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COMMON FIXED POINT OF MULTIVALUED MAPPINGS WITHOUT CONTINUITY

Abstract: In this paper, we prove a common fixed point theorem for single-valued and multivalued mappings on a metric space using the minimal type commutativity condition. We show that continuity of any mapping is not necessary for the existence of a common fixed point.

Key words: common fixed point, coincidence point, noncompatible maps.

1. Introduction

Fixed point theory for single-valued and multivalued mappings have been studied extensively and applied to diverse problems during the last few decades. The interest on this subject was enhanced after the publication of a paper by Nadler [9]. This theory provides techniques for solving a variety of applied problems in mathematical sciences and engineering (e.g. Kyzyska and Kubiaczyk [8], Sessa and Khan [16]).

Most of the fixed point theorems existing in the mathematical literature deal with compatible and continuous mappings. So it would be natural question: What about the mappings which are not compatible and continuous. Also it is known that there are so many discontinuous functions which have fixed point and the most surprising one is Dirichlet map defined on R (i. e. $fx = 1$ if $x \in Q$ and 0 otherwise, has 1 as a fixed point). These observations motivated several authors of the field to prove fixed point theorems for noncompatible, discontinuous mappings.

Sessa [15] introduced the concept of weakly commuting maps. Jungck [3] defined the notion of compatible maps in order to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true.

Kaneko [5] extended the concept of weakly commuting mappings for multivalued set up. Kaneko and Sessa [6] extended the concept of compatibility for single-valued mappings to the settings of single-valued and multivalued mappings.

Pant [10-13] initiated the study of noncompatible maps and introduced pointwise R-weak commutativity of mappings in [10]. He also showed that for single-valued mappings pointwise R-weak commutativity is equivalent to weak compatibility at the coincidence points.

Shahzad and Kamran [17], Singh and Mishra [18] have independently extended the idea of R-weak commutativity to the settings of single and multivalued mappings.

Pathak, Cho and Kang [14] introduced the concept of R-weakly commuting mappings of type A_g for single-valued mappings and showed that they are not compatible.

Recently, Kamaran [4] extended the concept of R-weakly commuting mappings of type A_g for multivalued mappings and introduced R-weakly commuting mappings of type A_T.

In their paper Kaneko [5] and Kaneko and Sessa [6] have assumed a pair of single valued and a multivalued mappings which are continuous at X. They have also remarked whether or not the continuity of the two mappings is realy needed in the proof.

Asad and Ahmad [1] extended a result of Fisher [2] for a single-valued mapping and two multivalued mappings under condition of weak commutativity or compatibility. They proved existence of a common fixed point by assuming the continuity of single-valued mapping only.

In this paper, we prove a common fixed point theorem for two pairs of single-valued and multivalued mappings by using the condition of R-weak commutativity of type A_T. We also show that existence of a common fixed point can be proved without assuming continuity of any mapping. We improve extend and generalize the results of Asad and Ahmad [1].

2. Preliminaries

Let (X, d) be a metric space and suppose that $CB(X)$ denotes the set of non-empty closed and bounded subsets of X.

For A, B in $CB(X)$ we denote

$D(A, B) = inf\{d(a, b) : a \in A, b \in B\}$

$D(x, A) = inf\{d(x, a) : a \in A$

$H(A, B) = max\{sup\{D(a, B) : a \in A\}, sup\{D(A, b) : b \in B\}\}.$

Kuratowski [7] showed that $(CB(X), H)$ is a metric space with the distance function H, moreover $(CB(X), H)$ is complete in the event that (X, d) is complete.

