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STABILITY OF GENERALIZED FUNCTIONAL-DIFFERENTIAL EQUATIONS

Abstract: The basic idea of this paper is to use Lyapunov functions to show that the trivial solution functional-differential equations of the form $\dot{x}(t) \in F(t, x_t)$ is stable and is asymptotically stable, and that of perturbed functional-differential equations of the form $\dot{x}(t) \in F(t, x_t) + G(t, x_t)$ is asymptotically stable.

Key words: multifunction, stability theory.

1. Introduction

Many papers have been devoted to the stability problem of functional-differential equations. For example, in the papers ([1], [5], [7]) stability problems of functional-differential equations in the form of $\dot{x}(t) = f(t, x_t)$ are investigated, where x_t denotes a mapping of $[-r, 0]$ into R^n defined by the formula $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$ where $x : [-r, T] \to R^n$, $r \geq 0$, $T > 0$.

The aim of this paper is to give some results concerning stability and asymptotic stability of the zero solution of functional-differential relations of the form

$$\dot{x}(t) \in F(t, x_t) \tag{1}$$

where F is a multivalued mapping with nonempty compact values in R^n and asymptotic stability of perturbed functional-differential equations of the form

$$\dot{x}(t) \in F(t, x_t) + G(t, x_t) \tag{2}$$

where perturbation G is a multivalued mapping with nonempty compact values in R^n.

The results obtain in this paper generelize some stability problems dealing with functional-differential equations in the form of $\dot{x}(t) = f(t, x_t)$ and inclusions of the form $\dot{x}(t) \in F(t, x_t) + G(t, x_t)$ that have been investigated by M.M. Hapajew, O.P. Filatow ([3]), M.A. Bouduries and J. Schinas ([2]), respectively.

2. Notations and Definitions

Let $r \geq 0$ and $T > 0$, and let C_T, C_0 denote the Banach spaces of all the continuous mappings of $[-r, T]$ and $[-r, 0]$, respectively, into R^n with the usual norms $\|x\| = \sup_{-r \leq t \leq T} |x(t)|$ and $|\!|x|\!| = \sup_{-r \leq t \leq 0} |x(t)|$ for $x \in C_T$ and $x \in C_0$, respectively, where $|\cdot|$ denotes the Eucdidean norm of n-dimensional linear vector space R^n.

Let L denote the Banach space of (equivalent classes of) Lebesgue integrable function defined on the interval $[0, \infty)$ into R. For every fixed $t \in [0, T]$ let x_t denote a mapping of $[-r, 0]$ into R^n defined by the formula $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$ where $x : [-r, T] \to R^n$ is given. Moreover, let Ω denote a collection of all nonempty compact subsets of the n-dimensional Eucdidean space R^n with the Hausdorff metric H defined by the formula

$$H(A, B) = \min\{\varepsilon > 0 : A \subset S_\varepsilon(B), \quad B \subset S_\varepsilon(A)\}$$

where $S_\varepsilon(U)$ is closed ε-neighbourhood of compact $U \in R^n$.

Throughout this paper we assume that the multivalued mapping $F : [0, \infty) \times C_0 \to \Omega$ satisfies the following conditions:

(i) $F(\cdot, u) : [0, \infty) \to \Omega$ is measurable for every fixed $u \in C_0$

(ii) $F(t, \cdot) : C_0 \to \Omega$ is Lipschitzean for every fixed $t \in [0, T]$, that is there exists a Lebesgue integrable function $K : [0, T] \to R^+$ such that $H(F(t, u),\ F(t, \overline{u})) \leq K(t) |\!|u - \overline{u}|\!|$, where H is the Hausdorff metric defined in Ω

(iii) There exists a Lebesqgue integrable function $m : [0, \infty) \to R^+$ such that $H(F(t, u), \{0\}) \leq m(t)$ for almost every $t \in [0, \infty)$ and any $u \in C_0$.

(iv) $0 \in F(t, \hat{0})$, where $\hat{0}$ is the null of C_0.

We state the following initial valued problem for (1):

$$(3) \qquad \begin{cases} \dot{x}(t) \in F(t, x_t) \ \text{ for almost every } \ t \in [0, \infty) \\ x(t) = \varphi(t) \ \text{ for } \ t \in [-r, 0] \end{cases}$$

where $\varphi \in C_0$ is a given absolutely continuous function.

An absolutely continuous function $x \in C_{[-r,\infty)}$ satisfying the above conditions in each closed interval $[-r, T] \subset [-r, \infty)(T > 0)$ will be called a solution of an initial value problem (3) on $[-r, \infty)$. In virtue of ([6]) the problem (3) has at least one solution.

