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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

Abstract: This paper is concerned with the impulsive delay differential equations with positive and negative coefficients

$$\begin{cases} x'(t) + p(t)x(t-\tau) - q(t)x(t-\sigma) = 0, \ t \geq t_0, \ \ t \neq t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\left(\int_{t_k-\tau}^{t_k} p(s+\tau)x(s)ds \right. \\ \qquad \left. - \int_{t_k-\sigma}^{t_k} q(s+\sigma)x(s)ds\right), \qquad k = 1, 2, 3, \cdots . \end{cases}$$

Sufficient conditions are obtained for every solution of the above equation tends to a constant as $t \to \infty$.

Key words: asymptotic behavior, Liapunov functional, delay differential equation, impulse, coefficients.

1. Introduction and and Preliminaries

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses but also represents a more natural framework for mathematical modelling of many real-world phenomena [1,2]. In recent years, there is increasing interest on the oscillation and stability theory of impulsive delay differential equations(see [3-10] and the references cited therein) and many results are obtained. However, the asymptotic behavior of solutions of impulsive delay differential equations is developing comparatively slowly [5,9].

In this paper, we consider the asymptotic behavior of solutions of the impulsive delay differential equation with positive and negative coefficients of the form

$$(1) \qquad \begin{cases} x'(t) + p(t)x(t-\tau) - q(t)x(t-\sigma) = 0, \ \ t \geq t_0, \ \ t \neq t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\left(\int_{t_k-\tau}^{t_k} p(s+\tau)x(s)ds \right. \\ \qquad \left. - \int_{t_k-\sigma}^{t_k} q(s+\sigma)x(s)ds\right), \qquad k = 1, 2, 3, \cdots , \end{cases}$$

where $\tau, \sigma \in [0,\infty)$, $p(t), q(t) \in C([t_0,\infty),[0,\infty))$, $t_0 < t_k < t_{k+1}$ with $\lim_{k\to\infty} t_k = \infty$, and b_k, $k = 1,2,\cdots$, are constants, $x(t_k^-)$ denotes the left-hand limit of $x(t)$ at $t = t_k$.

With equation (1), one associates an initial condition of the form

$$x_{t_0} = \varphi(s),\ s \in [-\rho, 0],\ \rho = \max\{\tau,\sigma\}, \tag{2}$$

where $x_{t_0} = x(t_0 + s)$ for $-\rho \le s \le 0$ and $\varphi \in C([-\rho,0],R)$.

A function $x(t)$ is said to be a solution of equation (1) satisfying the initial value condition (2) if $x(t)$ is defined on $[t_0-\rho,\infty)$ and satisfies

(i) $x(t) = \varphi(t-t_0)$ for $t_0 - \rho \le t \le t_0$, $x(t)$ is continuous for $t \ge t_0$ and $t \ne t_k$ $(k = 1,2,3,\cdots)$;

(ii) $x(t)$ is continuously differentiable for $t \ge t_0$, $t \ne t_k$, and $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^+) = x(t_k)$ for $k = 1,2,3\cdots$;

(iii) $x(t)$ satisfies (1).

Using the methods of steps as in the case without impulses, one can show the global existence and uniqueness of the solution of the initial value problem (1) and (2).

As is customary, a solution of (1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. Main Results

Theorem 1. *Assume that the following conditions hold:*

$$\tau \ge \sigma; \tag{3}$$

$$p^*(t) = p(t) - q(t+\sigma-\tau) > 0 \quad \text{for} \quad t \ge T_1 = t_0 + \tau - \sigma; \tag{4}$$

$$\lim_{t\to\infty} \int_{t-\tau}^{t-\sigma} q(s+\sigma)ds \ = \ \mu < 1; \tag{5}$$

$$\limsup_{t\to\infty} \left(\int_{t-\tau}^{t+\tau} p^*(s+\tau)ds + \frac{q(t+\sigma)}{p^*(t+\tau)} \int_{t-\tau}^{t} p^*(s+2\tau)ds + \int_{t-\tau}^{t-\sigma} q(s+\sigma)ds \right) \ < \ 2; \tag{6}$$

$$0 < b_k \le 1 \quad \text{for} \quad k = 1,2,3,\cdots \quad \textit{and} \quad \sum_{k=1}^{\infty}(1-b_k) < \infty. \tag{7}$$

Then every solution of (1) *tends to a constant as* $t \to \infty$.

