F A S C I C U L I M A T H E M A T I C I

Nr 35 2005

Aleksander Waszak

SOME REMARKS ON STRONG CONVERGENCE IN MODULAR SPACES OF SEQUENCES

Abstract: In this paper we study some connections between strong (A, φ)-summability of sequences and lacunary statistical convergence or lacunary strong convergence with respect to a modulus functions.

Key words: sequence spaces, modular spaces.

1. Introduction

In papers of J. Musielak [9], J. Musielak and W. Orlicz [11], W. Orlicz [13] and moreover [16] and [18] some modular spaces connected with strong (A, φ)-summability of sequences are considered and investigated.

In paper of A. Freedman, J. Somberg and M. Raphel [4] the spaces of lacunary strong convergence of sequences are introduced as the sets

$$N_\Theta = \left\{ x = (t_\nu) \, : \, \lim_{r\to\infty} \frac{1}{h_r} \sum_{\nu \in I_r} |t_\nu - s| \ \text{ for some } s \right\},$$

where $\Theta = (k_r)$ is a given lacunary sequence. The relation between I_r and k_r is mentioned in the part 2.

If $F = (f_n)$ is a given sequence of modulus functions (the notation of modulus function was introduced by H. Nakano [12]) and $A = (a_{n\nu})$ is a given matrix, then we may define the following sequence sets

$$N_\Theta(A, F) = \left\{ x = (t_\nu) \, : \, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n \in I_r} f_n \left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} t_\nu - s \right| \right) = 0 \ \text{ for some } s \right\},$$

$$N_\Theta^0(A, F) = \left\{ x = (t_\nu) \, : \, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n \in I_r} f_n \left(\left| \sum_{\nu=1}^{\infty} a_{n\nu} t_\nu \right| \right) = 0 \right\}.$$

Sequences x, which belong to $N_\Theta^0(A, F)$ are called lacunary strongly convergent to zero witch respect a modulus F, (for definition see [1], compare also [2], [3], [8] or [17]).

Throughout this paper it will be supposed that $s = 0$ and that we take the sequence (σ_n^φ), where $\sigma_n^\varphi(x) = \sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu(|t_\nu|)$ instead of the sequence $\left(\sum_{\nu=1}^{\infty} a_{n\nu}t_\nu\right)$.

Finally, the space $T_\Theta^0((A,\varphi),F)$ of lacunary strongly convergent to zero sequences is defined by the formula

$$T_\Theta^0((A,\varphi),F) = \left\{ x = (t_\nu) \,:\, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f_n\left(|\sigma_n^\varphi(x)|\right) = 0 \right\}.$$

2. Preliminaries

Let $A = (a_{n\nu})$ be an infinite matrix. The following assumptions on the matrix A will be used in some of our further considerations:

(a) is nonnegative i.e. $a_{n\nu} \geq 0$ for $n, \nu = 1, 2...,$

(b) for an arbitrary positive integer n (or ν) there exists a positive integer ν_0 (or n_0) such that $a_{n\nu_0} \neq 0$ (or $a_{n_0\nu} \neq 0$), respectively,

(c) there exist $\lim_{n\to\infty} a_{n\nu} = 0$ for $\nu = 1, 2, ...,$

(d) $\sup_n \sum_{\nu=1}^{\infty} a_{n\nu} \leq K < \infty,$

(e) $\sup_n a_{n\nu} \to 0$ as $\nu \to \infty$.

Let T, T_b , T_0, T_f denote spaces of all real sequences, bounded real sequences, real sequences convergent to zero and sequences with a finite number of elements different from zero, respectively. Sequences belonging to T will be denoted by $x = (t_\nu)$, $y = (s_\nu)$, $x_m = (t_\nu^m)$, $|x| = (|t_\nu|)$, $0 = (0)$. Moreover, we shall write e_p, e^q, e_p^q for the following sequences: $0, 0, ..., 1, 0, ...$ (with 1 at the p th place); $1, 1, ..., 1, 0, ...$ (with 1 at the first q places); $0, ..., 0, 1, ...1, 0, ...$ (with 1 at the p th, $(p+1)$ st, ..., $(p+q-1)$ st place), respectively.