By an absolutely continuous function on $[0, \infty)$ we mean a function that is absolutely continuous on every compact subinterval of $[0, \infty)$.

For any $t_0 \geq 0$ let $x(\cdot; t_0, \varphi)$ denote a solution of (3).

A trivial solution of the initial value problem (3) corresponding to $\varphi \equiv 0$ will be called:

(a) stable, if for each $\varepsilon > 0$ and any $t_0 \geq 0$ there exists $\sigma = \sigma(\varepsilon, t_0) > 0$ such that $\|\varphi\| < \sigma$, $\varphi \in C_0$ implies that $|x(t; t_0, \varphi)| < \varepsilon$, $t \geq t_0$ where $|\cdot|$ denotes the Euclidean norm in R^n,

(b) uniformly stable, if for each $\varepsilon > 0$ there exists a number $\sigma(\varepsilon) > 0$ such that the relation $t \geq t_0 \geq 0$ and $\varphi \in C_0$ then $\|\varphi\| < \sigma(\varepsilon)$ implies that $|x(t; t_0, \varphi)| < \varepsilon$. Notice that above σ can be chosen independently of t_0,

(c) asymptotically stable, if it is stable and, furthermore, there is $\sigma = \sigma(t_0) > 0$ such that for all $\varphi \in C_0$ satisfying $\|\varphi\| < \sigma$ we have $\lim_{t\to\infty} |x(t, t_0, \varphi)| = 0$.

Let us denote by K the class of all continuous, monotonically increasing real function on $[0, \infty)$ vanishing at 0.

We assume that there exist a continuous function $V : [0, \infty) \times C_0 \to R$ (called a Lyapunov function) with the following properties:

(h_i) $V(t, \hat{0}) = 0$ for $t \in [0, \infty)$

(h_{ii}) $V(t, \varphi) \geq \lambda_1(\|\varphi\|)$ for some $\lambda_1 \in K$, $t \geq 0$, $\varphi \in C_0$,

(h_{iii}) $V(t, \varphi) \leq \lambda_2(\|\varphi\|)$ for some $\lambda_2 \in K$, $t \geq 0$, $\varphi \in C_0$,

(h_{iv}) $V^*(t) = V(t, x(\cdot; t_0, \varphi)_t)$ is absolutely continous for each solution $x(\cdot; t_0, \varphi)$ of (3), where $x(\cdot; t_0, \varphi)_t(\theta) = x(t + \theta, t_0, \varphi)$ for $\varphi \in C_0$, $t_0 \geq 0$, and $\theta \in [-r, 0]$

(h_v) $\frac{d}{dt} V^*(t) = \lim\limits_{h\to 0^+} \frac{1}{h}(V(t+h, x(\cdot; t_0, \varphi)_{t+h}) - V(t, x(\cdot; t_0, \varphi)_t) \leq -\lambda_3(\|\varphi\|)$ for some $\lambda_3 \in K$, $\varphi \in C_0$.

3. Stability Theorems

Now we start presentation of some theorems concerning the stability of the trivial solution of (3). We will assume that $x(t) \equiv 0$ is the unique solution of (3).

Furthermore the solution $x(\cdot; t_0, \varphi)$ of (3) will be denote by $x(\cdot; \varphi)$. The first of the theorem concerning stability of the null solution (3) can be proved by the same technique as in the single-valued case [see [4] p.15].

Theorem 1. *Let $F : [0, \infty) \times C_0 \to \Omega$ satisfy the conditions* (i)–(iv) *and suppose that V is a Lyapunov function such that the conditions* (h_i), (h_{ii}) *and* (h_{iv}) *are satisfied. Then, a trivial solution of* (3) *is stable.*

Sketch of the proof.

1° Take arbitrary $\varepsilon > 0$. By the hypothes (h_{ii}) we have $V(t, \varphi) \geq \lambda_1(\|\varphi\|) = \lambda_1(\varepsilon) > 0$ for each $\varphi \in S_\varepsilon$ and $t \geq t_0 = 0$ where $S_\varepsilon = \{\varphi \in C_0 : \|\varphi\| = \varepsilon\}$.

2° Denote by $\alpha = \inf\{V(t,\varphi) : t \geq t_0 = 0, \varphi \in S_\varepsilon\} > 0$, $\alpha > 0$. For V is continuous with respect φ so we have $0 \leq V(0,\varphi) < \alpha$ for each $\varphi \in C_0$ such that $\|\varphi\| < \sigma$.

