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ON THE CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY RECORD VALUES WITH A RANDOM INDEX

Abstract: We give some characterizations of the exponential distribution based on the distributional properties and the expected values of record values; the index of record values has the geometric distribution.

Key words: record values, Laplace transform, IFRA, DFRA.

1. Introduction

Let X be a nonnegative random variable, and let $F(x) = P(X < x)$ be its distribution function. Let $\overline{F}(x) = 1 - F(x)$ be a survival function corresponding to X.

We say that F has increasing failure rate average ($F \in IFRA$) if$-\frac{1}{x}\ln\overline{F}(x)$ is nondecreasing in $x > 0$. Similarly, F has decreasing failure rate average ($F \in DFRA$) if $-\frac{1}{x}\ln\overline{F}(x)$ is nonincreasing in $x > 0$.
It is known (see [3]) that $F \in IFRA$ if and only if

$$\overline{F}(\alpha x) \geq [\overline{F}(x)]^{\alpha} \quad \text{for all} \quad 0 < \alpha < 1 \quad \text{and} \quad x > 0, \tag{1}$$

and $F \in DFRA$ if and only if

$$\overline{F}(\alpha x) \leq [\overline{F}(x)]^{\alpha} \quad \text{for all} \quad 0 < \alpha < 1 \quad \text{and} \quad x > 0. \tag{2}$$

We say that X is exponentially distributed if

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0, \quad \text{for some} \quad \lambda > 0. \tag{3}$$

We say that v is geometrically distributed if

$$P(v = k) = p(1-p)^{k-1}, \quad k = 1, 2, ..., \quad \text{for some} \quad 0 < p < 1. \tag{4}$$

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed random variables. Define the sequence of record times $(L(n), n \geq 1)$ in the following way $L(1) = 1$, $L(n) = \min\{j : X_j > X_{L(n-1)}\}$, $n \geq 2$.

Then the sequence $(R_n, n \geq 1)$, where $R_n = X_{L(n)}$, is called the sequence of record values of $(X_n, n \geq 1)$.

The following theorem is given in [5] (Theorem 4.5.2, p.129):

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed positive random variables with a continuous distribution function F. Assume that the limit $\lim_{x \to 0^+} \frac{F(x)}{x}$ exists and is finite. Moreover, assume that v is a geometric random variable independent of the sequence $(X_n, n \geq 1)$, and the condition (4) holds. The random variables X_1 and pR_v are identically distributed if and only if F is a distribution function of the exponential law.

Moreover, the following theorem given in [2] (Theorem 8.1, p.63) is valid:

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed nonnegative and nondegenerate random variables with a distribution function F. Assume that v is a geometric random variable independent of the sequence $(X_n, n \geq 1)$, and the condition (4) holds. The random variables X_1 and $p \sum_{j=1}^{v} X_j$ are identically distributed if and only if F is a distribution function of the exponential law.

We can obtain a characterization of the exponential distribution by a property of record R_v for a geometrically distributed v.

2. Results

Theorem 1. *Let $(X_n,\ n \geq 1)$ be a sequence of independent and identically distributed nonnegative random variables with a continuous distribution function $F \in IFRA$. Assume that $\lim_{x \to 0^+} \frac{F(x)}{x} = \lambda$, $0 < \lambda < \infty$. Moreover, assume that v is a geometric random variable independent of the sequence $(X_n, n \geq 1)$, and the condition (4) holds. The random variables $\sum_{i=1}^{v} X_i$ and R_v are identically distributed if and only if F is of the form (3).*

Proof. Let φ_1 and φ_2 be the Laplace transforms of $\sum_{i=1}^{v} X_i$ and R_v, respectively. We have for $s > 0$,

$$\varphi_1(s) = E\left[\exp\left(-s\sum_{i=1}^{v} X_i\right)\right] = \sum_{k=1}^{\infty} p(1-p)^{k-1}[\varphi(s)]^k = \frac{p\varphi(s)}{1 - q\varphi(s)},$$

where $q = 1 - p$, $\varphi(s) = E[\exp(-sX_1)]$. Because

$$F_{R_v}(y) = 1 - \left[\overline{F}(y)\right]^p \qquad \text{for} \quad y > 0 \quad ([5], \text{p.}130), \tag{5}$$

we obtain

$$\varphi_2(s) = E\left(e^{-sR_v}\right) = \int_0^{\infty} e^{-sy} p \left[\overline{F}(y)\right]^{p-1} dF(y) = 1 - s\int_0^{\infty} e^{-sy} \left[\overline{F}(y)\right]^p dy.$$

