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A STUDY OF BORNOLOGICAL PROPERTIES OF THE SPACE OF ENTIRE FUNCTIONS REPRESENTED BY MULTIPLE DIRICHLET SERIES

Abstract: The space of entire functions represented by Dirichlet series of several complex variables has been studied by S. Dauod [1]. M.D. Patwardhan [6] studied the bornological properties of the space of entire functions represented by power series. In this work we study the bornological aspect of the space Γ of entire functions represented by Dirichlet series of several complex variables. By $\overline{\Gamma}$ we denote the space of all analytic functions $\alpha(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$, having finite abscissa of convergence. We introduce bornologies onΓ and $\overline{\Gamma}$, and prove that $\overline{\Gamma}$is a convex bornological vector space which is the completion of the convex bornological vector space Γ.

Key words: convex bornological vector space, Dirichlet series.

1. Introduction

Let C be the ordinary complex plane equipped with its usual topology and Γ be the space of entire functions represented by Dirichlet series of two complex variables (s_1, s_2). (We consider the case of two variables for the sake of simplicity, though our results can be easily extended to any finite number of variables). Let

$$\alpha(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \tag{1.1}$$

where $a_{m,n} \in C$, $s_1, s_2 \in C^2$, $s_\omega = \sigma_\omega + it_\omega$, $\omega = 1, 2$, $0 < \lambda_1 < ... < \lambda_m \to \infty$ with m, $0 < \mu_1 < ... < \mu_n \to \infty$ with n and further (see [1])

$$\limsup_{m+n\to\infty} \frac{m+n}{\lambda_m + \mu_n} = D < +\infty. \tag{1.2}$$

Such a series is called a Dirichlet series of two complex variables. If $\lambda_m = \log m, \mu_n = \log n$, then $\alpha(s_1, s_2)$is a power series in (s_1, s_2). F.I. Geche [2] considered the general case when $(\lambda_m)_{m\geq 1}, (\mu_n)_{n\geq 1}$ are two increasing sequences of real numbers whose limits are infinity, he determined the nature of the region of convergence of the series (1.1) and the formula for the system of associated abscissas of convergence which depends substantially on whether $A < \infty$ or $A = \infty$, where$A = \sum_{i,j} | a_{i,j} |$. For example, V.P. Gromov [3] proved that if $\limsup_{m+n\to\infty} \frac{m+n}{\lambda_m+\mu_n} = 0$, then the formulae for the system of associated abscissa of convergence depend on $| a_{m,n} |$.

It is well known that the function $\alpha(s_1, s_2)$ is an analytic function in the hyperplane $\sigma_\omega < \tau(-\infty < \tau < \infty)$ where

$$\tau = \limsup_{m,n\to\infty} \frac{\log | a_{mn} |^{-1}}{\lambda_m + \mu_n}. \tag{1.3}$$

For each entire function $\alpha \,\varepsilon\, \Gamma$ we associate a real number $\| \alpha \|$ defined by

$$\| \alpha \| = \sup\{| a_{m,n} |^{\frac{1}{(\lambda_m+\mu_n)}}, \quad m, n \geq 1\},$$

satisfying for all $\alpha, \beta \,\varepsilon\, \Gamma$
(a) $\| 0 \| = 0$;
(b) $\| \alpha \| \geq 0$;
(c) $\| \alpha \| = 0 \iff \alpha \equiv 0$;
(d) $\| -\alpha \| = \| \alpha \|$;
(e) $\| \alpha + \beta \| \leq \| \alpha \| + \| \beta \|$.
Thus$\| \alpha \|$ is a total paranorm on Γ.

2. Definitions

In this section we give some definitions. We have

2.1
A bornology on a set X is a family **B** of subsets of X satisfying the following axioms:
(i) **B** is a covering of X , i.e. $X = \bigcup_{B \,\varepsilon\, \mathbf{B}} \mathbf{B}$;
(ii) **B** is hereditary under inclusion, i.e. if $A \in \mathbf{B}$ and B is a subset of X contained in A, then $B \in \mathbf{B}$;
(iii) **B** is stable under finite union.

A pair $(X, \mathbf{B})$ consisting of a set X and a bornology **B** on X is called a bornological space, and the elements of **B** are called the bounded subsets of X .

2.2
A base of a bornology $\mathbf{B}$ on X is any subfamily $\mathbf{B}_0$ of $\mathbf{B}$ such that every element of $\mathbf{B}$ is contained in an element of $\mathbf{B}_0$. A family $\mathbf{B}_0$ of subsets of X is a base for a bornology on X if and only if $\mathbf{B}_0$ covers X and every finite union of elements of $\mathbf{B}_0$ is contained in a member of $\mathbf{B}_0$. The collection of those subsets of X which are contained in an element of $\mathbf{B}_0$ defines a bornology $\mathbf{B}$ on X having $\mathbf{B}_0$ as a base. A bornology is said to be a bornology with a countable base if it possesses a base consisting of a sequence of bounded sets. Such a sequence can always be assumed to be increasing. For further definitions and notations, we shall refer to [4] and [5].

2.3
Let E be a vector space over the complex field C. A bornology $\mathbf{B}$ on E is said to be a vector bornology on E, if $\mathbf{B}$ is stable under vector addition, homothetic transformations and the formation of circled hulls or, in other words, if the sets A+B, λA, $\bigcup_{|\eta|\leq 1} \eta A$ belong to $\mathbf{B}$, whenever A and B belong to $\mathbf{B}$ and $\lambda \in C$. Any pair $(E, \mathbf{B})$ consisting of vector space E and a vector bornology $\mathbf{B}$ on E is called a bornological vector space.

2.4
A vector bornology on a vector space E is called a convex vector bornology if it is stable under the formation of convex hulls. Such a bornology is also stable under the formation of disked hulls, since the convex hull of a circled set is circled.A bornological vector space $(E, \mathbf{B})$ whose bornology $\mathbf{B}$ is convex is called a convex bornological vector space.

2.5
A separated bornological vector space $(E, \mathbf{B})$ (or a separated bornology $\mathbf{B}$) is the one in which $\{0\}$ is the only bounded vector subspace of E.

