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ON STRONG APPROXIMATION OF FUNCTIONS OF ONE AND TWO VARIABLES BY CERTAIN OPERATORS

Abstract: We investigate certain class of linear operators in polynomial weighted spaces of differentiable functions of one and two variables. We introduce strong differences of functions and operators and we give approximation theorems for them.
The present theorems show that strong approximation is more general than normal approximation.
Section I is devoted strong approximation of functions of one variable and Section II of functions of two variables.
This note is motivated by resultus on strong approximation connected with Fourier series ([5], [8])

Key words: linear operator, polynomial weighted space, strong approximation.

I. Strong approximation of functions of one variable

1. Introduction

1.1. Analogously as in [1] let $p \in N_0 := \{0, 1, 2, ...\}$,

$$w_0(x) := 1, \qquad w_p(x) := (1 + x^p)^{-1} \qquad \text{if} \quad p \geq 1, \tag{1}$$

for $x \in R_0 := [0, \infty)$, and let C_p be the polynomial weighted space of all real-valued functions f continuous on R_0 for which $w_p f$ is bounded in R_0 and the norm is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup \{w_p(x) \, |f(x)| \, : \, x \in R_0\}. \tag{2}$$

It is obvious that $C_p \subset C_q$ for $p < q$ and $\|f\|_q \leq \|f\|_p$ for $f \in C_p$.

Let $r \in N_0$ be a fixed number. Denote by C^r the set of all r-times differentiable functions $f \in C_r$ for which derivatives $f^{(m)} \in C_{r-m}$ for all $0 \leq m \leq r$. Clearly $C^0 \equiv C_0$.

In this paper we shall apply the modulus of continuity of $f \in C_0$, i.e.

(3) $$\omega(f;t) := \sup\{|f(x)-f(y)| \,:\, x,y \in R_0,\ |x-y| \le t\},\ t \ge 0.$$

It is well known ([9]) that if $f \in C_0$, then

(4) $$\omega(f;\lambda t) \le (\lambda+1)\,\omega(f;t) \quad \text{for} \quad \lambda, t \in R_0.$$

If $f \in C_0$ is uniformly continuous function, then $\lim_{t\to 0_+} \omega(f;t) = 0$.

We shall apply the following inequalities obtained from (1) for $p,q,r \in N_0$ and $p<q$:

(5) $$\frac{w_q(x)}{w_p(x)} \le 2, \quad (w_p(x))^r \le w_{pr}(x), \quad \big(w_p(x)\big)^{-r} \le 2^r \big(w_{pr}(x)\big)^{-1},$$

for $x \in R_0$. Moreover, if $p \in N$, then

(6) $$\frac{1}{w_p(t)} \le 2^p\left(\frac{1}{w_p(x)} + |t-x|^p\right) \quad \text{for all} \quad t,x \in R_0.$$

We shall denote by $M_k(a,b)$, $k \in N$, suitable positive constants depending only on indicated parameters a, b.

1.2. In [1] were proved direct and inverse approximation theorems for Szász-Mirakyan operators

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\, f\left(\frac{k}{n}\right)$$

and Baskakov operators

$$V_n(f;x) := \sum_{k=0}^{\infty}\binom{n-1+k}{k} x^k\,(1+x)^{-n-k}\, f\left(\frac{k}{n}\right),$$

$x \in R_0$, $n \in N = \{1,2,...\}$, of functions $f \in C_p$, $p \in N_0$.

In [6] was proved that the following modified Baskakov operators

$$V_{n;r}(f;x) := \sum_{k=0}^{\infty}\binom{n-1+k}{k} x^k\,(1+x)^{-n-k}\sum_{j=0}^{r}\frac{f^{(j)}\left(\frac{k}{n}\right)}{j!}\left(x-\frac{k}{n}\right)^j,$$

$(x \in R_0,\ n \in N)$ of r-times differentiable functions f have better approximation properties than $V_n(f)$.

1.3. In §2 of this section we shall introduce the class of linear operators of Baskakov and Kantrovich type ([3]) in the space C^r and we shall examine strong differences of these operators and $f \in C^r$. The main theorems will be given in §3.

The problem of strong approximation with the power $q > 0$ is well known for 2π-periodic functions and their Fourier series ([5], [8]).

In [7] is investigated the strong approximation of functions $f \in C_0$ by some linear operators. In this paper we shall examine this problem for functions $f \in C^r$ and introduced operators.

2. Definitions and lemmas

2.1. Analogously to [7] we denote by Ω the set of all infinite matrices $A = [a_{nk}(\cdot)]$, $n \in N$, $k \in N_0$, of functions $a_{nk} \in C_0$ and having properties:

(i) $\quad a_{nk}(x) \geq 0 \quad$ for $\quad x \in R_0,\ n \in N,\ k \in N_0,$

(ii) $\quad \sum_{k=0}^{\infty} a_{nk}(x) = 1 \quad$ for $\quad x \in R_0,\ n \in N,$

(iii) the series $\sum_{k=0}^{\infty} k^r a_{nk}(x)$ is uniformly convergent on R_0 for all $n, r \in N$ and its sum is a function belonging to C_r,

(iv) for every $r \in N$ there exists positive constant $M_1(r, A)$ independent on $x \in R_0$ and $n \in N$ such that for the functions

$$T_{n,r}(x; A) := \sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n} - x\right)^r, \qquad x \in R_0,\ n \in N, \tag{7}$$

(belonging to C_r) there holds the inequality

$$\|T_{n,2r}(\,\cdot\,; A)\|_{2r} \leq M_1(r, A)\, n^{-r}, \qquad n \in N. \tag{8}$$

2.2. Let $A \in \Omega$ and let $r \in N_0$. For $f \in C^r$ we define operators

$$L_{n;r}(f; A; x) := \sum_{k=0}^{\infty} a_{nk}(x)\, n \int_{I_{nk}} F_r(t, x) dt, \qquad x \in R_0, \quad n \in N, \tag{9}$$

where $I_{nk} := \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and

$$F_r(t, x) := \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!} (x - t)^j. \tag{10}$$

