(\lambda) & \tilde{X}_b(\lambda) \end{pmatrix} + R \begin{pmatrix} U_a(\lambda) & \tilde{U}_a(\lambda) \\ U_b(\lambda) & \tilde{U}_b(\lambda) \end{pmatrix},\end{aligned}$$

i.e., the eigenvalues of (E) by Theorem 1, are isolated. Furthermore, the multiplicity of an eigenvalue λ_0 is

$$\operatorname{def}\Lambda(\lambda_0) = \operatorname{ind} M(\lambda_0^+) - \operatorname{ind} M(\lambda_0^-),$$

since $X_b^{\#} = \begin{pmatrix} 0 & I \\ X_b(\lambda_0) & \tilde{X}_b(\lambda_0) \end{pmatrix}$ is invertible (the matrix $M(\lambda)$ is defined in Proposition). Hence, part (i) is proved.

Part (ii) – lower boundedness. Assume that (H_λ) satisfies (D) for all $\lambda \in \mathbb{R}$, and that (V_2)–(V_4) hold. For $\lambda \in \mathbb{R}$ define

$$\begin{aligned} M(\lambda) &:= R\left\{S + Q_b^{\#}(\lambda)\right\} R^T, \\ Q_b^{\#}(\lambda) &:= \begin{pmatrix} I & 0 \\ U_b(\lambda) & \tilde{U}_b(\lambda) \end{pmatrix} \begin{pmatrix} 0 & I \\ X_b(\lambda) & \tilde{X}_b(\lambda) \end{pmatrix}^{-1}. \end{aligned}$$

We pick $\lambda_0 \leq \underline{\lambda}$. Then $\mathcal{F}(\cdot;\lambda_0) > 0$ by the Comparison Theorem (Theorem 3 in the next section). Since (H_{λ_0}) is normal on $\mathbb{T}$, Proposition implies that $X_b(\lambda_0)$ and hence $X_b^{\#} = \begin{pmatrix} 0 & I \\ X_b(\lambda_0) & \tilde{X}_b(\lambda_0) \end{pmatrix}$ are invertible, and $M(\lambda_0) > 0$ on $\operatorname{Im} R$. It follows that $X_b^{\#}(\lambda)$ is invertible on some open interval J around λ_0. Moreover, the matrix $Q_b^{\#}(\lambda)$ defined above is strictly decreasing on J, by Lemma 5, and $\operatorname{ind} M(\lambda_0^+) = 0 = \operatorname{ind} M(\lambda_0^-)$. Now, we may apply the Index Theorem (Proposition) to obtain

$$\operatorname{def}\Lambda(\lambda_0) = \operatorname{ind} M(\lambda_0^+) - \operatorname{ind} M(\lambda_0^-) + \operatorname{def} X_b^{\#} = 0,$$

i.e., $\Lambda(\lambda_0)$ is invertible. This means in view of Theorem 1 that λ_0 is not an eigenvalue of (E). Therefore, if there exists an eigenvalue at all, there is the smallest one λ_1 and satisfies $\lambda_1 > \underline{\lambda}$. The proof is complete. ■

4. Auxiliary results

In this section we collect auxiliary results needed in our work. Recall that $\mathcal{U}(\lambda_0, \varepsilon)$ is the closed ε-interval around λ_0 (the center is removed).

Proposition 1 [Index Theorem [13, Theorem 3.4.1, Corollary 3.4.4]]. *Let $m \in \mathbb{N}$ and let there be given matrices $R, R^{\#}, X, U \in \mathbb{R}^{m\times m}$ with* $\operatorname{rank}(R^{\#}\ R) = \operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = m$ *and $R(R^{\#})^T = R^{\#}R^T$, $X^TU = U^TX$. Let $X(\lambda)$, $U(\lambda) \in \mathbb{R}^{m\times m}$ be matrices such that $X^T(\lambda)U(\lambda)$ are symmetric for all $\lambda \in \mathcal{U}(\lambda_0,\varepsilon)$, for some $\varepsilon > 0$, $X(\lambda) \to X$, $U(\lambda) \to U$ as $\lambda \to \lambda_0$, and $X(\lambda)$ is invertible for $\lambda \in \mathcal{U}(\lambda_0,\varepsilon)$. Suppose that $U(\lambda)X^{-1}(\lambda)$ decreases strictly on $[\lambda_0-\varepsilon, \lambda_0)$ and on $(\lambda_0, \lambda_0+\varepsilon]$, and denote*

$$\begin{gathered} M(\lambda) := R^{\#}R^T + RU(\lambda)X^{-1}(\lambda)R^T, \\ \Lambda(\lambda) := R^{\#}X(\lambda) + RU(\lambda), \qquad \Lambda := R^{\#}X + RU. \end{gathered}$$