2.6
A set P is said to be bornivorous if for every bounded set B there exists a $t \in C$, $t \neq 0$ such that $\mu B \subset P$ for all $\mu \in C$ for which $\mid \mu \mid \leq \mid t \mid$.

2.7
Let E be a bornological vector space. A sequence $\{x_n\}$ in E is said to be Mackey-convergent to a point $x \in E$ if there exists a decreasing sequence $\{t_n\}$ of positive real number tending to zero such that the sequence $\{\frac{x_n - x}{t_n}\}$ is bounded.

2.8
Let E be a vector space and let B be a disk in E not necessarily absorbent in E. We denote by E_B the vector space spanned by B, i.e. the space $\bigcup_{\lambda>0} \lambda B = \bigcup_{\lambda \in K} \lambda B$, where K be the real field $\mathbb{R}$ or the complex field C.

2.9
Let E be a separated convex bornological space . A sequence$\{x_n\}$ in E is said to be a bornologcal Cauchy sequence (or a Mackey-Cauchy sequence) in E if there exists a bounded disk $B \subset E$ such that $\{x_n\}$ is a Cauchy sequence in E_B.

3. The Space Γ

Let C denote the complex plane, and I be the set of positive integers. We write for $n \in I$,

$$C^n = \{(z_1, z_2, ..., z_n); \quad z_i \in C, \quad 1 \leq i \leq n\}.$$

We are concerned here with the space of entire functions from C^n to C under the usual pointwise addition and scalar multiplication. We shall denote by Γ the space of all entire functions $\alpha : C^2 \to C$, where
$\alpha(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$.

We define a bornology on Γ with the help of || . || defined in section 1. We denote by B_k the set $\{\alpha \,\varepsilon\, \Gamma : \| \alpha \| \leq k\}$. Then the family $\mathbf{B}_0 = \{B_k : k = 1, 2, ...\}$ forms a base for a bornology $\mathbf{B}$ on Γ. We now prove

Theorem 3.1. $(\Gamma, \boldsymbol{B})$ *is a separated convex bornological vector space with a countable base.*

Proof. Since the vector bornology **B** on the vector space Γ is stable under the formation of the convex hulls, it is a convex vector bornology. Since the convex hull of a circled set is circled, **B** is stable under the formation of disked hulls, and hence the bornological vector space $(\Gamma, \mathbf{B})$ is a convex bornological vector space. Now to show that$\{0\}$ is the only bounded vector subspaces of Γ, we must show that contains no bounded open set. Let $U(\varepsilon)$ denote the set of all $\alpha \,\varepsilon\, \Gamma$ such that $\| \alpha \| < \varepsilon$. To prove the result stated, it is enough to show that no $U(\varepsilon)$ is bounded. That is, given $U(\varepsilon)$, we have to show that there exists $U(\eta)$ for which there is no $c > 0$ such that $U(\varepsilon) \subset cU(\eta)$. For this purpose, take $\eta = \frac{\varepsilon}{4}$. Given c, we can find sufficiently large positive numbers λ_m and μ_n such that $\mid c \mid^{\frac{1}{(\lambda_m + \mu_n)}} < 2$. Let $\alpha = (\frac{\varepsilon}{2})^{(\lambda_m + \mu_n)} \exp(\lambda_m s_1 + \mu_n s_2)$. Then $\| \alpha \| = \frac{\varepsilon}{2}$; so $\alpha \,\varepsilon\, U(\varepsilon)$ and $\| \frac{\alpha}{c} \| = \sup_{m,n \geq 1} (\frac{\varepsilon}{2} \mid c \mid^{\frac{-1}{(\lambda_m + \mu_n)}}) > \frac{\varepsilon}{4} = \eta$, so that $c^{-1}\alpha$ does not belong to $U(\eta)$ i.e. α does not belong to $cU(\eta)$. This shows that $U(\varepsilon)$ is not bounded. Thus $\{0\}$ is the only bounded vector subspace of Γ, and hence $(\Gamma, \mathbf{B})$ is separated. Since **B** possesses a base consisting of increasing sequence of bounded sets, **B** is a bornology with a countable base. Thus $(\Gamma, \mathbf{B})$ is a separated convex bornological vector space with countable base. This proves Theorem 3.1. ■

Theorem 3.2. **B** *contains no bornivorous set.*

Proof. Suppose **B** contains a bornivorous set A. Then there exists a set $B_i \in \mathbf{B}$ such that $A \subset B_i$ and consequently B_i is also bornivorous. We now assert that if $i_1 > i$, then $tB_{i_1} \not\subseteq B_i$ for any $t \in C$ which leads to a contradiction. If $i_1 > i$, it is easy to see that $tB_{i_1} \not\subseteq B_i$ for any $t \in C$ such that $\mid t \mid \geq 1$. Now we prove that $tB_{i_1} \not\subseteq B_i$ for any $t \in C$ such that $\mid t \mid < 1$ also. Let $\mid t \mid < 1$. Since $i_1/i > 1$, we can choose m,n such that $1 < 1/\mid t \mid < (i_1/i)^{\lambda_m+\mu_n}$. Now let $a_{mn} \in C$ be such that $i^{\lambda_m+\mu_n}/\mid t \mid < \mid a_{mn} \mid \leq {i_1}^{\lambda_m+\mu_n}$ and let $\alpha = a_{m,n}\exp(\lambda_m s_1 + \mu_n s_2)$. Then $\parallel \alpha \parallel = \mid a_{mn} \mid^{1/(\lambda_m+\mu_n)} \leq i_1$ and hence $\alpha \in B_{i_1}$. Now $\parallel t\alpha \parallel = \parallel ta_{m,n}\exp(\lambda_m s_1 + \mu_n s_2) \parallel = \mid ta_{mn} \mid^{1/(\lambda_m+\mu_n)} > i$ and hence $t\alpha$ does not belong to B_i. Thus $tB_{i_1} \not\subseteq B_i$ for any $t \in C$. This proves Theorem 3.2. ■