A sequence of positive integers $\Theta = (k_r)$ is called lacunary if $k_0 = 0$, $k_r < k_{r+1}$ for all r and if $I_r = (k_{r-1}, k_r]$ then $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

In the following the quotient $\frac{k_r}{k_{r-1}}$ will be denoted by q_r , (compare [4]).

By a modulus function we understand the increasing function f from $[0,\infty)$ to $[0,\infty)$ such that: $f(x) = 0$ if and only if $x = 0$, $f(x+y) \leq f(x) + f(y)$ for $x, y \geq 0$ and is continuous from the right at 0. Throughout this paper the sequence (f_n), $n = 1, 2, ...$ of modulus functions will be denoted by F, (compare [12]).

By a φ-function we understand a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. The symbol $\varphi(|x|)$ means the function $\varphi(|x(t)|)$.

A φ-function φ is called non weaker than a φ-function ψ and we write $\psi \prec \varphi$ if there are constants $c, b, k, l > 0$ such that $c\psi(l\,u) \leq b\varphi(k\,u)$, (for all, large or small u, respectively).

φ-functions φ and ψ are called equivalent and we write $\varphi \sim \psi$ if there are positive constants b_1, b_2, c, k_1, k_2, l such that $b_1\varphi(k_1 u) \leq c\psi(l\,u) \leq b_2\varphi(k_2 u)$, (for all, large or small u, respectively).

A φ-function φ is said to satisfy the condition (Δ_2), (for all, large or small u, respectively) if for some constant $k > 1$ there is satisfied the inequality $\varphi(2u) \leq k\,\varphi(u)$.

In the following let $\varphi = (\varphi_\nu)$ and $\psi = (\psi_\nu)$ be two sequences of φ-functions. We say that relations between $\varphi = (\varphi_\nu)$ and $\psi = (\psi_\nu)$ hold if and only if these relations hold between φ-functions φ_ν and ψ_μ for every ν. For more properties of φ-functions see e.g. [7], [9], [10], [18], [19].

3. Spaces of strongly (A, φ)-summable sequences

For a given the sequence $\varphi = (\varphi_\nu)$ of φ-functions $\varphi_\nu(u)$ and the matrix $A = (a_{n\nu})$ we adopt the following notations:

$$\sigma_n^\varphi(x) = \sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right) \quad \text{for} \quad n = 1, 2, ...,$$

$$T_\varphi^0 = \left\{x \in T \,:\, \sigma_n^\varphi(x) < \infty \;\text{ for }\; n = 1, 2, ... \;\text{ and }\; \lim_{n\to\infty} \sigma_n^\varphi(x) = 0\right\},$$

$$T_\varphi = \left\{x \in T \,:\, \lambda x \in T_\varphi^0 \;\text{ for an arbitrary }\; \lambda > 0\right\}$$

$$T_\varphi^* = \left\{x \in T \,:\, \lambda x \in T_\varphi^0 \;\text{ for a certain }\; \lambda > 0\right\}.$$

Sequences x, which belong to T_φ^* are called strongly (A, φ)-summable to zero.

A list of the most interesting properties concerning the space T_φ^* is presented below, (compare also [11], [13], [16] or [18]).

(1) $T_\varphi \subset T_\varphi^0 \subset T_\varphi^*$.

(2) $T_f \subset T_\varphi$ if and only if the matrix A satisfies the condition (c).

(3) If the matrix A possesses the property (c), then e_p, e^q, $e_p^q \in T_\varphi$, if $\lim_{n\to\infty} a_{n\nu} = 0$ for $\nu = 1, 2, ...$ does not hold then we have $T_\varphi = T_\varphi^0 = T_\varphi^* = \{0\}$.

(4) If the matrix A possesses the property (d) then $T_b \cap T_\varphi^* = T_b \cap T_\psi^*$

and $T_b \cap T_\varphi = T_b \cap T_\varphi^*$ for an arbitrary two sequences φ and ψ of φ-functions.

(5) φ satisfies the condition (Δ_2) for large arguments if and only if $T_\varphi = T_\varphi^*$.