Proof. Let $x(t)$ be any solution of (1). We shall prove that the limit $\lim_{t\to\infty} x(t)$ exists and is finite. For this purpose, we rewrite (1) in the form

$$(8)\qquad \begin{cases} [x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds - \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds]' \\ \qquad\qquad + p^*(t+\tau)x(t) \;=\; 0, \qquad t \ge t_0, \quad t \ne t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\left(\int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds\right. \\ \qquad\qquad \left. + \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds\right), \quad k = 1,2,3,\cdots. \end{cases}$$

Now we introduce three functionals as

$$V_1(t) = \left[x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds - \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds\right]^2,$$

$$V_2(t) = \int_{t-\tau}^{t} p^*(s+2\tau)\int_s^t p^*(u+\tau)x^2(u)duds,$$

and

$$V_3(t) = \int_{t-\tau}^{t} p^*(s+2\tau)\int_s^t q(u+\sigma)x^2(u)duds.$$

In what follows, for the sake of convenience, when we write a functional inequality without specifying its domain of validity, we mean that it holds for all sufficiently large t.

As $t \ne t_k$, calculating $\frac{dV_1}{dt}$, $\frac{dV_2}{dt}$ and $\frac{dV_3}{dt}$ along the solution of (1), we have

$$(9)\qquad \begin{aligned} \frac{dV_1}{dt} &= 2\left[x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds - \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds\right] \\ &\quad \times(-p^*(t+\tau)x(t)) \\ &= -p^*(t+\tau)\left[2x^2(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)(2x(t)x(s))ds\right. \\ &\quad \left. - \int_{t-\tau}^{t} p^*(s+\tau)(2x(t)x(s))ds\right] \\ &\le -p^*(t+\tau)\left[2x^2(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x^2(t)ds\right. \\ &\quad - \int_{t-\tau}^{t} p^*(s+\tau)x^2(t)ds \\ &\quad \left. - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x^2(s)ds - \int_{t-\tau}^{t} p^*(s+\tau)x^2(s)ds\right], \end{aligned}$$

$$
\begin{aligned}
\frac{dV_2}{dt} &= \frac{d}{dt}\left(\int_{t-\tau}^{t}\left(p^*(s+2\tau)\int_s^t p^*(u+\tau)x^2(u)du\right)ds\right) \\
&= \int_{t-\tau}^{t}\frac{\partial}{\partial t}\left(p^*(s+2\tau)\int_s^t p^*(u+\tau)x^2(u)du\right)ds \\
&\quad + p^*(t+2\tau)\int_t^t p^*(u+\tau)x^2(u)du \\
&\quad - p^*(t+\tau)\int_{t-\tau}^{t} p^*(s+\tau)x^2(s)ds \\
&= p^*(t+\tau)x^2(t)\int_{t-\tau}^{t} p^*(s+2\tau)ds \\
&\quad - p^*(t+\tau)\int_{t-\tau}^{t} p^*(s+\tau)x^2(s)ds,
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
\frac{dV_3}{dt} &= q(t+\sigma)x^2(t)\int_{t-\tau}^{t} p^*(s+2\tau)ds \\
&\quad - p^*(t+\tau)\int_{t-\tau}^{t} q(s+\sigma)x^2(s)ds \\
&\le q(t+\sigma)x^2(t)\int_{t-\tau}^{t} p^*(s+2\tau)ds \\
&\quad - p^*(t+\tau)\int_{t-\tau}^{t-\sigma} q(s+\sigma)x^2(s)ds,
\end{aligned} \tag{11}
$$

Let $V(t) = V_1(t) + V_2(t) + V_3(t)$, then, by (9)-(11), we get

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt} \\
&\le -p^*(t+\tau)\Bigg[2x^2(t) - x^2(t)\int_{t-\tau}^{t+\tau} p^*(s+\tau)ds \\
&\quad - \frac{q(t+\sigma)}{p^*(t+\tau)}x^2(t)\int_{t-\tau}^{t} p^*(s+2\tau)ds \\
&\quad - x^2(t)\int_{t-\tau}^{t-\sigma} q(s+\sigma)ds\Bigg] \\
&= -p^*(t+\tau)x^2(t)\Bigg[2 - (\int_{t-\tau}^{t+\tau} p^*(s+\tau)ds \\
&\quad + \frac{q(t+\sigma)}{p^*(t+\tau)}\int_{t-\tau}^{t} p^*(s+2\tau)ds \\
&\quad + \int_{t-\tau}^{t-\sigma} q(s+\sigma)ds)\Bigg], \quad t \ne t_k.
\end{aligned} \tag{12}
$$