Lemma 2.1. [9] *Let $A, B \in\ CB(X)$, then for $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that $d(a, b) \leq\ H(A, B)+ \in$. If A and B are compact then one can find $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Definition 2.1. [5] *Let (X, d) be a metric space, $F : X \to CB(X)$ and $T : X \to X$. Then the pair F, T is said to be weakly commuting if for each $x \in X$, $TF(x) \in CB(X)$ and $H(FTx, TFx) \leq D(Tx, Fx)$.*

Definition 2.2. [6] *Let (X, d) be a metric space, $F : X \to CB(X)$ and $T : X \to X$. Then the pair F, T is said to be compatible if and only if $TFx \in CB(X)$ for each $x \in X$ and $H(FTx_n, TFx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Fx_n \to A \in CB(X)$ and $Tx_n \to t \in A$.*

Definition 2.3. [10] *The mappings $T : X \to X$ and $F : X \to CB(X)$ are said to be R-weakly commuting if, for given $x \in X$, $TFx \in CB(X)$ and there exists some positive real number R such that $H(TFx, FTx) \leq RD(Tx, Fx)$.*

Definition 2.4. [4] *The mappings $T : X \to X$ and $F : X \to CB(X)$ are said to be R-weakly commuting of type A_T at $x \in X$, if there exists some positive real number R such that*

$$D(TTx, FTx) \leq RD(Tx, Fx)$$

. Here T and F are R-weakly commuting of type A_T on X if the above inequality holds for all $x \in X$. If F is a single-valued self mapping on X this definition of R-weak commutativity reduces to that of Pathak, Cho and Kang [14].

Example 2.1. Let $X = [0, 1]$ and d be the usual metric on X. Define $T : X \to X$ by $Tx = \frac{x}{2}$ for all $x \in X$ and $F : X \to CB(X)$ by $Fx = [0, x]$ for all $x \in X$.
Let $\{x_n\}$ be a sequence in X such that $x_n \to 0$ as n $\to \infty$. Then $\lim_{n\to\infty} Tx_n = 0 \in \{0\} = \lim_{n\to\infty} Fx_n$ and $\lim_{n\to\infty} H(TFx_n,\ FTx_n) = 0$ so T and F are compatible.
We have $TTx = \frac{x}{4}$ and $FTx = [0, \frac{x}{2}]$. Therefore T and F are R-weakly commuting of type A_T.

Example 2.2. Let $X = [1, \infty)$ and d be the usual metric on X. Define $T : X \to X$ by $Tx = 2x$ for all $x \in X$ and $F : X \to CB(X)$ by $Fx = [1, x]$ for all $x \in X$.

Then $TTx = 4x$, $FTx = [1, 2x]$. Therefore $D(TTx, FTX) = 2D(Tx, Fx)$ for all $x \in X$ and the mappings T and F are R-weakly commuting of type A_T on X.

Further there exists no sequence $\{x_n\} in X$ such that condition of compatibility is satisfied.

In view of Examples 2.1 and 2.2 we observe the following:

Remark 2.1. Compatible maps are R-weakly commuting of type A_T but converse is not true in general.

Theorem A. [1] *Let (X,d) be a complete metric space, $F,G : X \to CB(X)$ and $T : X \to X$ such that the inequality*

$$H(Fx,Gy) \leq \alpha \frac{[D(Fx,Ty)]^2 + [D(Gy,Tx)]^2}{D(Fx,Ty) + D(Gy,Tx)} + \beta\, d(Tx,Ty)$$

holds for all $x, y \in X$, $x \neq y$, $Fx \neq Fy$, $Gx \neq Gy$; $\alpha, \beta \geq 0$, $2\alpha + \beta < 1$ whenever $D(Fx,Ty) + D(Gy,Tx) \neq 0$ and $H(Fx,Gy) = 0$ whenever $D(Fx,Ty) + D(Gy,Tx) = 0$ and
(I) $F(X) \cup G(X) \subseteq T(X)$,
(II) *$\{F,T\}$ and $\{G,T\}$ are weakly commuting,*
(III) *T is continuous at X.*
Then there exists a point z in X such that $z = Tz \in Fz \cap Gz$.

Theorem B. [1] *Let F, G and T be the same as defined in Theorem A and condition (II) is replaced by*
(II') *$\{F,T\}$ and $\{G,T\}$ are compatible pairs.*
Then there exists a point z in X such that $z = Tz \in Fz \cap Gz$.

