F A S C I C U L I M A T H E M A T I C I

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ON TWO SECOND ORDER HALF-LINEAR DIFFERENCE EQUATIONS

Abstract: In this paper, two second order half-linear difference equations are considered. By establishing their connections with a standard half-linear difference equation, we are able to obtain sufficient conditions for existence and nonexistence of eventually positive solutions.

Key words: gHalf-linear difference equation, neutral difference equation, eventually positive solution, eventually ω-positive solution.

1. Introduction

In this paper, we will be concerned with two delay difference equations of the form

(1) $$\Delta\left(x_n - x_{n-\tau}\right)^\alpha + q_n x_{n-\sigma}^\alpha = 0,$$

and

(2) $$\Delta\left[\Delta\left(x_{n-1} + p_{n-1}x_{n-1-\tau}\right)\right]^\alpha + \frac{q_n}{\tau^\alpha} x_{n-\sigma}^\alpha = 0,$$

where $\{p_n\}_{n=0}^\infty$ is a real sequence, $\{q_n\}_{n=0}^\infty$ is nonnegative and possesses a positive subsequence, α is a quotient of odd positive integers, τ is a positive integer and σ is a nonnegative integer.

The motivation for the study of these equations originates from the studies of the half-linear difference equation

(3) $$\Delta\left(\Delta x_{n-1}\right)^\alpha + q_n x_n^\alpha = 0,$$

(see [11-16] and the references cited therein), and its special case, namely the well known linear self-adjoint difference equation

$$\Delta^2 x_{n-1} + q_n x_n = 0,$$

which occurs in vibrating mechnical systems, loaded strings, etc. (see Cheng and Lu [17]). Indeed, when $\alpha = 1$, the equations (1) and (2) reduce to the equations

(4) $$\Delta\left(x_n - x_{n-\tau}\right) + q_n x_{n-\sigma} = 0,$$

and

(5) $$\Delta^2\left(x_{n-1} + p_n x_{n-1-\tau}\right) + \frac{q_n}{\tau^\alpha} x_{n-\sigma} = 0,$$

respectively. When $\tau = 1$ and $\sigma = 0$ in (1), and $p_n \equiv 0$, $\tau = 1$ as well as $\sigma = 0$ in (2), the equations (1) and (2) reduce to the half-linear second order difference equation (3). Therefore, it is reasonable that some of the properties of (3) will remain valid for the more general equations (1) and (2). It is also reasonable to expect relations exist between (3) and the more general equations. Indeed, nonoscillation and oscillation criteria for (4) and (5) have been obtained recently (see e.g. [1-10]) and the references cited therein.

In this paper, we will prove that the existence problems of eventually positive solutions for equations (1) and (3) are *equivalent.* This will allow us to deduce a large number of existence criteria from those for (3). In Section 3, eventually positive solutions and eventually ω-positive solutions of (2) are also considered.

Let $\mu = \max\{\tau, \sigma\}$. If a real sequence $x = \{x_n\}_{n=-\mu}^{\infty}$ satisfies the functional relation defined by (1), then it is said to be a solution of (1). Similarly, the solutions of equations (2), (4), (5) and (3) can be defined. Since (3)-(5) are recurrence relations, it is not difficult to see that the corresponding existence and uniqueness theorems can easily be formulated and proved by induction when appropriate initial conditions are given.

2. Existence criteria for equation (2)

We first consider a more general functional difference equation

(6) $$\Delta\left(x_n - p_n x_{n-\tau}\right)^\alpha + q_n f(x_{n-\sigma}) = 0,$$

and an associated functional inequality

(7) $$\Delta\left(x_n - p_n x_{n-\tau}\right)^\alpha + q_n f(x_{n-\sigma}) \le 0,$$

where α, τ, σ, p and q satisfy the same assumptions as stated before and $f : R \to R$ is a continuous nondecreasing function which satisfies $f(x) > 0$ for all $x > 0$.

First of all, a lemma is obtained. It is important for the proof of the following theorems.