If $r = 0$ and $f \in C_0$, then

$$L_{n;0}(f;A;x) := \sum_{k=0}^{\infty} a_{nk}(x)\, n \int_{I_{nk}} f(t)dt, \qquad x \in R_0, \quad n \in N. \tag{11}$$

The properties (i)-(iv) of A imply that $L_{n;r}(f;A)$, $n \in N$, is well defined for every $f(x) = x^p$, $p \in N_0$, and

$$L_{n;r}(1;A;x) = 1 \qquad \text{for} \quad x \in R_0,\ n \in N. \tag{12}$$

In Lemma 6 we shall prove that $L_{n;r}(f;A)$ are well defined for every $f \in C^r$. In Lemma 7 we shall show that for $f \in C^r$ and $L_{n;r}(f;A)$ there exist the following strong differences with the power $q > 0$:

$$H^q_{n;r}(f;A;x) := \left(\sum_{k=0}^{\infty} a_{nk}(x) \left| n \int_{I_{nk}} F_r(t,x)dt \right|^q \right)^{\frac{1}{q}}, \quad x \in R_0,\ n \in N. \tag{13}$$

In particular for $f \in C^0$ we have

$$H^q_{n;0}(f;A;x) := \left(\sum_{k=0}^{\infty} a_{nk}(x) \left| n \int_{I_{nk}} f(t)dt - f(x) \right|^q \right)^{\frac{1}{q}}. \tag{14}$$

By (9)-(12) we get

$$L_{n;r}(f;A;x) - f(x) := \sum_{k=0}^{\infty} a_{nk}(x)\, n \int_{I_{nk}} \left(F_r(t,x) - f(x)\right) dt, \tag{15}$$

for every $f \in C^r$, $r \in N_0$, and $x \in R_0$ and $n \in N$.

From (13)-(15) and properties (i) and (ii) of A we deduce that

$$|L_{n;r}(f;A;x) - f(x)| \le H^1_{n;r}(f;A;x) \tag{16}$$

and

$$H^p_{n;r}(f;A;x) \le H^q_{n;r}(f;A;x) \qquad \text{if} \quad 0 < p < q < \infty, \tag{17}$$

for every $f \in C^r$, $x \in R_0$ and $n \in N$.

2.3. First we shall prove some inequalities.

Lemma 1. *For every $A \in \Omega$ and $s \in N$ there exists $M_2(s,A) = const. > 0$ such that*

$$w_s(x) L_{n;0}\left(|t-x|^s;A;x\right) \le M_2(s,A)\, n^{-\frac{s}{2}}, \quad x \in R_0,\ n \in N. \tag{18}$$

Proof. From (11), (12) and (i) and by the Hölder inequality and (7) we get

$$L_{n;0}\left(|t-x|^s;A;x\right) \le \sum_{k=0}^{\infty} a_{nk}(x) \left(\left|\frac{k+1}{n}-x\right|^s + \left|\frac{k}{n}-x\right|^s \right) n \int_{I_{nk}} dt$$
$$\le (2^s+1) \sum_{k=0}^{\infty} a_{nk}(x) \left|\frac{k}{n}-x\right|^s + \frac{2^s}{n^s}$$
$$\le (2^s+1) \left(\sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n}-x\right)^{2s} \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} a_{nk}(x) \right)^{\frac{1}{2}} + \frac{2^s}{n^s}$$
$$= (2^s+1)\left(T_{n;2s}(x;A)\right)^{\frac{1}{2}} + 2^s n^{-s}, \qquad x \in R_0, \quad n \in N.$$

Next, by (1), (5) and (8) we obtain (18). ■

Lemma 2. *For every $A \in \Omega$, $p \in N_0$ and $s \in N_0$ there exists $M_3 = M_3(p,s,A) = const. > 0$ such that*

$$w_{p+s}(x)\, L_{n;0}\left(\frac{|t-x|^s}{w_p(t)}; A; x \right) \le M_3(s,A)\, n^{-\frac{s}{2}} \tag{19}$$

for all $x \in R_0$ and $n \in N$.

Proof. If $p = s = 0$, then (19) follows from (18). If $p \in N$ and $s \in N_0$, then by (11) and (6) we have

$$L_{n;0}\left(\frac{|t-x|^s}{w_p(t)}; A; x \right) \le \frac{2^p}{w_p(x)} L_{n;0}\left(|t-x|^s; A; x\right) +$$
$$+ 2^p L_{n;0}\left(|t-x|^{p+s}; A; x\right), \quad x \in R_0, \ n \in N,$$

which by (1) and Lemma 1 imply (19). ■

Applying (5), (6) and Lemma 2, we easily obtain

Lemma 3. *For every $A \in \Omega$, $p \in N_0$, $q \in N$ and $s \in N_0$ there exists $M_4 = M_4(p,q,s,A) = const. > 0$ such that*

$$w_{(p+s)q}(x)\, L_{n;0}\left(\left(\frac{|t-x|^s}{w_p(t)} \right)^q; A; x \right) \le M_4\, n^{-\frac{sq}{2}},$$

for $x \in R_0$ and $n \in N$.

Applying the Hölder inequality, we immediately obtain

Lemma 4. *Let $k \in N_0$ and $n \in N$ be fixed numbers. Then for every function h continuous on $I_{nk} = \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and $q > 1$ we have*

$$\left| n \int_{I_{nk}} h(t)dt \right|^q \le n \int_{I_{nk}} |h(t)|^q dt.$$

2.4. Now we shall give main lemmas.

Lemma 5. *For every $A \in \Omega$ and $p \in N_0$ there exists $M_5(p, A) = const. > 0$ such that*

$$\left\| L_{n;0}\left(\frac{1}{w_p(t)}; A; \cdot\right)\right\|_p \le M_5(p, A), \qquad n \in N, \tag{20}$$

and

$$\|L_{n;0}(f; A; \cdot)\|_p \le M_5(p, A)\, \|f\|_p, \qquad n \in N, \tag{21}$$

for every $f \in C_p$.

The formula (11) *and* (21) *and the property (i) of A show that $L_{n;0}(f; A)$, $n \in N$, is a positive linear operator from the space C_p into C_p.*

Proof. The inequality (20) follows from (19) with $s = 0$ and $p \in N_0$.