Then

$$\operatorname{ind} M(\lambda_0^-) := \lim_{\lambda\to\lambda_0^-} \operatorname{ind} M(\lambda), \quad \operatorname{ind} M(\lambda_0^+) := \lim_{\lambda\to\lambda_0^+} \operatorname{ind} M(\lambda)$$

both exist, $\Lambda(\lambda)$ is invertible for all $\lambda \in \mathcal{U}(\lambda_0, \delta)$ for some $\delta \in (0, \varepsilon)$, and

$$\operatorname{def}\Lambda = \operatorname{ind} M(\lambda_0^+) - \operatorname{ind} M(\lambda_0^-) + \operatorname{def} X.$$

Proposition 2 [Jacobi Condition [10, 11]] *Suppose* (D) *holds. Let* (X, U), $(\tilde{X}, \tilde{U})$ *be the special normalized conjoined bases of* (H) *at* a. *Then* $\mathcal{F} > 0$ *iff* (X, U) *has no focal points in* $(a, b]$ *and* $S + Q_b^{\#} > 0$ *on* $\operatorname{Im} R^T \cap \operatorname{Im} X_b^{\#}$, *where* $X^{\#} := \left(\begin{smallmatrix} 0 & I \\ X & \tilde{X} \end{smallmatrix}\right)$ *and* $Q^{\#}$ *is a certain* $2n \times 2n$*-matrix built up from* (X, U), $(\tilde{X}, \tilde{U})$. *Moreover, if* (H) *is normal on* $\mathbb{T}$, *then* $\mathcal{F} > 0$ *implies* X_b *(and hence* $X_b^{\#}$*) is invertible.*

Lemma 4. *Suppose that* $(X(\lambda), U(\lambda))$ *is a conjoined basis of* (H_λ) *for all* $\lambda \in \mathbb{R}$ *with* $\dot{X}_a(\lambda) = 0 = \dot{U}_a$, *i.e.,* X_a *and* U_a *are independent of* λ. *Then*

$$X_t^T(\lambda)\dot{U}_t(\lambda) - U_t^T(\lambda)\dot{X}_t(\lambda) = -\int_a^t \begin{pmatrix} X_\tau^\sigma(\lambda) \\ U_\tau(\lambda) \end{pmatrix}^T \dot{\mathcal{H}}_\tau(\lambda) \begin{pmatrix} X_\tau^\sigma(\lambda) \\ U_\tau(\lambda) \end{pmatrix} \Delta\tau$$

holds for all $t \in \mathbb{T}$ *and for all* $\lambda \in \mathbb{R}$.

Proof. In the computation below we skip the evaluation at $t \in \mathbb{T}$. Compare [5, Lemma 4]. We have

$$\begin{aligned} &\left\{X^T(\nu)\ [U(\lambda) - U(\nu)] - U^T(\nu)[X(\lambda) - X(\nu)]\right\}^\Delta \\ &\qquad = \left\{\begin{pmatrix} X(\nu) \\ U(\nu) \end{pmatrix}^T \begin{pmatrix} U(\lambda) \\ -X(\lambda) \end{pmatrix}\right\}^\Delta \\ &\qquad = \begin{pmatrix} -U(\nu) \\ X(\nu) \end{pmatrix}^{\Delta T} \begin{pmatrix} X^\sigma(\lambda) \\ U(\lambda) \end{pmatrix} - \begin{pmatrix} X^\sigma(\nu) \\ U(\nu) \end{pmatrix}^T \begin{pmatrix} -U(\lambda) \\ X(\lambda) \end{pmatrix}^\Delta \\ &\qquad = \begin{pmatrix} X^\sigma(\nu) \\ U(\nu) \end{pmatrix}^T \{\mathcal{H}(\nu) - \mathcal{H}(\lambda)\} \begin{pmatrix} X^\sigma(\lambda) \\ U(\lambda) \end{pmatrix}. \end{aligned}$$