4. The bornological dual of Γ

In this section we consider the properties of linear functionals defined on the space Γ We prove:

Theorem 4.1. *Every continuous linear functional f defined on Γ is of the form $f(\alpha) = \sum_{m,n=1}^{\infty} c_{mn}a_{mn}$, where $\alpha = a_{m,n}\exp(\lambda_m s_1 + \mu_n s_2)$, and $\{\mid c_{mn} \mid^{1/(\lambda_m+\mu_n)}\}$ is a bounded sequence.*
To prove the above result, we require the following

Lemma 4.1. *A necessary and sufficient condition that $f(\alpha) = \sum_{m,n=1}^{\infty} c_{mn}a_{mn}$ should be convergent for every sequence $\{a_{mn}\}$ satisfying*

$$\mid a_{mn} \mid^{1/(\lambda_m+\mu_n)} \to 0 \quad as \quad m,n \to \infty \tag{4.1}$$

is that $\{\mid c_{mn} \mid^{1/(\lambda_m+\mu_n)},\ m,n \geq 1\}$ should be bounded.

Proof. Suppose that $\{\mid c_{mn} \mid^{1/(\lambda_m+\mu_n)},\ m,n \geq 1\}$ is bounded. Then we can find $M > 0$ such that $\mid c_{mn} \mid^{1/(\lambda_m+\mu_n)} \leq M$ for $m,n \geq 1$. By (4.1), we can find m_0, n_0 such that

$$\mid a_{mn} \mid^{1/(\lambda_m+\mu_n)} \leq 1/2M \quad \text{for} \quad m \geq m_0\,,\ \ n \geq n_0.$$

Hence

$$\mid a_{mn}c_{mn} \mid \leq 1/2^{(\lambda_m+\mu_n)} \quad \text{for} \quad m \geq m_0\,,\ \ n \geq n_0.$$

Thus we have

$$
\begin{aligned}
\left|\sum_{m_0+1}^{\infty}\sum_{n_0+1}^{\infty} a_{mn}c_{mn}\right| &\le \sum_{m_0+1}^{\infty}\sum_{n_0+1}^{\infty} \mid a_{mn}c_{mn} \mid \\
&< \sum_{m_0+1}^{\infty}\sum_{n_0+1}^{\infty} 2^{-(\lambda_m+\mu_n)} \le \sum_{1}^{\infty}\sum_{1}^{\infty} 2^{-(\lambda_m+\mu_n)}.
\end{aligned}
$$

In view of (1.2) $-\lambda_m - \mu_n < \frac{-m-n}{D+\varepsilon}$, $m > m_0$, $n > n_0$
There fore

$$
\sum_{m,n=1}^{\infty} 2^{(-\lambda_m-\mu_n)} < \sum_{m,n=1}^{\infty} 2^{\frac{-m-n}{D+\varepsilon}} < \infty.
$$

Hence $\sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$ converges.

To prove the converse, suppose that the sequence $\{\mid c_{mn} \mid^{1/(\lambda_m+\mu_n)}\}$ is unbounded and let $p, q > 0$ be sufficiently large. Then we can find an increasing sequence of integers $\{m_p\}$ and $\{n_q\}$ such that

$$
\mid C_{m_p n_q} \mid \ge (p+q)^{(\lambda_{m_p}+\mu_{n_q})}, \quad p, q = 1, 2, ...
$$

Take

$$
a_{mn} = \begin{cases} 0, & \text{if } m \ne m_p \ \textit{and } n \ne n_q \\ 1/(p+q)^{(\lambda_m+\mu_n)}. & \text{if } m = m_p \ \textit{and } n = n_q, \quad p, q = 1, 2, ... \end{cases}
$$

Then

$$
\mid a_{mn} \mid^{1/(\lambda_m+\mu_n)} = \begin{cases} 0, & \text{if } m \ne m_p \ \textit{and } n \ne n_q \\ 1/(p+q). & \text{if } m = m_p \ \textit{and } n = n_q, \quad p, q = 1, 2, ... \end{cases}
$$

so that $\sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$ diverges. This proves Lemma 4.1. ■

Proof. of Theorem 4.1. Let $\alpha = \sum\limits_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$ and f be a continuous linear functional on Γ. Let $\alpha_{m,n} = \exp(\lambda_m s_1 + \mu_n s_2)$ and $f(\alpha_{m,n}) = c_{mn}$. Then

$$
\begin{aligned}
f(\alpha) &= \lim_{m,n \to \infty} f\left(\sum_{1}^{m}\sum_{1}^{n} a_{ij}\alpha_{ij}\right) \\
&= \lim_{m,n \to \infty} \sum_{1}^{m}\sum_{1}^{n} c_{ij}a_{ij}
\end{aligned}
$$

Hence for every $\alpha \in \Gamma$, $\sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$ converges and $f(\alpha) = \sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$. Hence by Lemma 4.1 $\{| c_{mn} |^{1/(\lambda_m+\mu_n)}\}$ is bounded. Conversely, suppose that $\{c_{mn}\}$ is a sequence of complex numbers such that $\{| c_{mn} |^{1/(\lambda_m+\mu_n)}\}$ is bounded. For any entire function $\alpha = \sum\limits_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$, $| a_{mn} |^{1/(\lambda_m+\mu_n)} \rightarrow 0$ as $m, n \rightarrow \infty$ and hence by Lemma 4.1, the series $\sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$ is convergent. We define the functional $f : \Gamma \rightarrow C$ by $f(\alpha) = \sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$, $\alpha \in \Gamma$. Then f is obviously linear on Γ. We shall show that it is continuous. For this purpose it is enough to show that if for any sequence of entire functions $| \alpha_p | \rightarrow 0$, as $p \rightarrow \infty$, then $f(\alpha_p) \rightarrow 0$. Thus, let

$$\alpha_p = \sum_{m,n=1}^{\infty} a^p_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \quad p = 1, 2, ...$$