As $t = t_k$, we have

$$
\begin{aligned}
(13)\quad V(t_k) &= \left[x(t_k) - \int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds - \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds\right]^2 \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} p^*(u+\tau)x^2(u)duds \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} q(u+\sigma)x^2(u)duds \\
&= [b_k x(t_k^-) + (1-b_k)\left(\int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds\right. \\
&\quad \left. + \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds\right) \\
&\quad - \int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds - \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds]^2 \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} p^*(u+\tau)x^2(u)duds \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} q(u+\sigma)x^2(u)duds \\
&= b_k^2\left[x(t_k^-) - \int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds - \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds\right]^2 \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} p^*(u+\tau)x^2(u)duds \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} q(u+\sigma)x^2(u)duds \\
&\le \left[x(t_k^-) - \int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds - \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds\right]^2 \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} p^*(u+\tau)x^2(u)duds \\
&\quad + \int_{t_k-\tau}^{t_k} p^*(s+2\tau)\int_s^{t_k} q(u+\sigma)x^2(u)duds \\
&= V(t_k^-),
\end{aligned}
$$

which, together with (6) and (12), implies

$$p^*(t+\tau)x^2(t) \in L^1(t_0, \infty),$$

and, hence, for any $\eta \geq 0$ we have

$$\lim_{t\to\infty} \int_{t-\eta}^{t} p^*(s+\tau)x^2(s)ds = 0. \tag{14}$$

By (6) and (14), we have

$$\begin{aligned} V_2(t) &= \int_{t-\tau}^{t} p^*(s+2\tau) \int_{s}^{t} p^*(u+\tau)x^2(u)duds \\ &\leq \int_{t-\tau}^{t} p^*(s+2\tau) \int_{t-\tau}^{t} p^*(u+\tau)x^2(u)duds \\ &= \int_{t}^{t+\tau} p^*(s+\tau) \int_{t-\tau}^{t} p^*(u+\tau)x^2(u)duds \\ &\leq 2\int_{t-\tau}^{t} p^*(u+\tau)x^2(u)du \to 0, \text{ as } t\to\infty, \end{aligned}$$

and

$$\begin{aligned} V_3(t) &= \int_{t-\tau}^{t} \left(p^*(s+2\tau) \int_{s}^{t} q(u+\sigma)x^2(u)du \right) ds \\ &= \int_{t-\tau}^{t} \left(\int_{t-\tau}^{u} p^*(s+2\tau)q(u+\sigma)x^2(u)ds \right) du \\ &= \int_{t-\tau}^{t} \left(q(u+\sigma) \int_{t-\tau}^{u} p^*(s+2\tau)ds \right) x^2(u)du \\ &\leq \int_{t-\tau}^{t} \left(q(u+\sigma) \int_{u-\tau}^{u} p^*(s+2\tau)ds \right) x^2(u)du \\ &\leq \int_{t-\tau}^{t} 2p^*(u+\tau)x^2(u)du \to 0, \text{ as } t\to\infty. \end{aligned}$$

On the other hand, by (6), (12) and (13), we see that $V(t)$ is eventually decreasing. Hence, the limit $\lim_{t\to\infty} V(t) = \alpha$ exists and is finite. So $\lim_{t\to\infty} V_1(t) = \alpha$, that is,

$$\lim_{t\to\infty} \left[x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds - \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds \right]^2 = \alpha. \tag{15}$$

Let

$$y(t) = x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds - \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds,$$

then

$$\begin{aligned} y(t_k) &= x(t_k) - \int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds - \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds \\ &= b_k x(t_k^-) + (1-b_k)\left(\int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds + \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds\right) \\ &\quad - \int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds - \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds \\ &= b_k\left[x(t_k^-) - \int_{t_k-\tau}^{t_k-\sigma} q(s+\sigma)x(s)ds - \int_{t_k-\tau}^{t_k} p^*(s+\tau)x(s)ds\right] \\ &= b_k y(t_k^-), \end{aligned}$$

moreover, in view of (8) and (15), we have

$$\begin{cases} y'(t) + p^*(t+\tau)x(t) = 0, \quad t \geq t_0, \quad t \neq t_k, \\ y(t_k) = b_k y(t_k^-), \; k = 1, 2, 3, \cdots, \end{cases} \tag{16}$$