Now, dividing by $\lambda - \nu$ and letting $\nu \to \lambda$ (observe that (3) is used) yields

$$\left\{X^T(\lambda)\dot{U}(\lambda) - U^T(\lambda)\dot{X}(\lambda)\right\}^\Delta = -\begin{pmatrix} X^\sigma(\lambda) \\ U(\lambda) \end{pmatrix}^T \dot{\mathcal{H}}(\lambda) \begin{pmatrix} X^\sigma(\lambda) \\ U(\lambda) \end{pmatrix}.$$

Integrating from a to t we get

$$-\int_a^t \begin{pmatrix} X_\tau^\sigma(\lambda) \\ U_\tau(\lambda) \end{pmatrix}^T \dot{\mathcal{H}}_\tau(\lambda) \begin{pmatrix} X_\tau^\sigma(\lambda) \\ U_\tau(\lambda) \end{pmatrix} \Delta\tau = X_\tau^T(\lambda)\dot{U}_\tau(\lambda) - U_\tau^T(\lambda)\dot{X}_\tau(\lambda)\Big|_a^t,$$

and $\dot{X}_a(\lambda) = 0 = \dot{U}_a$ yields the result. ■

Lemma 5. *Suppose that* $(X(\lambda), U(\lambda))$, $(\tilde{X}(\lambda), \tilde{U}(\lambda))$ *are normalized conjoined bases of* (H_λ) *for each* $\lambda \in \mathbb{R}$ *with* $\dot{X}_a(\lambda) = \dot{U}_a(\lambda) = 0 = \dot{\tilde{X}}_a(\lambda) = \dot{\tilde{U}}_a(\lambda)$. *Let* $t \in \mathbb{T}$, $t > a$. *Assume that* $X_t(\lambda)$ *is invertible for* λ *in some open interval* J. *For* $\lambda \in J$ *put*

$$Q_t(\lambda) := \begin{pmatrix} I & 0 \\ U_t(\lambda) & \tilde{U}_t(\lambda) \end{pmatrix} \begin{pmatrix} 0 & I \\ X_t(\lambda) & \tilde{X}_t(\lambda) \end{pmatrix}^{-1}.$$

Then (V_2) *implies that* $Q_t(\lambda)$ *decreases on* J. *Moreover,* (V_1) *and* (V_2) *imply* $Q_t(\lambda)$ *that decreases strictly on* J.

Proof. The proof is similar to the proof of [5, Lemma 5], so we sketch it only. Let $t \in \mathbb{T}$, $t > a$, and $\lambda \in J$. We apply Lemma 4 to

$$X_t^{\#}(\lambda) := \begin{pmatrix} 0 & I \\ X_t(\lambda) & \tilde{X}_t(\lambda) \end{pmatrix}, \quad U_t^{\#}(\lambda) := \begin{pmatrix} I & 0 \\ U_t(\lambda) & \tilde{U}_t(\lambda) \end{pmatrix}.$$

Then for $d \in \mathbb{R}^{2n \times 2n}$ it follows that

$$d^T \dot{Q}_t(\lambda) d = -\int_a^t \begin{pmatrix} x_\tau^\sigma \\ u_\tau \end{pmatrix}^T \dot{\mathcal{H}}_\tau \begin{pmatrix} x_\tau^\sigma \\ u_\tau \end{pmatrix} \leq 0,$$

where

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X(\lambda) & \tilde{X}(\lambda) \\ U(\lambda) & \tilde{U}(\lambda) \end{pmatrix} (X^{\#})^{-1}(\lambda) d,$$

and where we used (V_2), i.e., $\dot{\mathcal{H}}(\lambda) \geq 0$. Suppose that (V_1) and (V_2) hold with $d^T \dot{Q}_t(\lambda) d = 0$. Then $\dot{\mathcal{H}}_\tau(\lambda) \left(\begin{smallmatrix} x_\tau^\sigma \\ u_\tau \end{smallmatrix}\right) = 0$ for all $\tau \in [a,t]^\kappa$. Strict dense-normality implies $x_t = u_t \equiv 0$ on $\mathbb{T}$, i.e., $d = 0$. Thus, $\dot{Q}_t < 0$ follows. ∎