3. Main Results

Theorem 3.1. *Let (X,d) be a complete metric space. Let $S,T : X \to X$ and $F,G : X \to CB(X)$ satisfying the following conditions:*

$$F(X) \subseteq S(X), \qquad G(X) \subseteq T(X), \tag{3.1}$$

$$\begin{aligned} &\text{the pairs } \{F,T\} \text{ and } \{G,S\} \text{ are } R-\text{weakly commuting} \\ &of\ type\ A_T \text{ at coincidence points in } X, \end{aligned} \tag{3.2}$$

$$H(Fx,Gy) \ \leq \ \alpha \ \frac{[D(Fx,Sy)]^2 + [D(Gy,Tx)]^2}{D(Fx,\ Sy) + D(Gy,Tx)} + \beta d(Tx,Sy), \tag{3.3}$$

$x \neq y$, $Fx \neq Fy$, $Gx \neq Gy$ for all $x, y \in X$, $\alpha, \beta \geq 0$, $2\alpha + \beta < 1$, whenever $D(Fx,Sy) + D(Gy,Tx) \neq 0$ and $H(Fx,Gy) = 0$, whenever $D(Fx,Sy) + D(Gy,Tx) = 0$. Then there exists a point z in X such that $z = Tz = Sz \in Fz \cap Gz$.

Proof. Assume $\theta = \frac{\alpha+\beta}{1-\alpha}$. Let $x_0 \in X$ and y_1 be an arbitrary point in Fx_0. Choose $x_1 \in X$ such that $y_1 = Sx_1$. This is possible as $F(X) \subseteq S(X)$. By Lemma 2.1, we can find $y_2 \in Gx_1$ such that

$$d(y_1,y_2) \leq H(Fx_0,Gx_1) + \frac{1-\alpha}{1+\alpha}\,\theta.$$

Choose $x_2 \in X$ such that $y_2 = Tx_2$. This is also possible as $G(X) \subseteq T(X)$. Also we can find $y_3 \in Fx_2$ such that

$$d(y_2,y_3) \leq H(Fx_2, \qquad Gx_1) + \frac{1-\alpha}{1+\alpha}\theta^2.$$

Inductively, having selected $y_{2n} = Tx_{2n} \in Gx_{2n-1}$, choose $y_{2n+1} = Sx_{2n+1} \in Fx_{2n}$ such that

$$d(y_{2n+1}, y_{2n}) \leq H(Fx_{2n}, Gx_{2n-1}) + \frac{1-\alpha}{1+\alpha}\theta^{2n}.$$

Then having selected y_{2n+1}, choose $y_{2n+2} = Tx_{2n+2} \in Gx_{2n+1}$ such that

$$d(y_{2n+1}, y_{2n+2}) \leq H(Fx_{2n}, Gx_{2n-1}) + \frac{1-\alpha}{1+\alpha}\theta^{2n+1}.$$

Thus for $n \geq 1$, we have

$$\begin{aligned}
&d(y_{2n}, y_{2n+1}) \leq H(Fx_{2n}, Gx_{2n-1}) + \frac{1-\alpha}{1+\alpha}\theta^{2n} \\
&\leq \ \alpha\frac{[D(Fx_{2n}, Sx_{2n-1})]^2 + [D(Gx_{2n-1}, Tx_{2n})]^2}{D(Fx_{2n}, Sx_{2n-1}) + D(Gx_{2n-1}, Tx_{2n})} \\
&\quad + \beta d(Tx_{2n}, Sx_{2n-1}) + \frac{1-\alpha}{1+\alpha}\theta^{2n}, \\
&\leq \ \alpha[d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1})] + \beta d(y_{2n}, y_{2n-1}) + \frac{1-\alpha}{1+\alpha}\theta^{2n}.
\end{aligned}$$

So that

$$d(y_{2n}, y_{2n+1}) \ \leq \ \frac{\alpha+\beta}{1-\alpha}d(y_{2n}, y_{2n-1}) + \frac{\theta^{2n}}{1+\alpha} \ = \ \theta d(y_{2n}, y_{2n-1}) + \frac{\theta^{2n}}{1+\alpha}.$$