and

$$\lim_{t\to\infty} y^2(t) = \alpha. \tag{17}$$

If $\alpha = 0$, then $\lim\limits_{t\to\infty} y(t) = 0$. If $\alpha > 0$, then there exists a sufficiently large T_2 such that $y(t) \neq 0$ for $t \geq T_2$. Otherwise, there is a sequence $\tau_1, \tau_2, \cdots, \tau_k, \cdots$ with $\lim\limits_{k\to\infty} \tau_k = \infty$ such that $y(\tau_k) = 0$, so $y^2(\tau_k) \to 0$ as $k \to \infty$. It is a contradiction with $\alpha > 0$. Therefore, for any $t_k > T_2$, $t \in [t_k, t_{k+1})$ we have $y(t) > 0$ or $y(t) < 0$ because $y(t)$ is continuous on $[t_k, t_{k+1})$. Without loss of generality, we assume that $y(t) > 0$ on $[t_k, t_{k+1})$, it follows that $y(t_{k+1}) = b_{k+1}y(t_{k+1}^-) > 0$, thus $y(t) > 0$ on $[t_{k+1}, t_{k+2})$. By induction, we can conclude that $y(t) > 0$ on $[t_k, \infty)$. From (17), we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds - \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds\right] = \beta \tag{18}$$

must exist and is finite. In view of (16), we have

$$\begin{aligned} \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds &= y(t-\tau) - y(t) + \sum_{t-\tau<t_k<t} [y(t_k) - y(t_k^-)] \\ &= y(t-\tau) - y(t) - \sum_{t-\tau<t_k<t} (1-b_k)y(t_k^-). \end{aligned}$$

We let $t \to \infty$ and note that $\sum\limits_{k=1}^{\infty}(1-b_k) < \infty$, then we have

$$\lim_{t\to\infty} \int_{t-\tau}^{t} p^*(s+\tau)x(s)ds = 0. \tag{19}$$

By (18) and (19), we have

$$\lim_{t\to\infty}[x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds] = \beta. \tag{20}$$

Next, we shall prove that

$$\lim_{t\to\infty} x(t) = \frac{\beta}{1-\mu}. \tag{21}$$

To this end, we first show $|x(t)|$ is bounded. In fact, if $|x(t)|$ is unbounded, then there exists a sequence $\{s_n\}$ such that $s_n \to \infty, |x(s_n^-)| \to \infty$, as $n \to \infty$ and

$$|x(s_n^-)| = \sup_{t_0 \le t \le s_n} |x(t)|, \tag{22}$$

where, if s_n is not impulsive point then $x(s_n^-) = x(s_n)$. Thus, noticing (5) and (22), we have

$$\begin{aligned} |x(s_n^-) - \int_{s_n-\tau}^{s_n-\sigma} q(s+\sigma)x(s)ds| &\ge |x(s_n^-)| - \int_{s_n-\tau}^{s_n-\sigma} q(s+\sigma)|x(s)|ds \\ &\ge |x(s_n^-)| - |x(s_n^-)| \int_{s_n-\tau}^{s_n-\sigma} q(s+\sigma)ds \\ &= |x(s_n^-)|(1 - \int_{s_n-\tau}^{s_n-\sigma} q(s+\sigma)ds) \to \infty, \text{ as } n \to \infty, \end{aligned}$$

which contradicts (20). So $|x(t)|$ is bounded. Set

$$\limsup_{t\to\infty} x(t) = \omega_1, \quad \liminf_{t\to\infty} x(t) = \omega_2.$$

Choose two sequences $\{u_n\}$ and $\{v_n\}$ such that $u_n \to \infty$, $v_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} x(u_n) = \omega_1, \quad \lim_{n\to\infty} x(v_n) = \omega_2.$$

Since for $\delta > 0$ sufficiently small,

$$\begin{aligned} x(u_n) &= x(u_n) - \int_{u_n-\tau}^{u_n-\sigma} q(s+\sigma)x(s)ds + \int_{u_n-\tau}^{u_n-\sigma} q(s+\sigma)x(s)ds \\ &\le x(u_n) - \int_{u_n-\tau}^{u_n-\sigma} q(s+\sigma)x(s)ds + (\omega_1+\delta) \int_{u_n-\tau}^{u_n-\sigma} q(s+\sigma)ds, \end{aligned}$$

it follows, by (20) and (5), that

$$\omega_1 = \lim_{n\to\infty} x(u_n) \le \beta + (\omega_1 + \delta)\mu.$$

We let $\delta \to 0$ to obtain $\omega_1 \leq \beta + \omega_1\mu$, i.e., $\omega_1 \leq \frac{\beta}{1-\mu}$. Similarly, we can prove $\omega_2 \geq \frac{\beta}{1-\mu}$. Therefore $\omega_1 \leq \omega_2$, but clearly $\omega_1 \geq \omega_2$, which implies that $\omega_1 = \omega_2 = \frac{\beta}{1-\mu}$. Thus (21) hold. The proof is complete. ■

By Theorem 1, we have the following asymptotic behavior result immediately.