Lemma 1. *Let p_n be nonnegative for $n \geq 0$ and suppose there exists $N > 0$ such that*

$$p_{N+j\tau} \leq 1,\ j = 0, 1, 2, \ldots \tag{8}$$

Let $\{x_n\}$ be an eventually positive solution of (7). Set

$$y_n = x_n - p_n x_{n-\tau}. \tag{9}$$

Then $\Delta y_n \leq 0$ and $y_n > 0$ eventually.

Proof. In view of (9) and (7), Δy_n^α is eventually nonpositive and does not vanish identically for all large n. Thus y_n^α and y_n are of constant sign and $\xi_n^{\alpha-1}$ is positive for all large n, where $\xi_n \in [y_n, y_{n+1}]$ or $[y_{n+1}, y_n]$. Therefore, $\Delta y_n \leq 0$ eventually. Suppose to the contrary that y_n is eventually negative. Then there exists n_1 such that $x_n > 0, \Delta y_n \leq 0$ and $y_n \leq y_{n_1} < 0$ for $n \geq n_1$, that is,

$$x_n \leq y_{n_1} + p_n x_{n-\tau},\ n \geq n_1.$$

By choosing k^* so large that $N + k^*\tau \geq n_1$, we have

$$\begin{aligned} x_{N+k^*\tau+j\tau} &= y_{N+k^*\tau+j\tau} + p_{N+k^*\tau+j\tau} x_{N+k^*\tau+(j-1)\tau} \\ &\leq y_{n_1} + x_{N+k^*\tau+(j-1)\tau} \\ &\leq \ldots \leq j y_{n_1} + x_{N+k^*\tau} \end{aligned}$$

for $j \geq 0$. By letting j tend to infinity, we see that the right-hand side diverges to $-\infty$, which is contrary to our assumptions that $x_n > 0$ for $n \geq n_1$. The proof is complete. ■

Theorem 1. *Suppose that either (H1) $p_n + \sigma q_n > 0$ for $n \geq 0$, or (H2) $\sigma > 0$ and q_s does not vanish identically over any set of consecutive integers of the form $\{a, a+1, \ldots, a+\sigma\}$. Suppose further that $\{\overline{p}_n\}$ and $\{\overline{q}_n\}$ are two nonnegative sequences such that $\overline{p}_n \geq p_n$ and $\overline{q}_n \geq q_n$ for all large n, $F : R \to R$ is a continuous nondecreasing function which satisfies $F(x) \geq f(x)$ for $x > 0$, and there is an integer $N > 0$ such that $\overline{p}_{N+j\tau} \leq 1$ for $j \geq 0$. If the difference inequality*

$$\Delta\left(x_n - \overline{p}_n x_{n-\tau}\right)^\alpha + \overline{q}_n F(x_{n-\sigma}) \leq 0 \tag{10}$$

has an eventually positive solution $\{x_n\}$, so dose equation (6).

Proof. We may assume, in view of Lemma 1, that $x_n > 0$ and $z_n = x_n - \overline{p}_n x_{n-\tau} > 0$ for $n \geq T - \mu$. Then, in view of (10),

$$\Delta\left(z_n\right)^\alpha \leq -\overline{q}_n F(x_{n-\sigma}),\ n \geq T,$$

so that by summing from n to infinity, we obtain

$$y_n - \overline{p}_n y_{n-\tau} \ \geq \ \left[\sum_{k=n}^{\infty} \overline{q}_k F(y_{k-\sigma}) \right]^{\frac{1}{\alpha}}, \ n \geq T,$$

since $\lim_{k\to\infty} z_k \geq 0$ by Lemma 1. As a consequence, we see that

$$x_n - p_n x_{n-\tau} \geq \left[\sum_{k=n}^{\infty} q_k f(x_{k-\sigma}) \right]^{\frac{1}{\alpha}}, \ n \geq T. \tag{11}$$