Since $\{| c_{mn} |^{1/(\lambda_m+\mu_n)}\}$ is bounded, we can find M such that $| c_{mn} | \leq M^{(\lambda_m+\mu_n)}$, $m, n \geq 1$. Given ε, choose η so that $0 < \eta < 1/M$ and $\lambda M(1 + \frac{1}{1-\lambda M}) < \varepsilon$. Since $| \alpha_p | \rightarrow 0$, we can find a positive integer p_0 such that$| \alpha_p | \leq \eta$ for $p \geq p_0$. Hence

$$| f(\alpha_p) | \leq \eta M + \sum_{1}^{\infty} (\lambda M)^{\lambda_n} = \eta M(1 + \frac{1}{1 - \eta M}) < \varepsilon \quad \text{for } p \geq p_0.$$

Thus $f(\alpha_p) \rightarrow 0$ as $p \rightarrow \infty$. This completes the proof of Theorem 4.1. ■

The following result was given by Hogbe-Nlend [4]:

Lemma 4.2. *A linear functional $f : \Gamma \rightarrow C$ is bounded if and only if f maps every Mackey-convergent sequence to a bounded sequence in C. For the proof of above, we refer to [4, p.10].*

Next we prove

Lemma 4.3. *A linear functional f on Γ defined by $f(\alpha) = \sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$. is bounded if and only if $\lim\limits_{m,n \rightarrow \infty} | c_{mn} |^{1/(\lambda_m+\mu_n)} = 0$.*

Proof. Suppose $| c_{mn} |^{1/(\lambda_m+\mu_n)} \rightarrow 0$. Let $\{\alpha_q\}$ be a sequence in Γ such that $\alpha_q \longrightarrow 0$. Then there exists a positive constant J and a decreasing sequence t_q of scalars converging to zero such that $\| \alpha_q/t_q \| \leq J$, i.e. $| a^q_{mn} | \leq | t_q | J^{(\lambda_m+\mu_n)}$ $m, n \geq 1$. Since $| c_{mn} |^{1/(\lambda_m+\mu_n)} \rightarrow 0$, there exists

N such that $| c_{mn} |^{1/(\lambda_m+\mu_n)} \leq \frac{1}{2J}$ for all m, n and $m+n > N$. Hence $| c_{mn} | \leq \frac{1}{(2J)^{(\lambda_m+\mu_n)}}, \quad m+n \ > N$. Now

$$| f(\alpha_q) | = | \sum_{m,n=1}^{\infty} c_{mn} a_{mn}^q | \leq \sum_{m,n=1}^{\infty} | c_{mn} || a_{mn}^q | \leq \sum_{m,n=1}^{\infty} | c_{mn} || t_q | J^{(\lambda_m+\mu_n)}$$

$$\leq \sum_{m+n\leq N} | c_{mn} || t_q | J^{(\lambda_m+\mu_n)} + \sum_{m+n>N} | c_{mn} || t_q | J^{(\lambda_m+\mu_n)}$$

$$< O(1) \ + \ | t_q | \sum_{m+n>N} 2^{-(\lambda_m+\mu_n)}$$
$$\leq O(1) \ + \ | t_q | \sum_{m+n\geq 0} 2^{-(\lambda_m+\mu_n)}$$
$$\leq O(1) \ + \ | t_q | A < \infty,$$

where $A(< \infty)$ is independent of q. Since $t_q \to 0$ as $q \to \infty$, we get $| f(\alpha_q) | < \infty$ for all q sufficiently large. Thus the sequence $\{f(\alpha_q)\}$ is bounded. Hence f is bounded on every sequence which Mackey-converges to zero and consequently by Lemma 4.2, f is bounded. ■

Conversely, let f be such that $\limsup\limits_{m+n\to\infty} | c_{mn} |^{1/(\lambda_m+\mu_n)} = \rho > 0$. Then given $\eta > 0$ such that $\eta < \rho$, there exist divergent increasing sequences $\{m_q\}$ and $\{n_t\}$ of positive integers such that $| c_{mn} | > \eta^{(\lambda_m+\mu_n)}$ for all $m = m_q$, $n = n_t$. Choose $\pi \in \mathbb{R}$ such that $\pi > 1$ and $\pi\eta > 1$. Consider the sequence $\{\alpha_{mn}\}$ where $\alpha_{mn} = \pi^{m+n}\exp(\lambda_m s_1 + \mu_n s_2) \in \Gamma$ and define $t_{mn} \in C$ as $t_{mn} = \frac{1}{\pi^{(\lambda_m+\mu_n)}}$. Then $t_{mn} \to 0$ as $m, n \to \infty$ and $\| \alpha_{mn}/t_{mn} \| = \| \pi^{2(\lambda_m+\mu_n)}\exp(\lambda_m s_1 + \mu_n s_2) \| = | \pi |^2 < \infty$. Consequently $\alpha_{mn} \to 0$. But $f(\alpha_{mn}) = c_{mn}\pi^{(\lambda_m+\mu_n)}$ and

$$| f(\alpha_{m_q,n_t}) | = | c_{m_q,n_t} || \pi |^{\lambda_{m_q}+\mu_{n_t}} > \eta^{\lambda_{m_q}+\mu_{n_t}} | \pi |^{\lambda_{m_q}+\mu_{n_t}}$$

which is not bounded. Hence again by Lemma 4.2, f is not bounded. This proves sufficiency part and the proof of Lemma 4.3 is complete.

5. (σ_1, σ_2) - norms on Γ

We define, for each $\sigma_1, \sigma_2 < \infty$ and $\alpha \in \Gamma$, the expression $\| \alpha : \sigma_1, \sigma_2 \|$ by the equation

$$\| \alpha : \sigma_1, \sigma_2 \| = \sum_{m,n=1}^{\infty} | a_{m,n} | \exp(\lambda_m\sigma_1 + \mu_n\sigma_2). \tag{1}$$

It is easily seen that, for each pair σ_1, σ_2 (5.1) defines a norm on the class of entire functions represented by multiple Dirichlet series. We shall denote by $\Gamma(\sigma_1, \sigma_2)$ the space Γ endowed with this norm. We denote by $\mathbf{B}_{\sigma_1,\sigma_2}$ the bornology on Γ consisting of the sets bounded in the sense of the norm $\| \alpha : \sigma_1, \sigma_2 \|$. We now prove

Theorem 5.1. $\mathbf{B} = \bigcup_{0<\sigma_1,\sigma_2<\infty} \mathbf{B}_{\sigma_1,\sigma_2}$

Proof. Let $B \in \mathbf{B}$. Then there exists a constant J such that $\| \alpha \| < J$ for all $\alpha \in B$. Let now $\alpha = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \in \mathbf{B}$.