(6) Let the matrix A has properties (a)-(d); if $\psi \prec \varphi$ for large arguments then $T_\varphi^* \subset T_\psi^*$ and $T_\varphi \subset T_\psi$, if $\varphi \sim \psi$ for large arguments then $T_\varphi^* = T_\psi^*$ and $T_\varphi = T_\psi$.

4. Spaces of lacunary strongly convergent sequences

Let $\varphi = (\varphi_\nu)$ and $F = (f_n)$ be given sequences of φ-functions and modulus functions, respectively. Moreover, let a matrix A and a lacunary sequence Θ be given. We introduce the set $T_\Theta^0((A,\varphi),F)$ by the formula:

$$T_\Theta^0((A,\varphi),F) = \left\{ x = (t_\nu) \,:\, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f_n \left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right) \right) = 0 \right\}.$$

Moreover, let

$$T_\Theta((A,\varphi),F) = \left\{ x = (t_\nu) \,:\, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f_n \left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(\lambda\,|t_\nu|\right) \right) = 0 \right.$$
$$\left. \text{for an arbitrary } \lambda > 0 \right\},$$

$$T_\Theta^*((A,\varphi),F) = \left\{ x = (t_\nu) \,:\, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f_n \left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(\lambda\,|t_\nu|\right) \right) = 0 \right.$$
$$\left. \text{for a certain } \lambda > 0 \right\}.$$

The sequence x is said to be lacunary strong (A,φ)-convergent to zero with respect to a modulus F, if $x \in T_\Theta^0((A,\varphi),F)$.

Let us remark that in particulary we have:

1^0 If $\varphi_\nu(u) = u$ for all ν, then we obtain the set

$$T_\Theta^0((A,u),F) \equiv N_\Theta^0(A,F) \equiv \left\{ x = (t_\nu) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} f_n \left(\sum_{\nu=1}^{\infty} a_{n\nu}\,|t_\nu| \right) = 0 \right\},$$

(compare e.g. [1]).

2^0 If $f_n(v) = v$ for all n, then

$$T^0_\Theta((A,\varphi),\nu) \equiv T^0_\Theta((A,\varphi)) \equiv \left\{ x = (t_\nu) \,:\, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} \sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right) = 0 \right\}.$$

3^0 If $A = I$ and moreover $\varphi_\nu(u) = u$ and $f_n(v) = v$ for all ν and n, respectively, then we have the sequence space,

$$N^0_\Theta \equiv T^0_\Theta((I,u),\nu) \equiv \left\{ x = (t_n u) \,:\, \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} |t_n| = 0 \right\}$$

(compare [1]).

Theorem 1. *Let $\varphi = (\varphi_\nu)$ be a given sequence of φ-functions and let $F = (f_n)$ be a sequence of modulus functions. Then, for the usual definition of addition of sequences and multiplication by a scalar,*

(α) $T^0_\Theta((A,\varphi),F)$ *is a convex set,*

(β) $T^*_\Theta((A,\varphi),F)$ *is a linear space.*

Proof. We limit ourselves to the proof of the property (α). Suppose that $x = (t_\nu)$, $y = (s_\nu) \in T^0_\Theta((A,\varphi),F)$ and α, β are arbitrary numbers such that $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$. We have

$$\frac{1}{h_r} \sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|\alpha t_\nu + \beta s_\nu|\right) \right) \le \frac{1}{h_r} \sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right) \right) + \frac{1}{h_r} \sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|s_\nu|\right) \right).$$

Thus, $\alpha x + \beta y \in T^0_\Theta((A,\varphi),F)$. ∎

Theorem 2. *Let F and φ be sequences of modulus functions and φ-functions, respectively. Moreover let the matrix A and the sequence Θ be given. If*

$$w((A,\varphi),F) = \left\{ x = (t_\nu) \,:\, \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right) \right) = 0 \right\}$$

then the following relations are true:

(a) *If* $\liminf_r q_r > 1$, *then* $w((A,\varphi),F) \subseteq T^0_\Theta((A,\varphi),F)$.

(b) *If* $\limsup_r q_r < \infty$, *then* $T^0_\Theta((A,\varphi),F) \subseteq w((A,\varphi),F)$.