Let Ω be the set of all real sequences of the form $w = \{w_n\}_{n=T-\mu}^{\infty}$. Define an operator $S : \Omega \to \Omega$ by

$$(Sw)_n = 1, \ T - \mu \leq n \leq T - 1$$

and

$$(Sw)_n = \frac{1}{y_n} \left\{ p_n w_{n-\tau} y_{n-\tau} + \left[\sum_{k=n}^{\infty} q_k f(y_{k-\sigma}) \right]^{\frac{1}{\alpha}} \right\}, \ n \geq T.$$

Consider the following iteration scheme: $w^0 \equiv 1, w^{i+1} = Sw^i$ for $i = 0, 1, \ldots.$ Clearly, in view of (11),

$$0 \leq w_n^{i+1} \leq w_n^i \leq 1$$

for $n \geq T$ and $i = 0, 1, \ldots$. Thus, as $i \to \infty$, $\{w^i\}$ converges to some nonnegative sequence $\overline{w}$ which satisfies

$$\overline{w}_n x_n = p_n \overline{w}_{n-\tau} x_{n-\tau} + \left[\sum_{k=n}^{\infty} q_k f(\overline{w}_{k-\sigma} x_{k-\sigma}) \right]^{\frac{1}{\alpha}}, \ n \geq T.$$

Taking differences on both sides of the above equation, we see that $\{v_n\}$ defined by

$$v_n = \overline{w}_n x_n, \ n \geq T - \mu,$$

is an eventually nonnegative solution of (6). Finally, we will show that $\{v_n\}$ is eventually positive. To see this, note that $v_n = x_n > 0$ for $T - \mu \leq n \leq T - 1$. Suppose to the contrary that $v_n > 0$ for $T - \mu \leq n < n^*$ and $v_{n^*} = 0$, then

$$0 = v_{n^*} = p_{n^*} v_{n^*-\tau} + \left[\sum_{k=n^*}^{\infty} q_k f(v_{k-\sigma}) \right]^{\frac{1}{\alpha}},$$

so that $p_{n^*} = 0$ and $q_k f(v_{k-\sigma}) = 0$ for $k \geq n^*$.This is contrary to assumptions (H1) or (H2). The proof is complete. ∎

As an immediate consequence of Theorem 1, we have the following result.

Corollary 1. *Suppose that (8), and, either (H1) or (H2) hold. Then equation (6) has an eventually positive solution if, and only if, (7) has an eventually positive solution.*

Indeed, if we let F in Theorem 1 be f, then we see that the existence of an eventually positive solution of (7) implies the existence of an eventually positive solution of (6). The converse is trivially true.

We note that the assumptions (H1) or (H2) will be satisfied if either $p_n > 0$ or $q_n > 0$ for $n \geq 0$. Therefore, we have the following.

Corollary 2. *Equation (1) has an eventually positive solution if, and only if, the difference inequality*

$$\Delta\left(x_n - x_{n-\tau}\right)^\alpha + q_n x_{n-\sigma}^\alpha \leq 0$$

has an eventually positive solution.

Corollary 3. *Equation (3) has an eventually positive solution if, and only if, the difference inequality*

$$\Delta\left(\Delta x_{n-1}\right)^\alpha + Q_n x_n^\alpha \leq 0$$

has an eventually positive solution.

Theorem 2. *Let $p_n \equiv 1$. Then equation (6) has an eventually positive solution if, and only if, the difference inequality*

$$\Delta\left(\Delta x_{n-1}\right)^\alpha + Q_n f(x_n) \leq 0 \tag{12}$$

has an eventually positive solution.