By (11), (1) and (2) we have

$$\|L_{n;0}(f; A; \cdot)\|_p \le \|f\|_p \left\| L_{n;0}\left(\frac{1}{w_p(t)}; A; \cdot\right)\right\|_p,$$

for every $f \in C_p$, $p \in N_0$, and $n \in N$. Using (20), we obtain (21). ■

Lemma 6. *Let $A \in \Omega$ and $r \in N$. Then there exists $M_6(r, A) = const. > 0$ such that for every $f \in C^r$ and $n \in N$ we have*

$$\|L_{n;r}(f; A; \cdot)\|_r \le M_6(r, A) \sum_{j=0}^{r} \left\| f^{(j)} \right\|_{r-j}. \tag{22}$$

The formulas (9) *and* (10) *and the inequality* (22) *show that $L_{n;r}(f; A)$ is a linear operator from the space C^r into C_r.*

Proof. From (9), (10), (1) and (2) we deduce that

$$\begin{aligned} |L_{n;r}(f; A; x)| &\le \sum_{k=0}^{\infty} a_{nk}(x) \sum_{j=0}^{r} \frac{n}{j!} \int_{I_{nk}} \left| f^{(j)}(t) \right| |x-t|^j dt \\ &\le \sum_{j=0}^{r} \frac{1}{j!} \left\| f^{(j)} \right\|_{r-j} L_{n;0}\left(\frac{|t-x|^j}{w_{r-j}(t)}; A; x\right), \end{aligned}$$

for every $f \in C^r$, $x \in R_0$ and $n \in N$. Now applying Lemma 2, we easily obtain (22). ∎

Lemma 7. *Let $A \in \Omega$, $r \in N_0$ and $q > 0$. Then there exists $M_7 \equiv M_7(q, r, A) = const. > 0$ such that*

$$\left\| H_{n;r}^q(f; A; \cdot) \right\|_r \le M_7 \sum_{j=0}^{r} \left\| f^{(j)} \right\|_{r-j} \tag{23}$$

for every $f \in C^r$ and $n \in N$.

The formula (13) and (23) show that $H_{n;r}^q(f; A; \cdot) \in C_r$ for every $f \in C^r$ and $q > 0$.

Proof. First let $q \in N$. If $r = 0$, then by (1), (2), (14) and properties (i) and (ii) of A we get

$$\left\| H_{n;0}^q(f; A; \cdot) \right\|_0 \le 2\|f\|_0 \left(\sum_{k=0}^{\infty} a_{nk}(x) \right)^{\frac{1}{q}} = 2\|f\|_0, \quad n \in N.$$

If $r \in N$, then by (10), (1) and (2) we have

$$\begin{aligned} \left| n \int_{I_{nk}} F_r(t, x) dt - f(x) \right| \ \le\ & n \sum_{j=0}^{r} \frac{1}{j!} \left\| f^{(j)} \right\|_{r-j} \int_{I_{nk}} \frac{|x - t|^j}{w_{r-j}(t)} dt \\ & + \frac{\|f\|_r}{w_r(x)}, \quad x \in R_0,\ n \in N. \end{aligned} \tag{24}$$

Using (24) to (13) and by the Minkowski inequality and properties (i), (ii) of A, we get

$$\begin{aligned} H_{n;r}^q(f; A; x) \ \le\ & \sum_{j=0}^{r} \left\| f^{(j)} \right\|_{r-j} \left(\sum_{k=0}^{\infty} a_{nk}(x) \left(n \int_{I_{nk}} \frac{|x - j|^j}{w_{r-j}(t)} dt \right)^q \right)^{\frac{1}{q}} \\ & + \|f\|_r \frac{1}{w_r(x)}, \qquad x \in R_0,\ n \in N. \end{aligned}$$

Further, by (5) and Lemma 4 we have

$$\begin{aligned} w_r(x) H_{n;r}^q(f; A; x) \le 4 & \sum_{j=0}^{r} \left\| f^{(j)} \right\|_{r-j} \left(w_{rq}(x)\, L_{n;0} \left(\frac{|x - j|^{jq}}{w_{(r-j)q}(t)}; A; x \right) \right)^{\frac{1}{q}} \\ & + \|f\|_r, \qquad x \in R_0,\ n \in N, \end{aligned}$$

which by Lemma 3 and (2) yields (23) for $q \in N$.

If $0 < q \notin N$, then $[q]+1$ belongs to N and $q < [q]+1$ ($[q]$ denotes the integral part of q). This fact and (17) imply that

$$\left\| H^q_{n;r}(f;A;\cdot) \right\|_r \le \left\| H^{[q]+1}_{n;r}(f;A;\cdot) \right\|_r \qquad n \in N,$$

and by (23) with the power $[q]+1$ we obtain (23) for $0 < q \notin N$.

Thus the proof is completed. ■

3. Theorem and corollaries

3.1. Now we shall prove the main theorem.

Theorem 1. *Let be given $A \in \Omega$, $r \in N_0$ and $q > 0$. Then there exists $M_8 = M_8(q,r,A) = const. > 0$ such that for every $f \in C^r$ and $n \in N$ we have*

$$\left\| H^q_{n;r}(f;A;\cdot) \right\|_{r+1} \le M_8\, n^{-\frac{r}{2}}\, \omega\left(f^{(r)}; n^{-\frac{1}{2}} \right). \tag{25}$$

Proof. a) First let $r = 0$ and $q \in N$. For $f \in C_0$ we get from (14)

$$H^q_{n;0}(f;A;x) \le \left(\sum_{k=0}^{\infty} a_{nk}(x) \left(n \int_{I_{nk}} |f(t) - f(x)| dt \right)^q \right)^{\frac{1}{q}}$$

and by (3) and (4) we have

$$|f(t) - f(x)| \le \omega(f; |t-x|) \le \left(\sqrt{n}\, |t-x| + 1 \right) \omega\left(f; \frac{1}{\sqrt{n}} \right),$$

for $t, x \in R_0$ and $n \in N$. Consequently,

$$H^q_{n;0}(f;A;x) \le \omega\left(f; \frac{1}{\sqrt{n}} \right) \left(\sum_{k=0}^{\infty} a_{nk}(x) \left(n^{\frac{3}{2}} \int_{I_{nk}} |t-x| dt + 1 \right)^q \right)^{\frac{1}{q}}.$$