Then $| a_{m,n} || \exp(\lambda_m s_1 + \mu_n s_2) | \le J^{(\lambda_m+\mu_n)} | \exp(\lambda_m s_1 + \mu_n s_2) |$, $m, n \ge 1$. Thus if $e^{\sigma_1} < 1/2J$, $e^{\sigma_2} < 1/2J$, we have,

$$\sum_{m,n=1}^{\infty} | a_{m,n} | \, | \exp(\lambda_m s_1 + \mu_n s_2) | \le \sum_{m,n=1}^{\infty} J^{(\lambda_m+\mu_n)} \, | e^{s_1} |^{\lambda_m} \, | e^{s_2} |^{\mu_n}$$
$$\le \sum_{m,n=1}^{\infty} 2^{-(\lambda_m+\mu_n)} < \infty.$$

Hence if $0 < e^{\sigma_1}, e^{\sigma_2} < 1/2J$, then $B \in \mathbf{B}_{\sigma_1,\sigma_2}$ and so $\mathbf{B} \subset \bigcup_{0<\sigma_1,\sigma_2<\infty} \mathbf{B}_{\sigma_1,\sigma_2}$.
For the reverse inclusion let $B \in \mathbf{B}_{\sigma_1,\sigma_2}$, then there exists a constant k such that for all

$$\alpha \in B, \quad \| \alpha : \sigma_1, \sigma_2 \| \le k,$$

i.e. $\sum_{m,n=1}^{\infty} | a_{m,n} | \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n) \le k, \quad \text{for all } m, n$

i.e.

$$| a_{m,n} |^{1/(\lambda_m+\mu_n)} \le k^{1/(\lambda_m+\mu_n)} e^{-\sigma_1 (\frac{\lambda_m}{(\lambda_m+\mu_n)})} e^{-\sigma_2 (\frac{\mu_n}{(\lambda_m+\mu_n)})},$$
$$\le k^{1/(\lambda_m+\mu_n)}, \quad \text{since } \sigma_1, \sigma_2 > 0.$$

As $\lambda_m, \mu_n \to \infty$ as $m, n \to \infty$, we get

$$\sup_{m,n} \{ | a_{m,n} |^{1/(\lambda_m+\mu_n)} \} \le k' < \infty.$$

Hence $\alpha \in \mathbf{B}$. Thus $B \in \mathbf{B}$ and hence $\bigcup_{0<\sigma_1,\sigma_2<\infty} \mathbf{B}_{\sigma_1,\sigma_2} \subset \mathbf{B}$. This completes the proof of Theorem 5.1. ■

Now we prove

Lemma 5.1. *For a given sequence* $\{\alpha_q\}$ *of entire functions, the following are equivalent in* $(\Gamma, \boldsymbol{B})$.

(i) $\alpha_q \rightarrow 0$.
(ii) There exists a sequence $\{t_q\}$ *of positive real numbers tending to zero such that* $\{\alpha_q/t_q\}$ *is bounded.*

Proof. $(i) \Rightarrow (ii)$ is obviously true. To prove $(ii) \Rightarrow (i)$, let $\{\alpha_q\}$ be a sequence in Γ for which there exists a sequence $\{t_q\}$ of positive real numbers tending to zero and a constant J such that $\| \alpha_q/t_q \| \leq J$ for all q. Now there exists a positive number M such that $t_q \leq M$ for all q. Further, we can choose for each $i = 1, 2, 3, ...$ a q_i such that $t_q < 1/i$ for all $q \geq q_i$. Let us define a sequence $\{t_q^{'}\}$ as

$$t_q^{'} = \begin{cases} M, & \text{for all } q < q_i \\ 1/i. & \text{for all } q_i \leq q < q_{i+1}, \quad i = 1, 2, ... \end{cases}$$

Then $\{t_q^{'}\}$ is a decreasing sequence of positive real numbers tending to zero and further $t_q^{'} \geq t_q$ for all q. Hence

$$\begin{aligned} \| \alpha_q/t_q^{'} \| &= \| \alpha_q t_q/t_q t_q^{'} \| \\ &< A(t_q/t_q^{'})\| \alpha_q/t_q \| \leq J, \quad \text{where } A(t) = \max(1, | t |),\ t \in C \end{aligned}$$

Therefore $\alpha_q \rightarrow 0$. Hence $(ii) \Rightarrow (i)$ and the proof of Lemma 5.1 is complete. ■

Let $\{\alpha_q\}$ be a sequence of entire function in Γ. Then we have:

Theorem 5.2. $\alpha_q \rightarrow 0$ *in* Γ *if and only if* $\alpha_q(s_1, s_2) \rightarrow 0$ *uniformly in some finite hyper plane.*

Proof. Suppose $\alpha_q \rightarrow 0$ and

$$\alpha_q = \sum_{m,n=1}^{\infty} a_{m,n}^q \exp(\lambda_m s_1 + \mu_n s_2), \quad q = 1, 2, 3....$$

Then there exists a constant k and a sequence $\{t_q\}$ in C, tending to zero such that

$$\| \alpha_q/t_q \| \leq k \quad \text{for all } q,$$

i.e.