Lemma 6. *Let* $(X(\lambda), U(\lambda))$, $(\tilde{X}(\lambda), \tilde{U}(\lambda))$ *be the special normalized conjoined bases of* (H_λ) *at* a *for each* $\lambda \in \mathbb{R}$. *Assumptions* (V_1) *and* (V_2) *imply that for all* $\lambda_0 \in \mathbb{R}$ *there exists* $\varepsilon > 0$ *such that* $X_b(\lambda)$ *is invertible and* $Q_b(\lambda)$ *defined by*

$$Q_b(\lambda) := \begin{pmatrix} I & 0 \\ U_b(\lambda) & \tilde{U}_b(\lambda) \end{pmatrix} \begin{pmatrix} 0 & I \\ X_b(\lambda) & \tilde{X}_b(\lambda) \end{pmatrix}^{-1} \tag{8}$$

is strictly decreasing for all $\lambda \in \mathcal{U}(\lambda_0, \varepsilon)$.

Proof. Fix $\lambda_0 \in \mathbb{R}$ and let $(\hat{X}, \hat{U})$ be the conjoined basis of (H_{λ_0}) such that $(X(\lambda_0), U(\lambda_0))$ and $(\hat{X}, \hat{U})$ are normalized and $\hat{X}_b$ is invertible, see Lemma 1. Let $(\hat{X}(\lambda), \hat{U}(\lambda))$ be the conjoined basis of (H_λ) with $\hat{X}_a(\lambda) = \hat{X}_a$, $\hat{U}_a(\lambda) = \hat{U}_a$, $\lambda \in \mathbb{R}$. Due to continuity, $\hat{X}(\lambda)$ is invertible on some open interval around λ_0 and on that interval we have, by Lemma 5 with $(-\hat{X}(\lambda), -\hat{U}(\lambda))$ and $(X(\lambda), U(\lambda))$, that the matrix

$$\begin{pmatrix} I & 0 \\ -\hat{U}_b(\lambda) & U_b(\lambda) \end{pmatrix} \begin{pmatrix} 0 & I \\ -\hat{X}_b(\lambda) & X_b(\lambda) \end{pmatrix}^{-1} = \begin{pmatrix} \hat{X}_b^{-1}(\lambda) X_b(\lambda) & -\hat{X}_b^{-1}(\lambda) \\ -\hat{X}_b^{T-1}(\lambda) & \hat{U}_b(\lambda)\hat{X}_b^{-1}(\lambda) \end{pmatrix}$$

is strictly decreasing. Consequently, $\hat{X}_b^{-1}(\lambda) X_b(\lambda)$ is strictly decreasing as well. It follows that $X_b(\lambda)$ is invertible on $\mathcal{U}(\lambda_0, \varepsilon)$ for some $\varepsilon > 0$. Applying Lemma 5 again, the strict monotonicity of the matrix $Q_b(\lambda)$ in (8) follows. ■

Lemma 7. *Let $m \in \mathbb{N}$ and let be given real $m \times m$-matrices A, $\underline{A}$, B, $\underline{B}$, C , $\underline{C}$ such that the Hamiltonian matrices*

$$\mathcal{H} := \begin{pmatrix} -C & A^T \\ A & B \end{pmatrix}, \quad \underline{\mathcal{H}} := \begin{pmatrix} -\underline{C} & \underline{A}^T \\ \underline{A} & \underline{B} \end{pmatrix}$$

are symmetric. Suppose that

$$\underline{\mathcal{H}} \geq \mathcal{H}, \quad \operatorname{Ker} \underline{B} \subseteq \operatorname{Ker} B, \quad B(B^\dagger - \underline{B}^\dagger)B \geq 0$$

hold. Then

$$x^T C x + u^T B u \geq \underline{x}^T \underline{C} \underline{x} + \underline{u}^T \underline{B} \underline{u}$$

for all $x, u, \underline{x}, \underline{u} \in \mathbb{R}^m$ with $Bu - \underline{B}\underline{u} = (\underline{A} - A)x$. Moreover, there exists a matrix $E \in \mathbb{R}^{m \times m}$ such that

$$\underline{A} - A = (\underline{B} - B)E \quad \text{and} \quad E^T(\underline{B} - B)E \leq C - \underline{C}.$$

Proof. The proof is similar to the discrete case [5, Lemma 7], compare also the continuous case [13, Lemma 3.1.10]. ■

Remark 3. Observe that $\operatorname{Ker} \underline{B} \subseteq \operatorname{Ker} B$ from the above lemma is equivalent to $B = B\underline{B}^\dagger \underline{B} = \underline{B}\underline{B}^\dagger B$, see [3, Lemma A5, pg. 94] or [4, Remark 2(iii)].