(c) *If* $1 < \liminf_r q_r \le \limsup_r q_r < \infty$, *then* $T^0_\Theta((A,\varphi),F) = w((A,\varphi),F)$.

Proof. (a). Let us suppose that $x \in w((A,\varphi),F)$. There exists $\delta > 0$ such that $q_r > 1+\delta$ for sufficiently large r and we have $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ for sufficiently large r. Consequently,

$$\frac{1}{k_r}\sum_{n=1}^{k_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) \geq \frac{1}{k_r}\sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right)$$
$$\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right).$$

Finally, $x \in T^0_\Theta((A,\varphi),F)$.

(b). Let us remark that the condition $\limsup_r q_r < \infty$ implies that there exists a constant $M > 0$ such that $q_r < M$ for every r. If $x \in T^0_\Theta((A,\varphi),F)$ and $\varepsilon > 0$ is an arbitrary number, then there exists an index m_0 such that

$$H_m = \frac{1}{h_m}\sum_{n\in I_m} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) < \varepsilon$$

for every $m \geq m_0$. Thus, there exists a constant $L > 0$ such that $H_m \leq L$ for all m. Choosing an integer α such that $k_{r-1} < \alpha < k_r$ we obtain

$$I = \frac{1}{\alpha}\sum_{n=1}^{\alpha} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) \leq \frac{1}{k_{r-1}}\sum_{n=1}^{k_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) = I_1 + I_2,$$

where

$$I_1 = \frac{1}{k_{r-1}}\sum_{m=1}^{m_0}\sum_{n\in I_m} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right),$$

$$I_2 = \frac{1}{k_{r-1}}\sum_{m=m_0+1}^{\alpha}\sum_{n\in I_m} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right).$$

It is easily verified that

$$I_1 = \frac{1}{k_{r-1}}\left(\sum_{n\in I_1} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) + \ldots + \sum_{n\in I_{m_0}} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right)\right)$$

$$\leq \frac{1}{k_{r-1}}\left(h_1H_1 + \ldots + h_{m_0}H_{m_0}\right) \leq \frac{1}{k_{r-1}} m_0 k_{m_0}\sup_{1\leq i\leq m_0} H_i \leq \frac{m_0 k_{m_0}}{k_{r-1}} L.$$

Moreover, we have

$$\begin{aligned} I_2 &= \frac{1}{k_{r-1}} \sum_{m=m_0+1}^{\alpha} \sum_{n\in I_m} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) \\ &\le \varepsilon \frac{1}{k_{r-1}} \sum_{m=m_0+1}^{\alpha} h_m \le \varepsilon \frac{k_r}{k_{r-1}} = \varepsilon q_r < \varepsilon M. \end{aligned}$$

Thus, the following inequality holds $I \le \frac{m_0 k_{m_0}}{k_{r-1}} L + \varepsilon M$. Finally, $x \in w((A,\varphi),F)$. ■

Theorem 3. *Let the sequence Θ, the modulus functions F and two sequences of φ-functions φ and ψ be given. Suppose that the matrix A satisfies the conditions (a), (b) and (d) and let φ-functions φ and ψ satisfy the condition (Δ_2) for large u.*

- (α) *If $\psi \prec \varphi$ for large u, then $T^0_\Theta((A,\varphi),F) \subset T^0_\Theta((A,\psi),F)$.*
- (β) *If φ-function φ and ψ are equivalent for large u, then $T^0_\Theta((A,\varphi),F) = T^0_\Theta((A,\psi),F)$.*