$$| a_{m,n}^q/t_q | \leq k^{(\lambda_m+\mu_n)},\ m, n \geq 1.$$

If $s_\omega \in C,\ \omega = 1, 2$ such that $e^{\sigma_1}, e^{\sigma_2} < 1/2k$, then

$$| \alpha_q | = | \sum_{m,n=1}^{\infty} a_{m,n}^q \exp\left(\lambda_m s_1 + \mu_n s_2\right) | \leq \sum_{m,n=1}^{\infty} | a_{m,n}^q || e^{s_1} |^{\lambda_m} | e^{s_2} |^{\mu_n}$$

$$\leq \sum_{m,n=1}^{\infty} \mid t_q \mid k^{(\lambda_m+\mu_n)} \mid e^{s_1} \mid^{\lambda_m} \mid e^{s_2} \mid^{\mu_n} \leq \mid t_q \mid$$

Hence $\parallel \alpha_q(s) \parallel \to 0$ uniformly for all s_1, s_2 such that $e^{\sigma_1}, e^{\sigma_2} < 1/2k$. Conversely, suppose there exists $\sigma_0 < \tau$ such that $\alpha_q(s_1, s_2) \to 0$ uniformly for all $s_\omega = \sigma_\omega + it_\omega$, $\omega = 1, 2$ such that $\sigma_1, \sigma_2 < \sigma_0$. Then

$$\sup \mid \alpha_q \mid \ \to \quad 0 \quad \text{as}\, q \ \to \ \infty.$$

Now $\mid \alpha_q(s_1, s_2) \mid \ \leq \ \sup \mid \alpha_q(s_1, s_2) \mid$ for all s_1, s_2 such that $\sigma_1, \sigma_2 < \sigma_0$. Hence $\mid a^q_{m,n} \mid \exp(\lambda_m\sigma_1 + \mu_n\sigma_2) \leq \sup \mid \alpha_q(s_1, s_2) \mid$ i.e.

$$\left[\frac{\mid a^q_{m,n} \mid}{\sup \mid \alpha_q(s_1, s_2) \mid}\right] \leq \exp(\lambda_m\sigma_1 + \mu_n\sigma_2).$$

Let $t_q = \sup \mid \alpha_q(s_1 + s_2) \mid$. Then

$$\begin{aligned} \parallel \alpha_q/t_q \parallel &= \sup_{m,n\geq 1} \left\{ \frac{\mid a^q_{m,n} \mid^{1/(\lambda_m+\mu_n)}}{\mid \lambda_q \mid} \right\} \\ &\leq max\{1, \exp{(\sigma_1 + \sigma_2)}\} = A(\exp{(\sigma_1 + \sigma_2)}), \end{aligned}$$

and hence in view of Lemma 5.1, $\alpha_q \to 0$. This proves Theorem 5.2. ■

We now obtain some properties of linear functionals on Γ. We prove

Lemma 5.2. *In the topological dual Γ^o of Γ, every functional is of the form $f(\alpha) = \sum\limits_{m,n=1}^{\infty} c_{mn}a_{mn}$, $\alpha = \sum\limits_{m,n=1}^{\infty} a_{m,n}e^{(\lambda_m s_1+\mu_n s_2)}$, if and only if the sequence $\{\mid c_{mn} \mid e^{-(\lambda_m\sigma_1+\mu_n\sigma_2)}\}$ is bounded.*

Proof. Suppose that $f(\alpha)$ is continuous linear functional on Γ. Then there exists $k > 0$ such that $\mid f(\alpha) \mid \leq k \parallel \alpha : \sigma_1, \sigma_2 \parallel$ for every α. Let $\delta_{mn} = \exp{(\lambda_m s_1 + \mu_n s_2)}$ and $f(\delta_{mn}) = c_{mn}(m, n \geq 1)$. In Γ,

$$\alpha = \sum_{m,n=1}^{\infty} a_{m,n}\exp{(\lambda_m s_1 + \mu_n s_2)} = \lim_{m,n\ \to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\delta_{ij}.$$

Since f is continuous, we have

$$f(\alpha) = f(\lim_{m,n\ \to\ \infty} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} \exp{(\lambda_m s_1 + \mu_n s_2)}) = \lim_{m,n\ \to\ \infty} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} f(\delta_{ij})$$

$$= \lim_{m,n\ \to\ \infty} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}c_{ij} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}c_{mn}$$

Also $| c_{mn} | \leq k \parallel \delta_{mn} : \sigma_1, \sigma_2 \parallel = ke^{(\lambda_m \sigma_1 + \mu_n \sigma_2)}$. Hence $\{| c_{mn} | e^{-(\lambda_m \sigma_1 + \mu_n \sigma_2)}\}$ is bounded.

Conversely let $\{| c_{mn} | e^{-(\lambda_m \sigma_1 + \mu_n \sigma_2)}\}$ be bounded for all $m, n = 1, 2, ...$ Let f be defined by

$$f(\alpha) = \sum_{m,n=1}^{\infty} c_{mn} a_{mn}, \alpha = \sum_{m,n=1}^{\infty} a_{mn} \exp\left(\lambda_m s_1 + \mu_n s_2\right).$$

Then f is a linear functional and $| f(\alpha) | \leq \sum_{m,n=1}^{\infty} | c_{mn} || a_{mn} | < \sum_{m,n=1}^{\infty} k \times \exp(\lambda_m \sigma_1 + \mu_n \sigma_2) | a_{mn} | = k \parallel \alpha : \sigma_1, \sigma_2 \parallel \forall \alpha$ for some $k > 0$.

Hence $f(\alpha)$ is continuous on Γ. This proves Lemma 5.2. ■

Theorem 5.3. *The bornologocal dual* Γ^* *of* Γ *is the same as its topological dual* Γ^o.