Proof. Let $\{w_n\}$ be an eventually positive solution of (12). Then it is well known that $\{\Delta w_n\}$ is eventually positive and nonincreasing. Thus it is easy to see that there exists a sufficiently large integer T such that $0 < \tau \Delta w_{n-1} \leq w_{n-\tau}$ for $n \geq T$. Let

$$H_n = \begin{cases} \tau \Delta w_{n-1} & n \geq T \\ (n - T + \tau + 1)\, \Delta w_{T-1} & T - \tau \leq n < T \\ 0 & n < T - \tau \end{cases},$$

and let

$$x_n = w_{T-\tau} - \tau \Delta w_{T-1} + \sum_{i=0}^{\infty} H_{n-i\tau}$$

for $n \geq 0$. In view of the definition of H_n, it is clear that $0 < x_n < \infty$ for all $n \geq 0$, that

$$\max\{x_{T-\tau}, x_{T-\tau+1}, ..., x_{T-1}\} = w_{T-\tau} - \tau \Delta w_{T-1} + \tau \Delta w_{T-1} = w_{T-\tau},$$

and that

$$x_n - x_{n-\tau} = H_n = \tau \Delta w_{n-1},\ n \geq T.$$

For any integer n which satisfies $T + \sigma \leq n \leq T + \sigma + \tau - 1$, we see that

$$\begin{aligned} x_{n-\sigma} &= \tau \Delta w_{n-\sigma-1} + x_{n-\tau-\sigma} \leq \sum_{i=n-\tau-\sigma}^{n-\sigma-1} \Delta w_i + x_{n-\tau-\sigma} \\ &\leq \sum_{i=T-\tau}^{n-1} \Delta w_i + w_{T-\tau} = w_n - w_{T-\tau} + w_{T-\tau} = w_n. \end{aligned}$$

By induction, it is easily proved that for any integer n which satisfies $T + \sigma + k\tau \leq n \leq T + \sigma + k\tau - 1$ where $k = 0, 1, 2, ...$, we see that $x_{n-\sigma} \leq w_n$ is still valid. Thus

$$\Delta \left(x_n - x_{n-\tau}\right)^{\alpha} + q_n f(x_{n-\sigma}) \leq \tau^{\alpha} \Delta \left(\Delta w_{n-1}\right)^{\alpha} + q_n f(w_n) \leq 0$$

as desired.

We now show that the converse holds. Let $\{x_n\}$ be an eventually positive solution of (6). In view of Lemma 1, we see that there is an integer N_1 such that $x_{n-\tau} > 0, y_n = x_n - x_{n-\tau} > 0, \Delta y_n \leq 0$ for $n \geq N_1$. Let $N_2 = N_1 + \tau + \sigma$. For any $n \geq N_2$, there exists a positive integer k such that

$$N_1 + k\tau \leq n - \sigma < N_1 + (k+1)\,\tau$$

and

$$x_{n-\sigma} = x_{n-\sigma-k\tau} + \sum_{j=0}^{k-1} y_{n-\sigma-j\tau} \geq M + \sum_{j=0}^{k-1} y_{n-\sigma-j\tau},$$

where $M = \min\{x_{N_1-\tau}, x_{N_1-\tau+1}, ..., x_{N_1-1}\}$. Furthermore, since $\Delta y_n \leq 0$ for $n \geq N_1$, and since $n - k\tau + 1 \leq N_1 + 2\tau + \sigma = N_2 + \tau$,

$$\begin{aligned} \tau \sum_{j=0}^{k-1} y_{n-\sigma-j\tau} &= \tau y_{n-\sigma} + ... + \tau y_{n-\sigma-(k-1)\tau} \\ &\geq (y_n + ... + y_{n-\tau+1}) + ... + \left(y_{n-(k-1)\tau} + ... + y_{n-k\tau+1}\right) \\ &\geq \sum_{i=N_2+\tau}^{n} y_i \geq \sum_{i=n-k\tau+1}^{n} y_i \end{aligned}$$

so that

$$x_{n-\sigma} \geq M + \frac{1}{\tau} \sum_{i=N_2+\tau}^{n} y_i.$$

Let

$$z_n = M + \frac{1}{\tau} \sum_{i=N_2+\tau}^{n} y_i.$$

Then $z_n > 0$ for $n \geq N_2 + \tau$, and

$$\tau^\alpha \Delta \left(\Delta z_{n-1}\right)^\alpha + q_n f(z_n) = \Delta y_n^\alpha + q_n f\left(M + \frac{1}{\tau} \sum_{i=N_2+\tau}^{n} y_i\right) \leq \Delta y_n^\alpha + q_n f\left(x_{n-\sigma}\right) \leq 0$$