Applying the Minkowski inequality for sum and (12) and Lemma 4, we get

$$\begin{aligned} H^q_{n;0}(f;A;x) &\le \omega\left(f; \frac{1}{\sqrt{n}} \right) \left\{ \left(\sum_{k=0}^{\infty} a_{nk}(x) \left(n^{\frac{3}{2}} \int_{I_{nk}} |t-x| dt \right)^q \right)^{\frac{1}{q}} + 1 \right\} \\ &\le \omega\left(f; \frac{1}{\sqrt{n}} \right) \left\{ \sqrt{n} \left(L_{n;0}\left(|t-x|^q; A; x \right) \right)^{\frac{1}{q}} + 1 \right\} \end{aligned}$$

and further by (1), (5) and Lemma 1 we obtain

$$w_1(x)\, H^q_{n;0}(f;A;x) \le \omega\left(f; \frac{1}{\sqrt{n}} \right) \left\{ \sqrt{n}\ \left(w_q(x) L_{n;0}\left(|t-x|^q; A; x \right) \right)^{\frac{1}{q}} + 1 \right\}$$

$$\leq M_9(q,A)\,\omega\left(f;\frac{1}{\sqrt{n}}\right) \quad \text{for} \quad x\in R_0,\ n\in N.$$

From this and (2) follows (25) for $n\in N$, $r=0$ and $q\in N$.

b) Let $r\in N$ and $q\in N$. Similarly as in [3] and [4] we use the following modified Taylor formula for $f\in C^r$ at a point $t\in R_0$:

$$f(x)=\sum_{j=0}^{r}\frac{f^{(j)}(t)}{j!}(x-t)^j+\frac{(x-t)^r}{(r-1)!}\,Z_r(x,t), \quad x\in R_0, \tag{26}$$

where

$$Z_r(x,t):=\int_0^1(1-u)^{r-1}\left(f^{(r)}(t+u(x-t))-f^{(r)}(t)\right)du.$$

The formulas (13), (10) and (26) imply that

$$H^q_{n;r}(f;A;x)=\left(\sum_{k=0}^{\infty}a_{nk}(x)\left|n\int_{I_{nk}}\frac{(x-t)^r}{(r-1)!}\,Z_r(x,t)dt\right|^q\right)^{\frac{1}{q}}, \tag{27}$$

for every $f\in C^r$, $x\in R_0$ and $n\in N$. Since $f^{(r)}\in C_0$, we have by (3) and (4)

$$\begin{aligned} |Z_r(x,t)| &\leq \int_0^1(1-u)^{r-1}\,\omega\left(f^{(r)};u|x-t|\right)du \\ &\leq \omega\left(f^{(r)};|x-t|\right)\int_0^1(1-u)^{r-1}du \\ &\leq \frac{1}{r}\left(\sqrt{n}\,|t-x|+1\right)\,\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{28}$$

From (27) and (28) and by the Minkowski inequality and Lemma 4 we deduce that

$$\begin{aligned} H^q_{n;r}(f;A;x) &\leq \frac{1}{r!}\,\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right) \\ &\quad\times\left(\sum_{k=0}^{\infty}a_{nk}(x)\left(n\int_{I_{nk}}|t-x|^r\left(1+\sqrt{n}\,|t-x|\right)dt\right)^q\right)^{\frac{1}{q}} \\ &\leq \frac{1}{r!}\,\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right)\left\{\left(\sum_{k=0}^{\infty}a_{nk}(x)\left(n\int_{I_{nk}}|t-x|^r dt\right)^q\right)^{\frac{1}{q}}+\right. \end{aligned}$$

$$+\left(\sum_{k=0}^{\infty}a_{nk}(x)\left(n^{\frac{3}{2}}\int_{I_{nk}}|t-x|^{r+1}dt\right)^{q}\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \frac{1}{r!}\,\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right)\left\{\left(L_{n;0}\left(|t-x|^{rq};A;x\right)\right)^{\frac{1}{q}}+\right.$$

$$\left.+\sqrt{n}\left(L_{n;0}\left(|t-x|^{(r+1)q};A;x\right)\right)^{\frac{1}{q}}\right\}.$$

By (1), (5) and Lemma 1 we have

$$w_{r+1}(x)\left(L_{n;0}\left(|t-x|^{rq};A;x\right)\right)^{\frac{1}{q}}$$
$$\leq \left(w_{rq}(x)\,L_{n;0}\left(|t-x|^{rq};A;x\right)\right)^{\frac{1}{q}}\leq M_{10}(q,r,A,)\,n^{-\frac{r}{2}}$$

and analogously

$$w_{r+1}(x)\left(L_{n;0}\left(|t-x|^{(r+1)q};A;x\right)\right)^{\frac{1}{q}}\leq M_{11}(q,r,A)\,n^{-\frac{r+1}{2}},$$

for $x\in R_0$ and $n\in N$. Combining the above, we easily derive (25) for $n\in N$ and $q\in N$.

c) Let $r\in N_0$ and $0<q<\notin N$. Similarly as in the proof of Lemma 7 we have $q<[q]+1$ and by (17), (2) and (25) we can write

$$\left\|H^{q}_{n;r}(f;A;\cdot)\right\|_{r+1}\leq\left\|H^{[q]+1}_{n;r}(f;A;\cdot)\right\|_{r+1}\leq M_8\,n^{-\frac{r}{2}}\,\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right),$$

for every $f\in C^r$ and $n\in N$.