Proof. Let $x = (t_\nu) \in T^0_\Theta((A,\varphi),F)$. By assumption we have $\psi_\nu\left(|t_\nu|\right) \le b\varphi_\nu\left(c\,|t_\nu|\right)$ for $b, c, u_0 > 0$, $|t_\nu| > u_0$ and all ν. Let us denote $x = x^1 + x^2$, where $x^1 = (t_\nu^{(1)})$ and $t_\nu^{(1)} = t_\nu$ for $|t_\nu| < u_0$ and $t_\nu^{(1)} = 0$ for remaining values of ν. It is easily seen that $x^1 \in T^0_\Theta((A,\psi),F)$. Moreover, by the assumptions we get

$$\begin{aligned} \frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\psi_\nu\left(\left|t_\nu^{(2)}\right|\right)\right) &\le \frac{1}{h_r}\sum_{n\in I_r} f_n\left(b\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(c\left|t_\nu^{(2)}\right|\right)\right) \\ &\le \frac{L}{h_r}\sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(\left|t_\nu^{(2)}\right|\right)\right), \end{aligned}$$

where the constant L depends on the properties of F, φ and ψ. Finally, we obtain $x_2 = \left(t_\nu^2\right) \in T^0_\Theta((A,\psi),F)$ and consequently $x \in T^0_\Theta((A,\psi),F)$. By (α) we obtain $T^0_\Theta((A,\varphi),F) = T^0_\Theta((A,\psi),F)$. ■

Remark. Let us remark that the modulus functions f_n are continuous in the interval $[0,\infty)$. Moreover, it is easily verified that by the assumptions of matrix A and the function f_n we have that the sums

$$S^n_{pq} = a_{n,p} + a_{n,p+1} + \ldots + a_{n,p+q-1}$$

and $\sum_{n\in I_r} f_n\left(\max_{p\le\nu\le p+q-1} \varphi_\nu(1)\, S^n_{pq}\right)$ are bounded, and tend to zero as $n\to\infty$ and $r\to\infty$, respectively (compare [11], [16], [18]). Consequently we have $e_p, e^q, e^q_p \in T^0_\Theta((A,\varphi),F)$.

Theorem 4. *Let $F = (f_n)$ be a sequence of modulus functions such that are equicontinuous at 0 and $\sup_n f_n(1) < \infty$. Moreover, let the matrix $A = (a_{n\nu})$ and the sequence $\varphi = (\varphi_\nu)$ of φ-functions be given. The following inclusion hold:*

$$T_\Theta^0((A, \varphi)) \subseteq T_\Theta^0((A, \varphi), F).$$

Proof. Let $x \in T_\Theta^0((A, \varphi))$ for a given $\varepsilon > 0$ we choose $0 < \delta < 1$ such that $f_n(v) < \varepsilon$ for all n and every $v \in [0, \delta]$. We can write

$$\frac{1}{h_r} \sum_{n \in I_r} f_n \left(\sum_{\nu=1}^{\infty} a_{n\nu} \varphi_\nu \left(|t_\nu| \right) \right) = S_1 + S_2,$$

where $S_1 = \frac{1}{h_r} \sum_{n \in I_r} f_n \left(\sum_{\nu=1}^{\infty} a_{n\nu} \varphi_\nu \left(|t_\nu| \right) \right)$ and this sum is taken over $\left(\sum_{\nu=1}^{\infty} a_{n\nu} \varphi_\nu \left(|t_\nu| \right) \right) \leq \delta$, and $S_2 = \frac{1}{h_r} \sum_{n \in I_r} f_n \left(\sum_{\nu=1}^{\infty} a_{n\nu} \varphi_\nu \left(|t_\nu| \right) \right)$ and this sum is taken over $\left(\sum_{\nu=1}^{\infty} a_{n\nu} \varphi_\nu \left(|t_\nu| \right) \right) > \delta$. By definition of the modulus F we have $S_1 \leq \frac{1}{h_r} \sum_{n \in I_r} f_n(\delta) = \sum_{n \in I_r} f_n(\delta) < \varepsilon$ and moreover $S_2 \leq \frac{1}{\delta} \frac{1}{h_r} (\sup_n f_n(1)) \sum_{n \in I_r} \sum_{\nu=1}^{\infty} a_{n\nu} \varphi \left(|t_\nu| \right)$. Finally, we get $x \in T_\Theta^0((A, \varphi), F)$. ■