Proof. The proof follows immediately from the fact that a linear functional on a normed linear space is continuous if and only if it is bounded. On Γ^o we now define a map:

$$\parallel . : \frac{1}{2\sigma_1 \sigma_2} \parallel : \Gamma^o \to \mathbb{R},$$

$$\alpha = \sum_{m,n=1}^{\infty} a_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \to \parallel \alpha : \frac{1}{2\sigma_1 \sigma_2} \parallel$$

$$= \sum_{m,n=1}^{\infty} | a_{mn} | (2\, e_1^{\sigma})^{-\lambda_m} (2\, e^{\sigma_2})^{-\mu_n}.$$

By Lemma 5.2 it follows that $\alpha = \sum_{m,n=1}^{\infty} a_{mn} e^{(\lambda_m s_1 + \mu_n s_2)} \in \Gamma^o$ if and only if $\{| a_{mn} | e^{-(\lambda_m \sigma_1 + \mu_n \sigma_2)}\}$ is bounded. Consequently the function $\parallel . : \frac{1}{2\sigma_1 \sigma_2} \parallel$ is well defined and Γ^o becomes a normed linear space relative to $\parallel . : \frac{1}{2\sigma_1 \sigma_2} \parallel$. Denote by $\overline{\mathbf{B}}_{\sigma_1, \sigma_2}$ the canonical bornology of Γ^o with this norm which we call the $(\frac{1}{2\sigma_1 \sigma_2})$-norm. ■

6. The space $\overline{\Gamma}$

In this section we consider the set

$$\overline{\Gamma} = \{\beta = \sum_{m,n=1}^{\infty} b_{mn} \exp(\lambda_m s_1 + \mu_n s_2) : b_{mn} \in C$$

$$\text{and } \{| b_{mn} |^{1/(\lambda_m + \mu_n)}\} \text{ is bounded.}\}$$

A convex bornology $\overline{B}$ can be defined on $\overline{\Gamma}$ with the help of a function $\| \,.\, \| : \overline{\Gamma} \rightarrow \mathbb{R}$ defined in a similar fashion to that on Γ. We note that $\overline{B}$ when restricted to Γ gives B. Moreover,$\Gamma = \bigcup_{\sigma_1,\sigma_2<\infty} \Gamma(\sigma_1,\sigma_2)$, and as in the proof of Theorem 5.1, we have.

$$\overline{\mathbf{B}} = \bigcup_{\sigma_1,\sigma_2<\tau} \overline{\mathbf{B}}_{\sigma_1,\sigma_2}$$

We now prove

Theorem 6.1. $(\overline{\Gamma}, \overline{\boldsymbol{B}})$ *is Mackey-complete.*

Proof. We first observe that Lemma 5.1 holds for $\overline{\Gamma}$ also. Let thus $\{\alpha_k\}$ be a Mackey-Cauchy sequence in $\overline{\Gamma}$. Then there exists a sequence $\{z_{kp}\}$ of positive real numbers tending to zero, such that $\| \frac{\alpha_k-\alpha_p}{z_{kp}} \| \leq \omega$, where ω is some fixed positive number. Now we choose a sequence $\{t_{kp}\}$of positive numbers such that $t_{kp} \geq z_{kp}$ for all k,p and further such that $t_{k_1p_1} < t_{k_2p_2}$ whenever $k_1 \geq k_2$ and $p_1 \geq p_2$. For this, since $z_{kp} \rightarrow 0$, without loss of generality we can assume that $z_{kp} < 1$ for all k,p. Now we set $k_1 = 1, p_1 = 1$ and choose (k_i, p_i) inductively such that $k_i > k_{i-1}$, $p_i > p_{i-1}$ and $z_{kp} < 1/i$ for $k \geq k_i$, $p \geq p_i$. Define $\{t_{kp}\}$ as

$$t_{kp} = \frac{1}{\min(i,j)} \quad \text{if} \quad k_i \leq k < k_{i+1} \quad \text{and} \quad p_j \leq p < p_{j+1}.$$

It is easily seen that $\{t_{kp}\}$ is the required sequence. Moreover $t_{kp} \rightarrow 0$,and

$$\| \frac{\alpha_k - \alpha_p}{t_{kp}} \| \leq \| \frac{\alpha_k - \alpha_p}{z_{kp}} \| \leq \omega.$$

Hence there exists a positive integer N such that

$$| \frac{a^k_{mn} - a^p_{mn}}{t_{kp}} | \leq \omega \quad \text{for all} \quad m \geq 1 \quad \text{and} \quad n \geq 1, \quad k,p > N$$

i.e. for each fixed $m,n \geq 1, \{a^k_{mn}\}$ is a Cauchy sequences and hence there exist, $a_{mn}\, m,n \geq 1$ in $C \times C$ such that $a^k_{mn} \rightarrow a_{mn}$ as $k \rightarrow \infty$ for all $m \geq 1$ and $n \geq 1$.

Now $\frac{|a^k_{mn}-a_{mn}|^{1/(\lambda_m+\mu_n)}}{|t_{k,k+1}|} \leq \omega$ for all $m \geq 1$ and $n \geq 1$ i.e. $\| \frac{\alpha_k-\alpha}{t_{k,k+1}} \| \leq \omega$, where $\alpha = \sum_{m,n=1}^{\infty} a_{mn}\exp\left(\lambda_m s_1 + \mu_n s_2\right)$ and $t_{k,k+1} \rightarrow 0$. Hence $\alpha_k \rightarrow \alpha$. Now for k sufficiently large,

$$| a_{mn} |^{1/(\lambda_m+\mu_n)} = | a^k_{mn} - a_{mn} - a^k_{mn} |^{1/(\lambda_m+\mu_n)} < | a^k_{mn} - a_{mn} |^{1/(\lambda_m+\mu_n)}$$

$$+ \mid a^k_{mn} \mid^{1/(\lambda_m+\mu_n)} \leq \mid t_{k,k+1} \mid \omega + \mid a^k_{mn} \mid^{1/(\lambda_m+\mu_n)}.$$

Hence

$$\limsup_{m,n\to\infty}\{\mid a_{mn} \mid^{1/(\lambda_m+\mu_n)}\} \leq \limsup_{m,n\to\infty} \mid t_{k,k+1} \mid \omega + \limsup_{m,n\to\infty} \mid a^k_{mn} \mid^{1/(\lambda_m+\mu_n)}$$

$$+ \limsup_{m,n\to\infty} \mid a^k_{mn} \mid^{1/(\lambda_m+\mu_n)} \leq M\,\omega + \parallel \alpha_k \parallel < \infty,$$

where $M = \sup_k \mid t_{k,k+1} \mid < \infty$. Hence $\alpha \in \overline{\Gamma}$ and therefore $\overline{\Gamma}$ is Mackey-complete. ■