Now the proof is completed. ■

3.2. From Theorem 1 and (16) we can derive the following corollaries.

Corollary 1. *Suppose that A, r, q satisfy assumptions of Theorem 1. If $f\in C^r$ and $f^{(r)}$ is uniformly continuous on R_0, then*

$$\lim_{n\to\infty} n^{\frac{r}{2}}\left\|H^{q}_{n;r}(f;A;\cdot)\right\|_{r+1}=0$$

which implies that

$$\lim_{n\to\infty} n^{\frac{r}{2}}\,H^{q}_{n;r}(f;A;x)=0 \qquad \text{at every}\quad x\in R_0.$$

Corollary 2. *Let $A\in\Omega$ and $r\in N_0$. Then there exists $M_{12}(r,A)=$ const. >0 such that for every $f\in C^r$ and $n\in N$ we have*

$$\|L_{n;r}(f;A;\cdot)-f(\cdot)\|_{r+1}\leq\left\|H^{1}_{n;r}(;A;\cdot)\right\|_{r+1}\leq M_{12}(r,A)\,n^{-\frac{r}{2}}\,\omega\left(f^{(r)};\frac{1}{\sqrt{n}}\right).$$

Moreover if $f^{(r)}$ is uniformly continuous on R_0, then

$$\lim_{n\to\infty} n^{\frac{r}{2}} \, \|L_{n;r}(f;A;\cdot) - f(\cdot)\|_{r+1} = 0.$$

From this follows

$$\lim_{n\to\infty} n^{\frac{r}{2}} \, (L_{n;r}(f;A;x) - f(x)) = 0$$

at every $x \in R_0$.

Corollary 3. *The order of strong approximation of $f \in C^r$ by $L_{n;r}(f;A;\cdot)$ is independent on $q > 0$ and is dependent on $r \in N_0$. This order improves if r increases.*

3.3. Finaly we observe that the definition (9) contains the Szász-Mirakjan-Kantorovich operators

$$\tilde{S}_{n;r}(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \, n \int_{I_{nk}} F_r(t,x)dt,$$

and the Baskakov-Kantorovich operators

$$\tilde{V}_{n;r}(f;x) := \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k (1-x)^{-n-k} n \int_{I_{nk}} F_r(t,x)dt,$$

of functions $f \in C^r$, $r \in N_0$ ([6]). Hence the above Theorem 1 and corollaries concern also these operators.

II. Strong approximation of functions of two variables

In this section we shall introduce analogues of operators $L_{n;r}(f;A)$ for differentiable functions of two variables and we shall prove an analogue of Theorem 1. We shall use notation given in Section I.

4. Definitions and preliminary results

4.1. Let $p, q \in N_0$ and let

(29) $$w_{p,q}(x,y) := w_p(x)\, w_q(y), \qquad (x,y) \in R_0^2 = R_0 \times R_0,$$

where $w_p(\cdot)$ is defined by (1). Similarly as in Section I we denote by $C_{p,q}$ the set of all real-valued functions f continuous on R_0^2 for which $w_{p,q}f$ is bounded on R_0^2 and the norm is defined by

(30) $$\|f\|_{p,q} := \sup\left\{|f(x,y)|\, w_{p,q}(x,y) \,:\, (x,y) \in R_0^2\right\}.$$

We have $C_{p,q} \subseteq C_{r,s}$ if $p,q,r,s \in N_0$ and $p \le r$ and $q \le s$. Moreover for $f \in C_{p,q}$ we have $\|f\|_{r,s} \le \|f\|_{p,q}$.

For every fixed $r \in N_0$ we define the class $C^r\left(R_0^2\right)$ of all functions $f \in C_{r,r}$ r-times differentiable on R_0^2 which partial derivatives $f^{(m)}_{x^{m-i}y^i} \in C_{r-m,r-m}$, for all $0 \le i \le m \le r$. Clearly $C^0\left(R_0^2\right) \equiv C_{0,0}$.

Similarly to Section I we shall use the modulus of continuity $\omega(f;\cdot,\cdot)$ of function $f \in C_{0,0}$, i.e.

$$\omega(f;s;t) := \sup\left\{|f(x,y)-f(u,v)| \,:\, (x,y),(u,v) \in R_0^2,\right. \\ \left. |x-u| \le s,\ |y-v| \le t\right\}, \quad t,s \in R_0. \tag{31}$$

It is known ([9]) that for every $f \in C_{0,0}$, there holas the inequality

$$\begin{aligned} \omega(f;\lambda_1 s;\lambda_2 t) &\le (\lambda_1+1)\,\omega(f;s,0)+(\lambda_2+1)\,\omega(f;0,t) \\ &\le (\lambda_1+\lambda_2+2)\,\omega(f;s,t), \quad \lambda_1,\lambda_2,s,t \in R_0. \end{aligned} \tag{32}$$

Moreover $\lim\limits_{s,t\to 0^+} \omega(f;s,t) = 0$ if $f \in C_{0,0}$ is uniformly continuous on R_0^2.

4.2. Let $A, B \in \Omega$ be fixed matrices, $A = [a_{nk}(\cdot)]$, $B = [b_{nk}(\cdot)]$, and let $r \in N_0$ be fixed number. For $f \in C^r\left(R_0^2\right)$ we define operators

$$\begin{aligned} L_{n;r}(f;x,y) &\equiv L_{n;r}(f;A,B;x,y) \\ &:= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{njk}(x,y)\, n^2 \iint_{D_{njk}} \sum_{s=0}^{r} \frac{d^s f(t,z)}{s!}\,dtdz, \end{aligned} \tag{33}$$

for $(x,y) \in R_0^2$ and $n \in N$, where

$$p_{njk}(x,y) := a_{nj}(x)\, b_{nk}(y), \tag{34}$$

$$D_{njk} := \left\{(t,z) \,:\, \frac{j}{n} \le t \le \frac{j+1}{n},\ \frac{k}{n} \le z \le \frac{k+1}{n}\right\} \tag{35}$$

and $d^s f$ is the s-th differential of f, i.e.

$$d^s f(t,z) = \sum_{i=0}^{s} \binom{s}{i} f^{(s)}_{x^{s-i}y^i}(t,z)(x-t)^{s-i}(y-z)^i, \tag{36}$$

$d^0 f(t,z) \equiv f(t,z)$. If $r = 0$, then

$$\begin{aligned} L_{n;0}(f;x,y) &\equiv L_{n;0}(f;A,B;x,y) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{njk}(x,y)\, n^2 \iint_{D_{njk}} f(t,z)\,dtdz, \end{aligned} \tag{37}$$

for $f \in C_{0,0}$, $(x,y) \in R_0^2$ and $n \in N$.