Theorem 3 (Comparison Theorem)**.** *Suppose that* (V_2) *and* (V_3) *hold. Then $\mathcal{F}(\cdot; \lambda) > 0$ for all $\lambda \leq \underline{\lambda}$.*

Proof. Suppose $\mathcal{F}(\cdot; \underline{\lambda}) > 0$ and let $\lambda \leq \underline{\lambda}$. From (V_2) and (V_3) we have

$$B_t(\underline{\lambda}) \geq B_t(\lambda), \;\; \operatorname{Ker} B_t(\underline{\lambda}) \subseteq \operatorname{Ker} B_t(\lambda), \;\; B_t(\lambda)\left\{B_t^\dagger(\lambda) - B_t^\dagger(\underline{\lambda})\right\} B_t(\lambda) \geq 0.$$

Let (x, u) be admissible for $\mathcal{F}(\cdot; \lambda)$, i.e., $x_t^\Delta = A_t(\lambda)x_t^\sigma + B_t(\lambda)u_t$, $t \in \mathbb{T}^\kappa$, with $\left(\begin{smallmatrix} -x_a \\ x_b \end{smallmatrix}\right) \in \operatorname{Im} R^T$, and $x \not\equiv 0$. For $t \in \mathbb{T}^\kappa$ we define

$$\underline{u}_t := B_t^\dagger(\underline{\lambda})B_t(\lambda)u_t - \left\{I - B_t^\dagger(\underline{\lambda})B_t(\lambda)\right\} E_t x_t^\sigma,$$

where $E : \mathbb{T}^\kappa \to \mathbb{R}^{n\times n}$ is such that $A_t(\underline{\lambda}) - A_t(\lambda) = \{B_t(\underline{\lambda}) - B_t(\lambda)\}E_t$, by Lemma 7. Note also that $B_t(\underline{\lambda})B_t^\dagger(\underline{\lambda})B_t(\lambda) = B_t(\lambda)$ by Remark . Then (all functions evaluated at t)

$$\begin{aligned} B(\lambda)u - B(\underline{\lambda})\underline{u} &= B(\lambda)u - B(\underline{\lambda})B^\dagger(\underline{\lambda})B(\lambda)u \\ &\quad + \left\{B(\underline{\lambda}) - B(\underline{\lambda})B^\dagger(\underline{\lambda})B(\lambda)\right\} Ex^\sigma \\ &= \{B(\underline{\lambda}) - B(\lambda)\} Ex^\sigma = \{A(\underline{\lambda}) - A(\lambda)\} x^\sigma, \end{aligned}$$

so that

$$A(\underline{\lambda})x^\sigma + B(\underline{\lambda})\underline{u} = A(\lambda)x^\sigma + B(\lambda)u = x^\Delta,$$

i.e., $(x, \underline{u})$ is admissible for $\mathcal{F}(\cdot; \underline{\lambda})$. Applying Lemma 7 again we get

$$\begin{aligned} 0 < \mathcal{F}(x, \underline{u}; \underline{\lambda}) &= \int_a^b \left\{(x^\sigma)^T C(\underline{\lambda})x^\sigma + \underline{u}^T B(\underline{\lambda})\underline{u}\right\}_t \Delta t \\ + \begin{pmatrix} -x_a \\ x_b \end{pmatrix}^T S \begin{pmatrix} -x_a \\ x_b \end{pmatrix} &\le \int_a^b \left\{(x^\sigma)^T C(\lambda)x^\sigma + u^T B(\lambda)u\right\}_t \Delta t \\ + \begin{pmatrix} -x_a \\ x_b \end{pmatrix}^T S \begin{pmatrix} -x_a \\ x_b \end{pmatrix} &= \mathcal{F}(x, u; \lambda). \end{aligned}$$

Hence, $\mathcal{F}(\cdot; \lambda) > 0$ as well. ∎

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