Corollary 6.1. $\overline{\Gamma}$ *is complete.*

Proof. In view of Theorem 1 in [4, p.33] it is enough to show that $\overline{\mathbf{B}}$ is l^1 -disced. For this we show that $B_J \in \overline{\mathbf{B}}$ is l^1-disced. Let $\{t_i\}$ be a sequence of scalars such that $\sum_{i=1}^{\infty} \mid t_i \mid \leq 1$, and $\{\alpha_i\}$ be a sequence in B_J. Then

$$\begin{aligned}\parallel \alpha = \sum_{i=1}^{\infty} t_i\alpha_i \parallel &= \sup \{\mid \sum_{i=1}^{\infty} t_i a^i_{mn} \mid^{1/(\lambda_m+\mu_n)}\} \\ &\leq \sup \{(\sum_{i=1}^{\infty} \mid t_i \mid\mid a^i_{mn} \mid)^{1/(\lambda_m+\mu_n)}\} \\ &\leq \sup\{J(\sum_{i=1}^{\infty} \mid t_i \mid)^{1/(\lambda_m+\mu_n)}\} \leq J \qquad m,n \geq 1.\end{aligned}$$

Hence B_J is l^1 -disced and Corollary 6.1, follows. ■

Theorem 6.2. $(\Gamma, \boldsymbol{B})$ *is not complete.*

Proof. Consider the sequence $\alpha_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} 2^{-(i+j)} e^{is_1+js_2}$, $m,n \geq 1$. Then $\{\frac{\alpha_{mn}-\alpha_{xy}}{(1/2)^{xy}}, mn \geq xy\}$ is bounded in Γ. In other words,$\{\alpha_{mn}\}$ is a Mackey-Cauchy sequence in Γ and hence in $\overline{\Gamma}$. As $(\overline{\Gamma}, \overline{\mathbf{B}})$ is Mackey-complete, the Mackey-limit of $\{\alpha_{mn}\}$ exists in $\overline{\Gamma}$. In fact the Mackey-limit of $\{\alpha_{mn}\}$ in $\overline{\Gamma}$ is $\alpha = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (1/2)^{i+j} e^{is_1+js_2}$ as $\{(\alpha_{mn} - \alpha)/2^{-mn}\}$ is bounded in $\overline{\Gamma}$, and α does not belong to Γ.

We now claim that the Mackey-limit of the sequence $\{\alpha_{mn}\}$ does not exist in Γ. For otherwise, let $\alpha_{mn} \to \beta \in \Gamma$. Then $\alpha_{mn} \to \beta \in \overline{\Gamma}$. Hence $\beta = \alpha$ as $\overline{\Gamma}$ is a separated bornological vector space. This contradicts the fact that α does not belong to Γ. Hence $(\Gamma, \mathbf{B})$ is not Mackey-complete. This proves Theorem 6.2. ■

Lastly we have

Theorem 6.3. $(\overline{\Gamma}, \overline{\boldsymbol{B}})$ *is the Mackey-completion of* $(\Gamma, \boldsymbol{B})$.

Proof. Let $\alpha = \sum\limits_{m,n=1}^{\infty} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \in \overline{\Gamma}$.Then there exists a number h such that $| c_{mn} |^{1/(\lambda_m+\mu_n)} < h$ for all $m, n \geq 1$. Now we consider the sequence $\alpha_{qt} = \sum\limits_{m=1}^{q} \sum\limits_{n=1}^{t} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2)$, $q, t = 1, 2, ...$ in Γ. Then

$$\| \frac{\alpha - \alpha_{qt}}{(1/2)^{qt}} \| =$$

$$= \| \Big(\sum_{m,n=1}^{\infty} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) - \sum_{m=1}^{q} \sum_{n=1}^{t} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \Big) 2^{-(q+t)} \|$$

$$= \| \Big\{ \sum_{m=1}^{q} \sum_{n=1}^{\infty} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) + \sum_{m=q+1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \Big\} 2^{(q+t)}$$

$$- \Big\{ \sum_{m=1}^{q} \sum_{n=1}^{t} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \Big\} 2^{(q+t)} \|$$

$$= \| \Big\{ \sum_{m=1}^{q} \sum_{n=1}^{t} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) + \sum_{m=1}^{q} \sum_{n=t+1}^{\infty} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \Big\} 2^{(q+t)}$$

$$+ \Big\{ \sum_{m=q+1}^{\infty} \sum_{n=1}^{t} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) + \sum_{m=q+1}^{\infty} \sum_{n=t+1}^{\infty} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \Big\} 2^{(q+t)}$$

$$- \Big\{ \sum_{m=1}^{q} \sum_{n=1}^{t} c_{mn} \exp(\lambda_m s_1 + \mu_n s_2) \Big\} 2^{(q+t)} \|$$

$$= \| \sum_{m=1}^{q} \sum_{n=t+1}^{\infty} \frac{c_{mn} \exp(\lambda_m s_1 + \mu_n s_2)}{(1/2)^{q+t}} + \sum_{m=q+1}^{\infty} \sum_{n=1}^{t} \frac{c_{mn} \exp(\lambda_m s_1 + \mu_n s_2)}{(1/2)^{q+t}}$$

$$+ \sum_{m=q+1}^{\infty} \sum_{n=t+1}^{\infty} \frac{c_{mn} \exp(\lambda_m s_1 + \mu_n s_2)}{(1/2)^{q+t}} \| < 3h2^{(D+1)} < \infty.$$

Hence $\{\alpha_{qt}\} \to \alpha$ in Γ.Thus every $\alpha \in \overline{\Gamma}$ can be written as the Mackey-limit of sequence $\{\alpha_{qt}\}$ in Γ. This proves Theorem 6.3. ■

The following corollary is immediate from Theorems 6.1 and 6.3.

Corollary 6.2. $(\overline{\Gamma}, \overline{\boldsymbol{B}})$ *is the completion of* $(\Gamma, \boldsymbol{B})$.

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