Obviously we can set $B \equiv A$ to (33) and (37).

From (33) and properties of $A, B \in \Omega$ we deduce that

$$L_{n;r}(1; A, B; x, y) \equiv \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{nj}(x)\, b_{nk}(y) = 1, \tag{38}$$

for all $(x, y) \in R_0^2$, $n \in N$ and $r \in N_0$.

Similarly to Section I we shall prove that $L_{n;r}(f; A, B)$ is a linear operator from the space $C^r(R_0^2)$ into $C_{r,r}$ and for $f \in C^r(R_0^2)$ there exist strong differences with the power $q > 0$:

$$\begin{aligned} &H_{n;0}^q(f; x, y) = H_{n;r}^q(f; A, B; x, y) := \\ &:= \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p_{njk}(x, y) \left| n^2 \iint_{D_{njk}} \sum_{s=0}^{r} \frac{d^s f(t, z)}{s!} dtdz - f(x, y) \right|^q \right)^{\frac{1}{q}} \end{aligned} \tag{39}$$

for $(x, y) \in R_0^2$ and $n \in N$. In particular for $f \in C_{0,0}$ we have

$$\begin{aligned} &H_{n;0}^q(f; x, y) = \\ &= \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p_{njk}(x, y) \left| n^2 \iint_{D_{njk}} f(t, z) dtdz - f(x, y) \right|^q \right)^{\frac{1}{q}}. \end{aligned} \tag{40}$$

4.3. From (33) - (37) and by (11) we deduce that if $f \in C_{0,0}$ and $f(x, y) = f_1(x)\, f_2(y)$ for $(x, y) \in R_0^2$, then

$$L_{n;0}(f; A, B; x, y) = L_{n;0}(f_1; A; x)\, L_{n;0}(f_2; B; y) \tag{41}$$

for $(x, y) \in R_0^2$ and $n \in N$.

Analogusly to (16) and (17) we have

$$|L_{n;r}(f; A, B; x, y) - f(x, y)| \le H_{n;r}^1(f; A, B; x, y), \tag{42}$$

$$H_{n;r}^p(f; A, B; x, y) \le H_{n;r}^q(f; A, B; x, y) \qquad \text{if } q > p > 0, \tag{43}$$

for $f \in C^r\left(R_0^2\right)$, $(x, y) \in R_0^2$, $n \in N$ and $r \in N_0$.

Applying (29), (41), Lemma 1 and Lemma 2, we immediately obtain the following

Lemma 8. *Let $A, B \in \Omega$ and $p_1, p_2, s_1, s_2 \in N_0$ and $q \in N$. Then there exists positive constant $M_{13} = M_{13}(p_1, p_2, s_1, s_2, q, A, B)$ such that*

$$w_{(p_1+s_1)q,(p_2+s_2)q}(x, y) L_{n;0}\left(\left(\frac{|t - x|^{s_1} |z - y|^{s_2}}{w_{p_1,p_2}(t, z)} \right)^q ; x, y \right) \le M_{13}\, n^{-\frac{(s_1+s_2)q}{2}}$$

for all $(x, y) \in R_0^2$ and $n \in N$.

Using the Hölder inequality, we obtain

Lemma 9. *Let h be a function continuous on D_{njk} defined by* (35). *Then for every $q \in N$ we have*

$$\left| n^2 \iint_{D_{njk}} h(t,z)dtdz \right|^q \le n^2 \iint_{D_{njk}} |h(t,z)|^q dtdz.$$

Lemma 10. *Suppose that $A, B \in \Omega$ and $r \in N_0$. Then there exists $M_{14} \equiv M_{14}(r,A,B) = const. > 0$ such that for every $f \in C^r\left(R_0^2\right)$ and $n \in N$ we have*

$$\|L_{n;0}(f;\cdot,\cdot)\|_{0,0} \le \|f\|_{0,0} \quad \text{if} \quad r = 0, \tag{44}$$

$$\|L_{n;r}(f;\cdot,\cdot)\|_{r,r} \le M_{14} \sum_{s=0}^{r} \sum_{i=0}^{s} \left\| f^{(s)}_{x^{s-i}y^i} \right\|_{r-s,r-s} \tag{45}$$

The formulas (33) *and* (37) *and inequalities* (44) *and* (45) *show that $L_{n;r}(f; A,B)$, $n \in N$, are linear operators from the space $C^r\left(R_0^2\right)$ into $C_{r,r}$.*

Proof. If $r = 0$, then by (37), (38), (29) and (30) we immediately obtain (44). If $r \in N$, then $f^{(s)}_{x^{s-i}y^i} \in C_{r-s,r-s}$ and by (30) and (33) - (37) we get

$$\begin{aligned} |L_{n;r}(f;x,y)| \le \sum_{s=0}^{r} \frac{1}{s!} \sum_{i=0}^{s} \binom{s}{i} \left\| f^{(s)}_{x^{s-i}y^i} \right\|_{r-s,r-s} \\ \times L_{n;0}\left(\frac{|t-x|^{s-i}|z-y|^i}{w_{r-s,r-s}(t,z)}; x, y \right). \end{aligned}$$

Further by (29), (1), (5) and Lemma 8 we get

$$\begin{aligned} & w_{r,r}(x,y)\ |L_{n;r}(f;x,y)| \\ & \qquad \le M_{15} \sum_{s=0}^{r} \sum_{i=0}^{s} \left\| f^{(s)}_{x^{s-i}y^i} \right\|_{r-s,r-s} \frac{w_{r,r}(x,y)}{w_{r-i,r-s+i}(x,y)}\, n^{-\frac{s}{2}} \\ & \qquad \le 2M_{15} \sum_{s=0}^{r} \sum_{i=0}^{s} \left\| f^{(s)}_{x^{s-i}y^i} \right\|_{r-s,r-s}, \end{aligned}$$

for all $(x,y) \in R_0^2$ and $n \in N$, where $M_{15} \equiv M_{15}(r,A,B) = const. > 0$. From the above and (30) follows (45). ■