as required. The proof is complete. ■

We remark that when $\alpha = 1$, Theorem 1 and Theorem 2 are Theorem 1 and Theorem 2 in [6] respectively. We remark further that when $p_n \equiv 1$, Corollary 1 and Theorem 2 are valid if the assumption $\sigma \geq 0$ is replaced by $\sigma \leq 0$.

Corollary 4. *The following statements are equivalent:*
(a) The equation (12) has an eventually positive solution.
(b) The equation

$$\Delta \left(x_n - x_{n-\tau}\right)^\alpha + q_n f(x_n) = 0 \tag{13}$$

has an eventually positive solution.
(c) The equation

$$\Delta \left(\Delta z_{n-1}\right)^\alpha + Q_n f(z_{n-\sigma}) = 0 \tag{14}$$

has an eventually positive solution.

Corollary 5. *The following statements are equivalent:*
(a) The equation (1) has an eventually positive solution.
(b) The equation (3) has an eventually positive solution.
(c) The equation

$$\Delta \left(\Delta z_{n-1}\right)^\alpha + Q_n z_{n-\sigma}^\alpha = 0 \tag{15}$$

has an eventually positive solution.
(d) The equation

$$\Delta \left(x_n - x_{n-\tau}\right)^\alpha + q_n x_n^\alpha = 0 \tag{16}$$

has an eventually positive solution.

Corollary 6. *Equation (1) has an eventually positive solution if, and only if, the equation*

$$\Delta \left(\Delta z_{n-1}\right)^\alpha + Q_n z_n^\alpha = 0 \tag{17}$$

has an eventually positive solution.

For the sake of convenience, we will write

$$G(x,y) = \left(1 + x^{1/y}\right)^y - 1. \tag{18}$$

Let us consider the following functional inequality

$$\sum_{k=n}^{\infty} y_{k+1} G(y_k, \alpha) + \frac{1}{\tau^\alpha} \sum_{k=n}^{\infty} q_{k+1} \le y_n,\ n = 0,1,2... \ . \tag{19}$$

Let us define the mappings T and $S : l^\infty \to l^\infty$ as follows:

$$(Tz)_n = \sum_{k=n}^{\infty} z_{k+1} G(z_k, \sigma) + \frac{1}{\tau^\alpha} \sum_{k=n}^{\infty} q_{k+1},\ n = 0,1,2,...,$$

and

$$(Su)_n = \sum_{k=n}^{\infty} u_{k+1} G(u_k, \sigma),\ n = 0,1,2,... \ .$$

Define

$$z^{(0)} = 0, \tag{20}$$

$$z_n^{(m+1)} = \left(Tz^{(m)}\right)_n\ n \ge 0, m = 0,1,2,..., \tag{21}$$

$$\psi_n^{(0)} = \frac{1}{\tau^\alpha} \sum_{k=n}^{\infty} q_{k+1},\ n = 0,1,2,..., \tag{22}$$

$$\psi_n^{(1)} = \left(S\psi^{(0)}\right)_n,\ n = 0,1,2,..., \tag{23}$$

and

$$\psi_n^{(m+1)} = \left(S(\psi^{(0)} + \psi^{(m)}\right)_n,\ n \ge 0, m = 1,2,.... \tag{24}$$

By Theorem 2 and Theorems 1, 2, and 3 in [15], we see that the following statements are equivalent:

(a) The equation (1) has an eventually positive solution.

(b) The funtional inequality (19) has an eventually positive solution.

(c) The sequence $\left\{z^{(m)}\right\}$ defined by (20) and (21) is well defined and pointwise convergent.

(d) The sequence $\left\{\psi^{(m)}\right\}$ defined by (22), (23) and (24) is well defined and pointwise convergent.