By virtue of the equality $\phi_1(s) = \phi_2(s)$ (for $s > 0$), we get on simplification

$$-\frac{\varphi(s) - \varphi(0)}{s} \frac{1}{1 - q\varphi(s)} = \int_0^\infty e^{-sy} \left[\overline{F}(y)\right]^p dy. \tag{6}$$

Taking limits of both sides of (6) as s goes to 0+, we have

$$-\varphi'(0)\frac{1}{p} = \int_0^\infty \left[\overline{F}(y)\right]^p dy. \tag{7}$$

Writing $EX_1 = \int_0^\infty \overline{F}(y)dy = -\varphi'(0)$ we get from (7)

$$\int_0^\infty \overline{F}(y)dy = p\int_0^\infty \left[\overline{F}(y)\right]^p dy. \tag{8}$$

Substituting $y = z/p$ in the integral on the right-hand side of (8) we get

$$\int_0^\infty \left\{\overline{F}(y) - \left[\overline{F}\left(\frac{y}{p}\right)\right]^p\right\} dy = 0. \tag{9}$$

Let $F \in IFRA$. Then the inequality (1) holds. Hence

$$\overline{F}(y) - \left[\overline{F}\left(\frac{y}{p}\right)\right]^p \geq 0 \qquad \text{for} \quad y > 0.$$

Therefore

$$\overline{F}(y) = \left[\overline{F}\left(\frac{y}{p}\right)\right]^p \tag{10}$$

for almost all (with respect to the Lebesgue measure) $y > 0$ and a fixed $0 < p < 1$. Since $\lim_{x\to 0+} \frac{F(x)}{x} = \lambda$, $0 < \lambda < \infty$, it follows from (10) that $\overline{F}(x) = \exp(-\lambda x)$, $x > 0$, $\lambda > 0$, (see [5], p.130).

Now suppose that X_1 has distribution function (3). Then from (5) we obtain that $F_{R_v}(y) = 1 - e^{-\lambda p y}$ for $y > 0$, $\lambda > 0$, $0 < p < 1$. It is known ([4], p. 70) that the random variable $\sum_{i=1}^{v} X_i$ is exponentially distributed with the scale parameter $p\lambda$. Therefore R_v and $\sum_{i=1}^{v} X_i$ are identically distributed. ∎

Remark 1. Theorem 1 is also true if the condition "$F \in IFRA$" is replaced by "$F \in DFRA$ and $EX_1 < \infty$". Here in the proof we use the inequality (2).

Theorem 2. *Let $(X_n,\ n \geq 1)$ be a sequence of independent and identically distributed nonnegative random variables with a continuous distribution function F. Assume that $\lim_{x\to 0+} \frac{F(x)}{x} = \lambda$, $0 < \lambda < \infty$. Let*

v_1 and v_2 be two integer-valued random variables distributed independently of the sequence $(X_n,\ n \geq 1)$. Suppose that $P(v_1 = k) = p_1(1-p_1)^{k-1}$, $P(v_2 = k) = p_2(1-p_2)^{k-1}$, $k = 1, 2, \ldots$, $0 < p_1 < 1$, $0 < p_2 < 1$, $p_1 \neq p_2$. The random variables

$$p_1 R_{v_1} \qquad \text{and} \qquad p_2 R_{v_2}$$

are identically distributed if and only if F is of the form (3)*.*

Proof. From (5) we obtain

$$F_{pR_v}(y) \;=\; 1 - \left[\overline{F}\left(\frac{y}{p}\right)\right]^p \qquad \text{for} \quad y > 0. \tag{11}$$

Let $p_1 R_{v_1}$ and $p_2 R_{v_2}$ have the same distribution and $p_1 < p_2$. Then

$$\left[\overline{F}\left(\frac{z}{p_1}\right)\right]^{p_1} \;=\; \left[\overline{F}\left(\frac{z}{p_2}\right)\right]^{p_2} \qquad \text{for} \quad z > 0. \tag{12}$$

Substituting $z = yp_2$ in (12) we get

$$\overline{F}(y) \;=\; \left[\overline{F}\left(\frac{y}{\frac{p_1}{p_2}}\right)\right]^{\frac{p_1}{p_2}} \qquad \text{for} \quad y > 0,$$

i.e. the equation (10) for a fixed $p = p_1/p_2$, $0 < p < 1$. Since $\lim_{x \to 0+} \frac{F(x)}{x} = \lambda$, $0 < \lambda < \infty$, it follows that F is of the form (3). The same holds if $p_1 > p_2$. Now let X_1 be exponentially distributed with distribution function (3). Then from (11) we conclude that the random variables $p_1 R_{v_1}$ and $p_2 R_{v_2}$ have the same distribution function (3). ■