Remark. Let us remark that in the case $A = I$, $f_n(\nu) = \nu^{\frac{1}{n+1}}$, for $n \geq 1$ and $\nu > 0$, and convex φ-functions φ_ν, we may choose the sequence $x = (t_\nu)$ by the formulas: $t_\nu = \varphi_\nu^{-1}(h_r)$ if $\nu = k_r$ for some $r \geq 1$ and $t_\nu = 0$ otherwise. Then we have

$$\frac{1}{h_r} \sum_{n \in I_r} f_n \left(\sum_{\nu=1}^{\infty} \varphi_\nu \left(|t_\nu| \right) \right) = \frac{1}{h_r} f_{k_r}(h_r) = (h_r)^{-1} (h_r)^{\frac{1}{k_r+1}} \to 0, \quad \text{as } r \to \infty$$

and $\frac{1}{h_r} \sum_{n \in I_r} \sum_{\nu=1}^{\infty} a_{n\nu} \varphi_\nu \left(|t_\nu| \right) = \frac{1}{h_r} h_r \to 1$ as $r \to \infty$. Thus $x \in T_\Theta^0((A, \varphi), F)$ but $x \notin T_\Theta^0((A, \varphi))$.

5. Some remarks on lacunary (A, φ)-statistical convergence

Let $\Theta = (k_r)$ be a lacunary sequence, and let the matrix $A = (a_{n\nu})$, the sequence $x = (t_\nu)$, the sequence φ of φ-functions $\varphi_\nu(u)$ and a positive number ε be given. We adopt the following notation

$$K_\Theta^r((A, \varphi), \varepsilon) = \left\{ n \in I_r : \sum_{\nu=1}^{\infty} a_{n\nu} \varphi_\nu \left(|t_\nu| \right) \geq \varepsilon \right\}.$$

The sequence x is said to be lacunary (A,φ)-statistically convergent to a number zero if for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r}\, \mu\left(K_\Theta^r\left((A,\varphi),\varepsilon\right)\right) = 0,$$

where $\mu\left(K_\Theta^r\left((A,\varphi),\varepsilon\right)\right)$ denotes the number of elements belonging to the set $K_\Theta^r\left((A,\varphi),\varepsilon\right)$. The set of all lacunary (A,φ)-statistical convergent sequences is denoted by $S_\Theta((A,\varphi))$,

$$S_\Theta\left((A,\varphi)\right) = \left\{x = (t_\nu) \,:\, \lim_{r\to\infty} \frac{1}{h_r} \mu\left(K_\Theta^r\left((A,\varphi),\varepsilon\right)\right) = 0\right\},$$

(compare [2], [4], [5], [6], [15] and [17]).

Theorem 5. *If $\psi \prec \varphi$ and $\varphi \in (\Delta_2)$ for large arguments then*

$$S_\Theta\left((A,\psi)\right) \subset S_\Theta\left((A,\varphi)\right).$$

Proof. The assumptions imply that

$$\sum_{\nu=1}^{\infty} a_{n\nu}\psi_\nu\left(|t_\nu|\right) \le b \sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(c\,|t_\nu|\right) \le Lb \sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right),$$

for $b, c > 0$, $n \in N$, where the constant L depends on the properties of φ. Consequently we obtain

$$\mu\left(K_\Theta^r\left((A,\varphi),\varepsilon\right)\right) \le \mu\left(K_\Theta^r\left((A,\psi),\varepsilon\right)\right)$$

and

$$\lim_{r\to\infty} \frac{1}{h_r}\, \mu\left(K_\Theta^r\left((A,\varphi),\varepsilon\right)\right) \le \lim_{r\to\infty} \frac{1}{h_r}\, \mu\left(K_\Theta^r\left((A,\psi),\varepsilon\right)\right).$$

■

Corollary. *If $\psi \sim \varphi$ and $\varphi, \psi \in (\Delta_2)$ for large arguments then*

$$S_\Theta\left((A,\varphi)\right) = S_\Theta\left((A,\psi)\right).$$

Theorem 6. *Let Θ, F and φ be given. Suppose that the sequence (f_n) is pointwise convergent.*

(α) *If $\lim_n f_n(\nu) > 0$ for $\nu > 0$ then $T_\Theta^0((A,\varphi),F) \subset S_\Theta^0((A,\varphi))$ for every matrix A.*