In [14], Li and Yeh have proved that the equation

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha}+\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{n+1}\right)^{\alpha+1} z_n^{\alpha}=0$$

has an eventually positive solution. Thus using Theorem 2, we see that the equation

$$\Delta\left(x_n-x_{n-\tau}\right)^{\alpha}+\tau^{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{1}{n+1}\right)^{\alpha+1} x_{n-\sigma}^{\alpha}=0$$

also has an eventually positive solution.

By means of Theorem 2 and Corollary 1 in [17], we easily establish the following Hille-Wintner type comparison theorem: Suppose that

$$\sum_{k=n}^{\infty} q_{k+1} \leq \sum_{k=n}^{\infty} \overline{q}_{k+1},\ n \geq 0.$$

If the equation

$$\Delta\left(x_n-x_{n-\tau}\right)^{\alpha}+\overline{q}_n x_{n-\overline{\sigma}}^{\alpha}=0$$

has an eventually positive solution, so dose equation (1).

By Theorem 2, the Corollaries and the results of nonexistence and existence of eventually positive solutions for (3) and (17) (see e.g. [11-16]), it is now clear that further existence criteria for (1) can be established.

3. Existence Criteria for Equation (3)

As in the previous section, we begin by considering a more general equation

$$\Delta\left[\Delta\left(x_{n-1}+p_{n-1} x_{n-1-\tau}\right)\right]^{\alpha}+\frac{q_n}{\tau^{\alpha}} f\left(x_{n-\sigma}\right)=0, \tag{25}$$

where $\alpha, \tau, \sigma, p_n, q_n$ satisfy the same assumptions as stated in Section 1, and $f : R \to R$ is a continuous nondecreasing function which satisfies $f(x) > 0$ for all $x > 0$. For the sake of convenience, we will set

$$Q_n=\frac{q_n}{\tau^{\alpha}}.$$

Theorem 3. *Suppose that*

$$0 \leq p_n < 1, \tag{26}$$

$$f(xy) \geq K f(x) f(y) \tag{27}$$

for $x > 0, y > 0$ and some $K > 0$, and let (25) has an eventually positive solution. Then equation

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} + KQ_n f(1 - p_{n-\sigma}) f(z_{n-\sigma}) = 0 \tag{28}$$

also has an eventually positive solution.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (25), say, $x_n > 0$ and $x_{n-\sigma} > 0$ for $n \geq n_0 \geq 0$. Set

$$z_n = x_n + p_n x_{n-\tau}. \tag{29}$$

By the condition (26), we have

$$z_n > 0,\ n \geq n_1 \geq n_0. \tag{30}$$

and from equation (25), it follows that

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} \leq 0,\ n \geq n_1 + 1. \tag{31}$$

We claim that $(\Delta z_n)^{\alpha} > 0$ eventually. Otherwise, since $\{q_n\}$ has a positive subsequence, $(\Delta z_n)^{\alpha} < 0$ and $\Delta z_n < 0$ for $n \geq n_2 \geq n_1$. which together with (31) gives

$$z_{n+1} \leq z_{n_2} + \Delta z_{n_2}(n - n_2 + 1)$$

and hence we see that $\lim_{n\to\infty} z_n = -\infty$. But this contradicts (30). In this case, in view of (25), we have

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} + Q_n f(z_{n-\sigma} - p_{n-\sigma} z_{n-\sigma-\tau}) \leq 0$$

eventually. By (27) and (31), we have

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} + KQ_n f(1 - p_{n-\sigma}) f(z_{n-\sigma}) \leq 0$$

eventually. By Corollary 3, we see that equation (25) has an eventually positive solution. The proof is complete. ■

Corollary 7. *Suppose the conditions (26) and (27) hold and suppose (25) has an eventually positive solution. Then the equation*

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} + KQ_n f(1 - p_{n-\sigma}) f(z_n) = 0$$

has an eventually positive solution.