Similarly we can prove the following

Lemma 11. *Let $A, B \in \Omega$, $r \in N_0$ and $q > 0$. Then for every $f \in C^r\left(R_0^2\right)$ and $n \in N$ we have $H^q_{n;r}(f; A, B) \in C_{r,r}$.*

5. Theorems and corollaries

5.1. First we shall prove analogue of Theorem 1 for $r = 0$.

Theorem 2. *Suppose that $A, B \in \Omega$ and $q > 0$. Then there exists $M_{16} \equiv M_{16}(q, A, B) = const. > 0$ such that for every $f \in C_{0,0}$ we have*

$$\left\| H^q_{n;0}(f; A, B) \right\|_{1,1} \le M_{16}\, \omega\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right), \quad n \in N. \tag{46}$$

Proof. a) Let $q \in N$. By Lemma 11 we have $H^q_{n;0}(f; A, B) \in C_{0,0}$ for every $f \in C_{0,0}$ and $n \in N$. By (31), (32) and (35) we have

$$\begin{aligned} \left| n^2 \iint_{D_{njk}} f(t,z)dtdz - f(x,y) \right| &\le n^2 \iint_{D_{njk}} |f(t,z) - f(x,y)| dtdz \\ &\le n^2 \iint_{D_{njk}} \omega\left(f; |t-x|, |z-y|\right) dtdz \\ &\le n^2\, \omega\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \iint_{D_{njk}} \left(\sqrt{n}\,|t-x| + \sqrt{n}\,|z-y| + 2\right) dtdz. \end{aligned}$$

Using this inequality to (40) and by the Minkowski inequality, Lemma 9 and (38), (41), (11) and (12) we get

$$\begin{aligned} &H^q_{n;0}(f; A, B; , x, y) \le \omega\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \\ &\times \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p_{njk}(x,y) \left| n^2 \iint_{D_{njk}} \left(\sqrt{n}\,|t-x| + \sqrt{n}\,|z-y| + 2\right) dtdz \right|^q \right)^{\frac{1}{q}} \\ &\le \omega\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \Big\{ n^{\frac{1}{2}} \left(L_{n;0}\left(|t-x|^q; A, B; x, y\right)\right)^{\frac{1}{q}} \\ &\qquad\qquad + n^{\frac{1}{2}} \left(L_{n;0}\left(|z-y|^q; A, B; x, y\right)\right)^{\frac{1}{q}} + 2 \Big\} \\ &= \omega\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \Big\{ n^{\frac{1}{2}} \left(L_{n;0}\left(|t-x|^q; A; x\right)\right)^{\frac{1}{q}} \\ &\qquad\qquad + n^{\frac{1}{2}} \left(L_{n;0}\left(|z-y|^q; B; y\right)\right)^{\frac{1}{q}} + 2 \Big\}, \end{aligned}$$

which by (29), (5) and Lemma 1 implies that

$$w_{1,1}(x,y) H^q_{n;0}(f; A, B; x, y) \le M_{16}(q, A, B)\, \omega\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)$$

for $(x,y)\in R_0^2$ and $n\in N$. From this and (30) follows (46) for $q\in N$.

b) If $0<q\notin N$, then arguing as in the proof of Theorem 1 and applying (43) and (46) for the power $[q]+1$, we obtain (46) for $0<q\notin N$ and we complete the proof. ■

Theorem 3. *Suppose that $A,B\in\Omega$, $r\in N$ and $q>0$. Then there exists $M_{17}\equiv M_{17}(q,r,A,B)=const.>0$ such that for every $f\in C^r\left(R_0^2\right)$ and $n\in N$ we have*

$$\left\|H_{n;r}^q(f;A,B)\right\|_{r+1,r+1}\le M_{17}\, n^{-\frac{r}{2}}\sum_{j=0}^{r}\omega\left(f_{x^{r-i}y^i}^{(r)};\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right). \tag{47}$$

Proof. First we consider $q\in N$. For $f\in C^r\left(R_0^2\right)$ we use the following modified Taylor formula at a point $(t,z)\in R_0^2$:

$$\begin{aligned} f(x,y)&=\sum_{s=0}^{r}\frac{d^sf(t,z)}{s!}\\ &\quad+\frac{1}{(r-1)!}\int_0^1(1-u)^{r-1}\left(d^rf(\tilde{x},\tilde{y})-d^rf(t,z)\right)du, \end{aligned}\tag{48}$$

$(x,y)\in R_0^2$, where $(\tilde{x},\tilde{y}):=(t+u(x-t),z+u(y-z))$ and $d^rf(\tilde{x},\tilde{y})$ and $d^rf(t,z)$ are the r-th differentials of f with $\Delta x=x-t$ and $\Delta y=y-z$ ([2]). By (48) and (35) we can write

$$\begin{aligned} &\left|n^2\iint_{D_{njk}}\sum_{s=0}^{r}\frac{d^sf(t,z)}{s!}dtdz-f(x,y)\right|\\ &\quad\le n^2\iint_{D_{njk}}\left|\sum_{s=0}^{r}\frac{d^sf(t,z)}{s!}dtdz-f(x,y)\right|dtdz\\ &\quad\le\frac{n^2}{(r-1)!}\iint_{D_{njk}}\left(\int_0^1(1-u)^{r-1}\left|d^rf(\tilde{x},\tilde{y})-d^rf(t,z)\right|du\right)dtdz. \end{aligned}\tag{49}$$