(β) *If moreover $\varphi = (\varphi_\nu)$ is a sequence of convex φ-functions then the inclusion $T_\Theta^0((A,\varphi),F) \subset S_\Theta((A,\varphi))$ implies that $\lim_n f_n(\nu) > 0$ for $\nu > 0$.*

Proof. (α). Let ε be a positive number and let $x \in T^0_\Theta((A,\varphi),F)$. If $\lim\limits_n f_n(\nu) > 0$, then there exists $\alpha > 0$ such that $f_n(\nu) > \alpha$ for $\nu > \varepsilon$ and for all n. We have

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) \geq \frac{1}{h_r}\sum_{n\in I_r^1} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right)$$
$$\geq \frac{1}{h_r}\sum_{n\in I_r^1} f_n(\varepsilon) \geq \frac{1}{h_r}\alpha\,\mu\left(K^r_\Theta((A,\varphi),\varepsilon)\right),$$

where $I_r^1 = \left\{n \in I_r : \sum\limits_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right) \geq \varepsilon\right\}$. Finally $x \in S_\Theta((A,\varphi))$.

(β). Let us suppose that $\lim\limits_n f_n(\nu) > 0$ does not hold. Then there exists a positive number α such that $\lim\limits_n f_n(\alpha) = 0$. We can select a lacunary sequence $\Theta = (k_r)$ such that $f_n(\alpha) < \frac{1}{2^r}$ for any $n > k_{r-1}$. In the following, we take $A = I$ and we can select the sequence $x = (t_\nu)$ by the formulas: $t_\nu = \varphi_\nu^{-1}(\alpha)$ for $k_{k-1} < \nu \leq \frac{1}{2}\left(k_{r-1}+k_r\right)$, and $t_\nu = 0$ for $\frac{1}{2}\left(k_{r-1}+k_r\right) < \nu \leq k_r$. It is easily verified that $\sum\limits_{n\in I_r} f_n\left(\left|\sum\limits_{\nu=k_{k-1}+1}^{k_r}\varphi_\nu\left(\varphi_\nu^{-1}(\alpha)\right)\right|\right) < \left(k_r - k_{r-1}\right)\frac{1}{2^r}$ and $\sigma_n^\varphi(x) \sum\limits_{\nu=k_{k-1}+1}^{k_r}\varphi_\nu\left(t_\nu\right) = \frac{k_r-k_{r-1}}{2}\alpha$. Finally, we have $x \in T^0_\Theta((A,\varphi),F)$, but $x \notin S_\Theta((A,\varphi))$. ■

Theorem 7. *Let Θ, F and φ be given.*

(α) *If $\lim\limits_\nu \sup\limits_n f_n(\nu) < \infty$ then $S_\Theta((A,\varphi)) \subset T^0_\Theta((A,\varphi),F)$ for every matrix A.*

(β) *If moreover $\varphi = (\varphi_\nu)$ is a sequence of convex φ-functions then the inclusion $S_\Theta((A,\varphi)) \subset T^0_\Theta((A,\varphi),F)$ implies that $\sup\limits_\nu \sup\limits_n f_n(\nu) < \infty$.*

Proof. (α). Let $x \in S_\Theta((A,\varphi))$. Let us denote $h(\nu) = \sup\limits_n f_n(\nu)$, $h = \sup\limits_\nu h(\nu)$, $I_r^1 = K^r_\Theta((A,\varphi),\varepsilon)$ and $I_r^2 = \left\{\nu \in I_r : \sum\limits_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right) < \varepsilon\right\}$. Thus, we have

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) \leq \frac{1}{h_r}\sum_{n\in I_r^1} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) +$$
$$+\frac{1}{h_r}\sum_{n\in I_r^2} f_n\left(\sum_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu\left(|t_\nu|\right)\right) \leq \frac{1}{h_r}h\mu\left(K^r_\Theta((A,\varphi),\varepsilon)\right) + h(\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, we obtain that $x \in T^0_\Theta((A,\varphi))$.