3° Let $x(\cdot,\varphi)$ be a solution of (3) corresponding to φ satisfying $\|\varphi\| < \sigma$. There we show that for every $t \geq t_0 = 0$ we have $|x(t,\varphi)| < \varepsilon$.

The proof this fact we use the method contradiction and we get that $V(0,\varphi) \geq \alpha$.

Theorem 2. *Suppose $F : [0,\infty) \times C_0 \to \Omega$ satisfies the conditions* (i)–(iv) *and let V be a Lyapunov function such that the hypoteses* (h_i)-(h_v) *are satisfied. Then the trivial solution of* (3) *is asymptotically stable.*

Proof. Let us observe that, by Theorem 1. the trivial solution of (3) is stable. Hence, it suffices to show that $\lim_{t\to\infty} |x(t,\varphi)| = 0$ for each $\varphi \in C_0$ satisfying $\|\varphi\| < \sigma(\varepsilon)$, where $\sigma(\varepsilon)$ is a number determinated in the definition of the stability.

We have

$$\lambda_1(\|\varphi\|) \leq V(t,\varphi) \leq \lambda_2(\|\varphi\|) \quad \text{for} \quad \lambda_1,\lambda_2 \in K, \;\; t \geq 0, \;\; \varphi \in C_0.$$

Take arbitrary $\eta > 0$. By continuity of λ_2 and with fact that $\lambda_2(0) = 0$, it follows that there exists a number $\sigma_1(\eta) < \eta$ such that $\lambda_2(\|\varphi\|) < \lambda_1(\eta)$ for each $\varphi \in C_0$ satisfying $\|\varphi\| < \sigma_1(\eta)$. For each $\varphi \in C_0$ such that $\|\varphi\| = \eta$ we obtain $\lambda_1(\eta) = \lambda_1(\|\varphi\|) \leq V(t,\varphi)$ for $t \geq 0$.

Thus $\lambda_1(\eta) \leq \inf\{V(t,\varphi) : \varphi \in C_0, \|\varphi\| = \eta, t \geq 0\}$. Hence, it follows

$$V(t,\varphi) \leq \lambda_2(\|\varphi\|) < \lambda_1(\eta) \leq \inf_{\|\varphi\|=\eta} \{V(t,\varphi) : t \geq 0\}$$

for $t \geq 0$ and $\|\varphi\| = \eta$.

Now, we prove that there exists $t^* > 0$ such that for each $t > t^*$ there holds the estimation $\|x(\cdot,\varphi)\| < \eta$ for every $\varphi \in C_0$ satisfying $\|\varphi\| < \sigma(\varepsilon)$ ($\eta > 0$ is given above).

Let us observe that there exists $t^* > 0$ such that $\|x(\cdot,\varphi)_{t^*}\| < \sigma_1(\eta)$, where $\|x(\cdot,\varphi)_{t^*}\| = \sup_{-r\leq\theta\leq 0} |x(t^* + \theta,\varphi)|$.

Indeed, if is not true, then, for each $t \geq 0$ it will be $\|x(\cdot,\varphi)_t\| \geq \sigma_1(\eta)$. By the stability of the trivial solution of (3) it follows that $\|\varphi\| < \sigma(\varepsilon)$ implies $\sigma_1(\eta) \leq \|x(\cdot,\varphi)_t\| \leq \bar{\varepsilon}$, where $0 < \bar{\varepsilon} < \varepsilon$. In virtue of the hypotheses (h_v) there exists a function $\lambda_3 \in K$ such that for all $\varphi \in C_0$ and $t \geq 0$ we have $\frac{d}{dt}V(t,x(\cdot,\varphi)_t) \leq -\lambda_3(\|\varphi\|)$. But $\|x(\cdot,\varphi)_t\| \in [\sigma_1(\eta),\bar{\varepsilon}]$ and λ_3 are continuous. Then $\alpha = \inf \lambda_3(\|x(\cdot,\varphi)_t\|) > 0$ where inf is taken after all $\|x(\cdot,\varphi)_t\| \in [\sigma_1(\eta),\bar{\varepsilon}]$ and by the definition λ_3 is reach in the point $\sigma_1(\eta)$.

Thus $\|x(\cdot,\varphi)_t\| \in [\sigma_1(\eta),\bar{\varepsilon}]$ implies that $-\lambda_3(\|x(\cdot,\varphi)_t\|) \leq -\alpha$ for each $t \geq 0$.