Next, by (36), (31) and (32), we have

$$\begin{aligned} &\left|d^rf(\tilde{x},\tilde{y})-d^rf(t,z)\right|\\ &\quad\le\sum_{i=0}^{r}\binom{r}{i}\left|f_{x^{r-i}y^i}^{(r)}(\tilde{x},\tilde{y})-f^{(r)}(t,z)\right||x-t|^{r-i}|y-z|^i\\ &\quad\le\sum_{i=0}^{r}\binom{r}{i}\omega\left(f_{x^{r-i}y^i}^{(r)};u|x-t|,|y-z|\right)|x-t|^{r-i}|y-z|^i\\ &\quad\le\sum_{i=0}^{r}\binom{r}{i}\omega\left(f_{x^{r-i}y^i}^{(r)};\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)\left(\sqrt{n}\,|t-x|+\sqrt{n}\,|z-y|+2\right)\\ &\qquad\times|t-x|^{r-i}|z-y|^i, \end{aligned}\tag{50}$$

for $0 \le u \le 1$, $(x,y) \in R_0^2$, $(t,z) \in D_{njk}$ and $n \in N$. Using (49) and (50) to (39) and by the Minkowski inequality for sum and by (37) we get

$$(51)\quad H^q_{n;r}(f;A,B;x,y) \le \frac{1}{r!}\sum_{i=0}^{r}\binom{r}{i}\omega\left(f^{(r)}_{x^{r-i}y^i};\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)$$
$$\times\left\{\sqrt{n}\left(L_{n;0}\left(|t-x|^{(r-i+1)q}|z-y|^{iq};A,B;x,y\right)\right)^{\frac{1}{q}}\right.$$
$$+\sqrt{n}\left(L_{n;0}\left(|t-x|^{(r-i)q}|z-y|^{(i+1)q};A,B;x,y\right)\right)^{\frac{1}{q}}$$
$$\left.+2\left(L_{n;0}\left(|t-x|^{(r-i)q}|z-y|^{iq};A,B;x,y\right)\right)^{\frac{1}{q}}\right\}$$
$$:=\frac{1}{r!}\sum_{i=0}^{r}\binom{r}{i}\omega\left(f^{(r)}_{x^{r-i}y^i};\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)\sum_{p=1}^{3}Z_{n,p}(x,y)$$

for $(x,y) \in R_0^2$ and $n \in N$. Applying (5) and Lemma 8 with $s_1 = s_2 = 0$, we can write

$$(52)\quad w_{r+1,r+1}(x,y)Z_{n,1}(x,y)$$
$$\le \frac{w_{r+1}(x)}{w_{r-i+1}(x)}\frac{w_{r+1}(y)}{w_i(y)}n^{\frac{1}{2}}\left(M_{13}n^{-\frac{(r+1)q}{2}}\right)^{\frac{1}{q}} \le 4M_{13}^{\frac{1}{2}}n^{-\frac{r}{2}},$$

for $(x,y) \in R_0^2$, $n \in N$ and $0 \le i \le r$. Analogously we obtain

$$(53)\quad w_{r+1,r+1}(x,y)Z_{n,p}(x,y) \le M_{18}\,n^{-\frac{r}{2}}, \qquad p = 2,3,$$

for $(x,y) \in R_0^2$, $n \in N$ and $0 \le i \le r$, where $M_{18} \equiv M_{18}(i,q,r,A,B) = const. > 0$.

From (51) - (53) and (30) we immediately obtain the desired inequality (47) for $q \in N$.

If $q \notin N$, then reasoning as in the proof of Theorem 1 and applying (43) and (47) for $[q]+1$, we obtain (47) for $0 < q \notin N$.

Now the proof is completed. ∎

5.2. Theorem 2, Theorem 3 and (42) imply the following analogues of Corollaries 1-3.

Corollary 4. *Suppose that $A, B \in \Omega$, $r \in N_0$ and $q > 0$. Then for every $f \in C^r\left(R_0^2\right)$ having partial derivatives $f^{(r)}_{x^{r-i}y^i}$, $0 \le i \le r$, uniformly continuous on R_0^2 we have*

$$\lim_{n\to\infty} n^{\frac{r}{2}}\left\|H^q_{n;r}(f;A,B)\right\|_{r+1,r+1} = 0.$$

Consequently,

$$\lim_{n\to\infty} n^{\frac{r}{2}} H^q_{n;r}(f; A, B; x, y) = 0.$$

at every $(x, y) \in R_0^2$.

Corollary 5. *Let* A, B, r *and* f *satisfy assumptions of Corollary 4. Then*

$$\lim_{n\to\infty} n^{\frac{r}{2}} \|L_{n;r}(f; A, B) - f\|_{r+1,r+1} = 0,$$

which implies that

$$\lim_{n\to\infty} n^{\frac{r}{2}} (L_{n;r}(f; A, B; x, y) - f(x, y)) = 0$$

at every $(x, y) \in R_0^2$.

Corollary 6. *The order of strong approximation of* $f \in C^r\left(R_0^2\right)$ *by operators* $L_{n;r}(f; A, B)$ *is independent on* $q > 0$ *and is dependent on* $r \in N_0$. *This order of strong approximation improves if* r *increases.*

Remarks. 1. Analogously to Section I (3.3) can be defined define analogues of Szász-Mirakyan and Baskakov operators of functions $f \in C^r\left(R_0^2\right)$. We can easily verify that Theorem 2, Theorem 3 and Corollaries 4-6 concern also these operators.

2. By inequalilies (16) and (42) we deduce that results given in above theorems on strong approximation are more general than suitable results on normal approximation.

References

[1] M. Becker, Global approximation theorems for Szász-Mirakyan operators in exponential weight spaces, *Indiana Univ. Math. J.,* (1978), 127-142.

[2] G.M. Fichtenholz, *Calculus,* Vol. 1, Warsaw, 1964.

[3] Z. Finta, Modified Kantorovich polynomials, *Math. Balkanica,* 13(3-4) (1999), 205-211.

[4] G.H. Kirov, A generalization of the Bernstein polynomials, *Math. Balkanica,* 6(2)(1992), 147-153.

[5] L. Leindler, *Strong Approximation by Fourier Series,* Akad. Kiado, Budapest 1985.

[6] L. Rempulska, Z. Walczak, On modified Baskakov operators, *Procedings of A. Razmadze Math. Inst.,* 133(2003), 109-117.

[7] L. Rempulska, M. Skorupka, On strong approximation of functions by certain operators, *Math. J. of Okayama Univ.,* (in print).

[8] R. Taberski, Strong summability of double Fourier series, *Bull. Acad. Polon. Sci., Ser. Sci. Math.,* 11(1969), 719-726.

[9] A.F. Timan, *Theory of Approximation of Functions of Real Variable,* New York, 1963.

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