Corollary 8. *Suppose the condition (26) holds and suppose (2) has an eventually positive solution, then the equation*

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} + Q_n(1 - p_{n-\sigma}) z_n^{\alpha} = 0 \tag{32}$$

also has an eventually positive solution.

Before the following theorems are obtained, we first give a definition. An eventually positive solution $\{x_n\}$ that satisfies $\liminf_{n\to\infty} x_n = \omega > 0$ is called eventually ω-positive.

Theorem 4. *Suppose that (27) hold and $-1 < p^* \le p_n \le 0$. Let $\{x_n\}$ be an eventually ω-positive solution of (25). Then the equation*

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} + KQ_n f(1-p_{n-\sigma}) f(z_{n-\tau-\sigma}) = 0 \tag{33}$$

has an eventually positive solution.

Proof. In view of (25), we see that

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha} = -Q_n f(x_{n-\sigma}) \le 0 \text{ eventually.}$$

Furthermore, we see that $(\Delta z_n)^{\alpha}$ and hence Δz_n and z_n are of one sign eventually.

Consider first the case that $\Delta z_n < 0$ eventually. If $z_n < 0$ for all large n, then there is a positive number β such that $z_n \le -\beta$ for $n \ge n_0 \ge 0$. Hence

$$x_n = -p_n x_{n-\tau} + z_n \le x_{n-\tau} + z_n \le x_{n-\tau} - \beta,\ n \ge n_0.$$

By induction, it is easy to see that

$$x_{k\tau+n_0} \le z_{k\tau+n_0} + x_{(k-1)\tau+n_0} \le z_{k\tau+n_0} + z_{(k-1)\tau+n_0} + x_{(k-2)\bar{\tau}n_0} \le \ldots \le -k\beta + x_{n_0}$$

for $k = 0, 1, 2, \ldots$. Note that by taking k to infinity, the right hand side is negative, which is a contradiction.

We have thus shown that $\Delta z_n > 0$ and $z_n < 0$ eventually. Suppose to the contrary that $z_n < 0$ eventually. Then $\lim_{n\to\infty} z_n = \overline{\beta} \le 0$. However, since

$$x_{n+\tau} - z_{n+\tau} = -p_{n+\tau} x_n \le -p^* x_n,$$

thus by taking limit inferior on both sides, we obtain

$$\omega - \overline{\beta} \le -p^* \omega,$$

which is a contradiction. Thus, we see that $z_n > 0$ eventually.

Now we will prove that $\Delta z_n > 0$ eventually. Suppose to the contrary that $\Delta z_n < 0$ eventually. Then there is some $M \ge 0$ such that $z_n > 0, \Delta z_n < 0$ and $\Delta\left(\Delta z_n\right)^{\alpha} \le 0$ for $n \ge M$. In this case, we have

$$\Delta z_n \le \Delta z_M,\ n \ge M$$

and

$$z_{n+1} \le z_M + \Delta z_M (n - M + 1).$$

Note that by taking n to infinity, the right hand side is negative, which is a contradiction.

In view of (25), we see that

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha}+Q_{n} f\left(z_{n-\sigma}-p_{n-\sigma} x_{n-\tau-\sigma}\right)=0.$$

Since $z_n \leq x_n$ and f is a non-decreasing function, we see that

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha}+Q_{n} f\left(z_{n-\sigma}-p_{n-\sigma} z_{n-\tau-\sigma}\right)=0.$$

Note that $\Delta z_n > 0$ eventually. Then we get that

$$\Delta\left(\Delta z_{n-1}\right)^{\alpha}+K Q_{n} f\left(1-p_{n-\sigma}\right) f\left(z_{n-\tau-\sigma}\right) \leq 0$$

eventually. The completes the proof by Corollary 3. ■

As an immediate corollary of Theorem 2 and Theorem 4 we have the following result: Suppose the conditions of Theorem 4 hold and suppose equation (2) has an eventually ω-positive solution. Then equation (32) has an eventually positive solution.

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