(β). Let us suppose that $\sup\limits_\nu \sup\limits_n f_n(\nu) = \infty$. Then we choose the increasing sequence (ν_r) such that $f_{k_r}(\nu_r) \geq h_r$, for $r \geq 1$. We can take the matrix $A = I$ and the sequence $x = (t_\nu)$ defined by the formulas: $t_\nu = \varphi^{-1}_{k_r}(\nu_r)$ for $\nu = k_r$ (and for some $r = 1, 2, ...$) and $t_\nu = 0$ otherwise. Finally, since $\mu\left(K^r_\Theta((I,\varphi),\varepsilon)\right)$ is the finite number and $\sum\limits_{n\in I_r} f_n(\sum\limits_{\nu=k_{k-1}+1}^{k_r} \varphi_\nu(|t_\nu|)) \geq h_r$ for every r, then we obtain $x \in S_\Theta((A,\varphi))$ but $x \notin T^0_\Theta((A,\varphi),F)$. ■

Theorem 8. *Suppose that the matrix A is regular and that the modulus functions $F = (f_n)$ are bounded. Then the condition $x \in T_0$ implies $x \in S_\Theta((A,\varphi))$.*

Proof. If $x = (t_\nu) \in T_0$, by regularity of A we have $\lim\limits_{n\to\infty} \sum\limits_{\nu=1}^{\infty} a_{n\nu}\varphi_\nu(|t_\nu|) = 0$. Thus, by the definition of statistical (A,φ)-convergence, we obtain $\lim\limits_{n\to\infty} \frac{1}{h_r}\mu\left(K^r_\Theta((A,\varphi),\varepsilon)\right) = 0$ and $x \in S_\Theta((A,\varphi))$. ■

References

[1] T. Bilgin, Lacunary strong A-convergence with respect to a sequence modulus functions, *Preprint of Department of Mathematics University of 100. Yil, Van Turkey,* (2001), 1-5.

[2] J.S. Connor, On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull.,* 32(1989), 194-198.

[3] A. Esi, The A-statistical and strongly $(A-p)$-Cesàro convergence of sequences, *Pure and Appl. Math. Sci.,* 43(1996), 89-93.

[4] A.R. Freedman, J.J. Sember, M. Raphel, Some Cesàro-type summability spaces, *Proc. London Math. Soc.,* 37(1978), 508-520.

[5] J. Frydy, On statistical convergence, *Analysis,* 5(1985), 301-313.

[6] J. Frydy, C. Orhan, Lacunary statistical convergence, *Pacific J. Math.,* 160(1993), 45-51.

[7] A. Kufner, O. John, S. Fučik, *Function spaces,* Prague, 1977.

[8] I.J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Camb. Phil. Soc.,* 100(1986), 161-166.

[9] J. Musielak, Orlicz spaces and modular spaces, *Lecture Notes in Math. 1034,* Springer Verlag. Berlin-Heidelberg-New York-Tokyo 1983

[10] J. Musielak, W. Orlicz, On modular spaces, *Studia Math.,* 18(1959), 49-65.

[11] J. Musielak, W. Orlicz, On modular spaces of strongly summable sequences, *Studia Math.,* 22(1962), 127-144.

[12] H. Nakano, Concave modulars, *J. Math. Soc. Japan,* 5(1953), 29-49.

[13] W. Orlicz, On some spaces of strongly summable sequences, *Studia Math.,* 22(1963), 331-336.

[14] S. Pehlivan, B. Fisher, On some sequence spaces, *Indian J. Pure Appl. Math.,* 25(1994), 1067-1071.

[15] T. Salat, On statistically convergent sequences of real numbers, *Math. Slovaca,* 2(1980), 139-150.

[16] A. Waszak, On spaces of strongly summable sequences with an Orlicz metric, *Comment. Math.,* 11(1968), 229-246.

[17] A. Waszak, On the strong convergence in some sequence spaces, *Fasc. Math.,* 33(2002), 125-137.

[18] A. Waszak, D. Żurakowski, Musielak-Orlicz spaces of sequences connected with strong summability, *Fasc. Math.,* 28(1998), 157-165.

[19] D. Żurakowski, On completeness of some Musielak-Orlicz spaces, *Fasc. Math.,* 24(1994), 13-17.

Aleksander Waszak
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87, 61-614 Poznań, POLAND

Received on 12.11.2004 and, in revised from, on 15.12.2004