Hence, and by virtue of the hypothese (h_v) we have $V(t, x(\cdot, \varphi)_t) - V(0, \varphi) \leq -\alpha t$ for $t \geq 0$ and $\| x(\cdot, \varphi)_t \| \in [\sigma_1(\eta), \bar{\varepsilon}]$

Therefore $V(t, x(\cdot, \varphi)_t) < 0$ for $\| x(\cdot, \varphi)_t \| \in [\sigma_1(\eta), \bar{\varepsilon}]$ and $t > \frac{1}{\alpha} V(0, \varphi)$. It contradicts the property (h_{ii}) of function V. Hence there exists a point $t^* < \frac{1}{\alpha} V(0, \varphi)$ such that $\| x(\cdot, \varphi)_{t^*} \| < \sigma_1(\eta) < \eta$.

Let us observe that there exists $t_1 > t^*$ such that $|x(t_1, \varphi)| = \eta$ and $|x(t, \varphi)| < \eta$ for $t \in [t^*, t_1)$.

In a similar way as in the proof of Theorem 1. we can obtain a contradiction.

Therefore for every $t_1 \geq t^*$ we get $|x(t, \varphi)| < \eta$.

The proof this complete. ■

4. Stability of Perturbed Multivalued Differential Equations

We shall be concerned with the perturbed multivalued differential equations of the form

$$\dot{x}(t) \in F(t, x_t) + G(t, x_t) \tag{4}$$

where the perturbation $G : [0, \infty) \times C_0 \to \Omega$ satisfies the following condition: $G(\cdot, \cdot) \to \Omega$ is an upper semi-continuous for $(t, u) \in [0, \infty) \times C_0$.

Let $\langle \cdot, \cdot \rangle$ stand for the inner product in R^n. Given $z \in R^n$ and a bounded subset A of R^n, we define $(z, A) = \sup\{(z, y) : y \in A\}$. From now on, we assume that the Lyapunov function $V : [0, \infty) \times R^n \to R$ is continuous on the $[0, \infty) \times R^n$ and $V(t, 0) = 0$ for all $t \in [0, \infty)$. Following ([4], p.22, Th. 1.4) and using the hypothesis (h_i), (h_{ii}) and conditions

$$(\mathrm{h}_{vi}) \qquad \frac{d}{dt} V^*(t) \leq -\lambda_3(V(t, x)) \quad \text{for some} \quad \lambda_3 \in K, \ (t, x) \in [0, \infty) \times R^n$$

the asymptotic stability of the null solution of (3) can be proved by the same technique as in the single – valued case. We consider the following hypotheses

(h_1) $V(t, x) \geq \lambda_1(|x|)$ for $\lambda_1 \in K$, $(t, x) \in [0, \infty) \times R^n$,

(h_2) $\frac{\partial V(t,x)}{\partial t} + \left\langle \frac{\partial V(t,x)}{\partial x}, F(t, x_t) \right\rangle \leq -\lambda_3(V(t, x))$ for $\lambda_3 \in K$, $(t, x) \in [0, \infty) \times R^n$ where we mean $V'(t, x_t) = \frac{\partial V(t,x)}{\partial t} + \left\langle \frac{\partial V(t,x)}{\partial x}, F(t, x_t) \right\rangle$ for inclusion (3), of course $V' : [0, \infty) \times C_0 \to R$,

(h_3) $\| x_t \| \left| \frac{\partial V(t,x)}{\partial x} \right| \leq \lambda_2(V(t, x))$ for $\lambda_2 \in K$, $(t, x) \in [0, \infty) \times R^n$.

Now we will present a theorem concerning stability of the trivial solution of (4)

Theorem 3. *Suppose that* (h_1), (h_2), (h_3) *hold, where* $\lambda_1, \lambda_2, \lambda_3 \in K$ *and there exists* $\rho \neq 0$ *such that for* $\lambda_2(l) \leq \rho\lambda_3(l)$ *for every* $l \geq 0$. *If* $H(G(t,u),\{0\}) \leq \gamma \| u \|$ *for all* $(t,u) \in [0,\infty) \times C_0$ *and* $\gamma < \rho^{-1}$ *then the null solution* (4) *is asymptotically stable.*