Theorem 2. *The conditions of Theorem 1 implies that every oscillatory solution of* (1) *tends to zero as* $t \to \infty$.

Proof. Let $x(t)$ be any oscillatory solution of (1), by Theorem 1, $x(t)$ tends to a constant ω as $t \to \infty$ and $x(t)$ is bounded. We let

$$\limsup_{t\to\infty} x(t) = \omega_1, \quad \liminf_{t\to\infty} x(t) = \omega_2,$$

then $\omega_1 = \omega_2 = \omega$. On the other hand, noting that $x(t)$ is oscillatory, we have $\omega_1 \geq 0$ and $\omega_2 \leq 0$. Thus, $\omega = \omega_1 = \omega_2 = 0$. The proof of Theorem 2 is complete. ■

In Theorem 1, taking $q(t) \equiv 0$, we have

Corollary 1. *Assume that*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t+\tau} p(s+\tau)ds < 2;$$

$$0 < b_k \leq 1 \quad \text{and} \quad \sum_{k=1}^{\infty}(1-b_k) < \infty.$$

Then every solution of the equation

$$\begin{cases} x'(t) + p(t)x(t-\tau) = 0, \quad t \geq t_0, \quad t \neq t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\int_{t_k-\tau}^{t_k} p(s+\tau)x(s)ds, \\ \qquad\qquad k = 1, 2, 3, \cdots \end{cases}$$

tends to a constant as $t \to \infty$.

In Theorem 1, taking $b_k \equiv 1$, we get

Corollary 2. *Assume that the conditions (3)-(6) hold. Then every solution of the equation*

$$x'(t) + p(t)x(t-\tau) - q(t)x(t-\sigma) = 0, \quad t \geq t_0$$

tends to a constant as $t \to \infty$.

Theorem 3. *The conditions in Theorem* 1 *together with*

$$\int_{t_0}^{\infty} p^*(t)dt = \infty \tag{23}$$

implies that every solution of (1) *tends to zero as* $t \to \infty$.

Proof. By theorem 2, we only have to prove that every nonoscillatory solution of (1) tends to zero as $t \to \infty$. Without loss of generality, let $x(t)$ be an eventually positive solution of (1), we shall prove $\lim\limits_{t\to\infty} x(t) = 0$. As in the proof of Theorem 1, we can rewrite (1) in the form of (16). Integrating from t_0 to t on both sides of (16) produces

$$\int_{t_0}^{t} p^*(s+\tau)x(s)ds = y(t_0) - y(t) - \sum_{t_0<t_k<t} (1-b_k)y(t_k^-).$$

By using (18) and $\sum\limits_{k=1}^{\infty}(1-b_k) < \infty$, we have

$$\int_{0}^{\infty} p^*(s+\tau)x(s)ds < \infty,$$

which, together with (23), yields $\liminf\limits_{t\to\infty} x(t) = 0$. On the other hand, by Theorem 1, we see that $\lim\limits_{t\to\infty} x(t)$ exists. Therefore, $\lim\limits_{t\to\infty} x(t) = 0$. The proof is complete. ∎

Example. The impulsive differential equation

$$\begin{cases} x'(t) + \frac{2}{(t-1)^\alpha}x(t-2) - \frac{1}{t^\alpha}x(t-1) = 0, \quad t \ge 2, \quad t \ne k, \\ x(k) = \frac{k^\beta - 1}{k^\beta}x(k^-) + \frac{1}{k^\beta}\left(\int_{k-2}^{k} \frac{2}{(s+1)^\alpha}x(s)ds - \int_{k-1}^{k} \frac{1}{(s+1)^\alpha}x(s)ds\right), \quad k = 3,4,5,\cdots, \end{cases}$$

where $\alpha > 1$ and $\beta > 1$, satisfies all conditions of Theorem 1. Therefore, every solution of this equation tends to a constant as $t \to \infty$.

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