Theorem 3. *Assume that the assumptions of Theorem 2 are satisfied. Let $E(p_i R_{v_i}) < \infty$ for $i = 1, 2$ and $F \in IFRA$ (or $F \in DFRA$). Then X_1 has the distribution function defined in* (3) *if and only if*

$$E(p_1 R_{v_1}) \;=\; E(p_2 R_{v_2}). \tag{13}$$

Proof. If X_1 has the exponential distribution function (3), then

$$E(p_1 R_{v_1}) \;=\; E(p_2 R_{v_2}) \;=\; \frac{1}{\lambda}.$$

Now let us suppose that the condition (13) is satisfied. Because

$$E(p_i R_{v_i}) \;=\; \int_0^\infty \overline{F}_{p_i R_{v_i}}(y)\,dy \qquad \text{for} \quad i = 1, 2,$$

formula (13) can be written as follows

$$\int_0^\infty \left\{ \left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1} - \left[\overline{F}\left(\frac{y}{p_2}\right)\right]^{p_2} \right\} dy \;=\; 0. \tag{14}$$

Let $p_1 < p_2$ and $F \in IFRA$. From (1), for $\alpha = p_1/p_2$, we have

$$\overline{F}\left(\frac{p_1}{p_2} z\right) \;\geq\; \left[\overline{F}(z)\right]^{\frac{p_1}{p_2}}, \qquad z > 0.$$

Substituting $y = p_1 z$ in the above inequality we obtain

$$\left[\overline{F}\left(\frac{y}{p_2}\right)\right]^{p_2} \;\geq\; \left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1} \qquad \text{for} \quad y > 0.$$

Therefore $[\overline{F}(y/p_1)]^{p_1} - [\overline{F}(y/p_2)]^{p_2}$ does not change sign. From (14) we obtain

$$\left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1} \;=\; \left[\overline{F}\left(\frac{y}{p_2}\right)\right]^{p_2} \qquad \text{for almost all} \quad y > 0.$$

Next, analogously as in the proof of Theorem 2, we get that F has the form (3). If $F \in DFRA$, then in the proof we use (2). ■

Theorem 4. *Assume that the assumptions of Theorem 2 are satisfied. Then X_1 has the distribution function defined in* (3) *if and only if*

$$r_{p_1 R_{v_1}}(y) \;=\; r_{p_2 R_{v_2}}(y) \qquad \text{for} \quad y > 0, \tag{15}$$

where r is the failure rate.

Proof. By formula (11) we get

$$\overline{F}_{p_1 R_{v_1}}(y) \;=\; \left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1} \qquad \text{for} \quad y > 0.$$

Hence the density function of $p_1 R_{v_1}$ is of the form

$$f_{p_1 R_{v_1}}(y) \;=\; \left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1 - 1} f\left(\frac{y}{p_1}\right) \qquad \text{for} \quad y > 0,$$

and the failure rate

$$r_{p_1 R_{v_1}}(y) \;=\; \frac{f\left(\frac{y}{p_1}\right)}{\overline{F}\left(\frac{y}{p_1}\right)} \qquad \text{for} \quad y > 0.$$

The condition (15) can be written as

$$\frac{f\left(\frac{y}{p_1}\right)}{\overline{F}\left(\frac{y}{p_1}\right)} = \frac{f\left(\frac{y}{p_2}\right)}{\overline{F}\left(\frac{y}{p_2}\right)}, \qquad y > 0,$$

or equivalently by derivative

$$\left(-\ln\left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1}\right)' = \left(-\ln\left[\overline{F}\left(\frac{y}{p_2}\right)\right]^{p_2}\right)', \qquad y > 0.$$

Hence

$$-\ln\left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1} = -\ln\left[\overline{F}\left(\frac{y}{p_2}\right)\right]^{p_2} + C, \qquad y > 0. \tag{16}$$

Taking limits of both sides of (16) as y goes to $0+$, we have $C = 0$. Then

$$\left[\overline{F}\left(\frac{y}{p_1}\right)\right]^{p_1} = \left[\overline{F}\left(\frac{y}{p_2}\right)\right]^{p_2} \qquad \text{for} \quad y > 0.$$

Next, analogously as in the proof of Theorem 2, we get that F has the form (3).

If X_1 has the distribution function (3), then the random variables $p_1 R_{v_1}$ and $p_2 R_{v_2}$ are identically distributed. Therefore the equality (15) is true. ■

Note that the characterization of the exponential distribution by the distributional properties of the random variable $vX_{1:v}$, where v has the geometric distribution, was considered by Ahsanullah [1].

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