Proof. In virtue of the hypotheses (h_1), (h_2) and (h_3) we have

$$\begin{aligned}
\frac{d}{dt}V(t,x_t) &= \frac{\partial V(t,x)}{\partial t} + \left\langle \frac{\partial V(t,x)}{\partial x}, \dot{x}(t) \right\rangle \\
&\leq \frac{\partial V(t,x)}{\partial t} + \left\langle \frac{\partial V(t,x)}{\partial x}, F(t,x_t) \right\rangle + \left\langle \frac{\partial V(t,x)}{\partial x}, G(t,x_t) \right\rangle \\
&\leq -\lambda_3(V(t,x)) + \left\langle \frac{\partial V(t,x)}{\partial x}, G(t,x_t) \right\rangle \\
&\leq -\lambda_3(V(t,x)) + \left| \frac{\partial V(t,x)}{\partial x} \right| |G(t,x_t)| \\
&\leq -\lambda_3\left(V(t,x)\right) + \left| \frac{\partial V(t,x)}{\partial x} \right| \gamma \| x_t \| \\
&\leq -\lambda_3\left(V(t,x)\right) + \lambda_2\left(V(t,x)\right)\gamma \\
&\leq -\lambda_3\left(V(t,x)\right) + \rho\gamma\lambda_3\left(V(t,x)\right) \\
&= (\gamma\rho - 1)\lambda_3\left(V(t,x)\right) = (1-\gamma\rho)\lambda_3\left(V(t,x)\right)
\end{aligned}$$

Since $1 - \gamma\rho > 0$ there is

$$\frac{d}{dt}V(t,x_t) \leq -\lambda_4(V(t,x)), \quad \text{where} \quad \lambda_4 = (1-\gamma\rho)\lambda_3 \in K. \tag{5}$$

Now we can proceed as in the proof given in [4] (p. 22, Th. 1.4) in case of the equation $\frac{dx}{dt} = f(t,x)$. The condition (h_{iv}) holds true, so by inequality (5) it follows that the null solution (4) is asymptoticaly stable. This completes the proof. ∎

Now we give an example where the zero solution of functional-differential equation of the form $\dot{x}(t) = F(t,x_t)$ is asymptotical stable.

Consider the scalar equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x(t-r) \tag{$*$}$$

where $a(t)$ and $b(t)$ are continuous functions, $0 < a \leq a(t) < \infty$, $|b(t)| \leq b < \frac{4}{3}a$.

One can choose $V(t,x_t) = \frac{1}{2}x^2(t)$, $\lambda_1(|x(t)|) = \frac{1}{4}x^2(t)$, $\lambda_2(|x(t)|) = x^2(t)$. Furthermore for $t \geq 0$, $r \geq 0$ we have $x(t-r) \leq \| x_t \|$ and let $\frac{1}{2}x^2(t) = \| x_t \|^2$. Then

$$\begin{aligned}V'(t,x_t) &= x(t)x'(t) = [-a(t)x(t) + b(t)x(t-r)]x(t)\\ &= -a(t)x^2(t) + b(t)x(t)x(t-r)\\ &\leq -ax^2(t) + \frac{b}{2}[x^2(t) + x^2(t-r)] \leq -(a - \frac{b}{2})x^2(t) + \frac{b}{2}\|x_t\|^2\\ &= -\frac{1}{2}(2a-b)x^2(t) + \frac{b}{2}\|x_t\|^2 \leq -(2a - \frac{3}{2}b)\|x_t\|^2 < 0.\end{aligned}$$

We can see that the condition of the Theorem 2 are satisfied. Therefore, the zero solution of $(*)$ is asymptotically stable.

References

[1] ANDRIEJEW A.C., Ob ustojcziwosti nieawtonojanowo funkcjonalno–differencialnowo urawnienija, *Doklady akademii nauk,* 356(2)(1997), 151–153.
[2] BOUDURIES M.A., SCHINAS J., On the application of Lyapunov's second method to multivalued differential equations, *Rev. Roum. Math. Pures Et Appl.,* t. XXVIII, 10(1983), 949–952.
[3] HAPAJEW M.M., FILATOW O.P., *Ob ustojcziwosti differencijalnych wkliuczienij s mnogoznacznymi wozmuszczenijami,* Matimaticzieskije zamietki, 43(3) (1988), 205-211.
[4] HALANAY A.H., *Differential Equations,* cademic Press, New York - San Francisko-London, (1966).
[5] IGNAIYEV A.O., *On the asymptotic stability in functional differential equations,* Proceedings of the American Mathematical Society, 127(6)(1999), 1753–1761.
[6] JANIAK T., ŁUCZAK-KUMOREK E., KISIELEWICZ M., Existence theorem for functional differential relations, *Demonstratio Math.,* 12(2)(1979), 411–420.
[7] KNIAZISZCZE L.B., SZAWIEL H.A., Niemonotonnyje funkcjonały Lapunowa i ocienki rieszenij differencialnych urawnienij s zapazdywaniem, *Differencialnyje urawnienija,* 33(2)(1997), 205–211.

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