Similarly we can show that

$$d(y_{2n}, y_{2n-1}) \ \leq \ \theta \ d(y_{2n-1}, y_{2n-2}) + \frac{\theta^{2n-1}}{1+\alpha}.$$

Combining the above inequalities we have

$$d(y_{n+1}, y_{n+2}) \leq \theta^2 d(y_n, y_{n-1}) + 2\frac{\theta^{n+1}}{1+\alpha} \leq ...... \leq \theta^n d(y_1, y_2) + n\frac{\theta^{n+1}}{1+\alpha}.$$

Routine calculation shows that $\{y_n\}$ is a Cauchy sequence. Since X is complete therefore there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Since $F(X) \subseteq S(X)$, there exists a point $p \in X$ such that $Sp = z$. Since $z = \lim_{n\to\infty} Sx_{2n+1}$ and $Sx_{2n+1} \in Fx_{2n}$, therefore $D(Fx_{2n}, Sx_{2n+1}) \to 0$ as $n \to \infty$. By (3.3), we have

$$\begin{aligned}
&D(Gp, Sp) \leq H(Fx_{2n}, Gp) + D(Fx_{2n}, Sp) \\
&\leq \ \alpha\frac{[D(Fx_{2n}, Sp)]^2 + [D(Gp, Tx_{2n})]^2}{D(Fx_{2n}, Sp) + D(Gp, Tx_{2n})]} \\
&\quad + \beta d(Tx_{2n}, Sp) + D(Fx_{2n}, Sp) \\
&\leq \ \alpha[D(Fx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, Sp) + D(Gp, Tx_{2n})] \\
&\quad + \beta d(Tx_{2n}, Sp) + D(Fx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, Sp).
\end{aligned}$$

On letting $n \to \infty$ the above inequality yields $D(Gp, z) \leq \alpha D(Gp, z)$, which is a contradiction. Therefore $z = Sp \in Gp$ that is $p \in X$ is a coincidence point of S and G. Since S and G are R weakly commuting of type A_T at coincidence point, therefore there exists some real number R such that $D(SSp, GSp) \leq RD(Sp, Gp)$, which gives $Sz \in Gz$.

Since $G(X) \subseteq T(X)$, there exists a point $q \in X$ such that $Tq = z$. Since $z = \lim_{n\to\infty} Tx_{2n}$ and $Tx_{2n} \in Gx_{2n-1}$. Therefore $D(Gx_{2n-1}, Tx_{2n}) \to 0$ as $n \to \infty$. By (3.3), we have

$$\begin{aligned} D(Fq, Tq) &\leq H(Fq, Gx_{2n-1}) + D(Gx_{2n-1}, Tq), \\ &\leq \alpha \frac{[D(Fq, Sx_{2n-1})]^2 + [D(Gx_{2n-1}, Tq)]^2}{D(Fq, Sx_{2n-1}) + D(Gx_{2n-1}, Tq)} \\ &\quad + \beta d(Tq, Sx_{2n-1}) + D(Gx_{2n-1}, Tq) \\ &\leq \alpha[D(Fq, Sx_{2n-1}) + D(Gx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Tq)] \\ &\quad + \beta d(Tq, Sx_{2n-1}) + D(Gx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Tq). \end{aligned}$$

On letting n $\to \infty$ the above inequality yields $D(Fq, z) \leq \alpha D(Fq, z)$, which is a contradiction. Therefore $z = Tq \in Fq$ that is $q \in X$ is a coincidence point of T and F. Since T and F are R-weakly commuting of type A_T at coincidence point, therefore there exists some real number R such that $D(TTq, FTq) \leq RD(Tq, Fq)$, which gives $Tz \in Fz$. By (3.3), we have

$$\begin{aligned} d(Sx_{2n+1}, Sz) &\leq H(Fx_{2n}, Gz) \\ &\leq \alpha \frac{[D(Fx_{2n}, Sz)]^2 + [D(Gz, Tx_{2n})]^2}{D(Fx_{2n}, Sz) + D(Gz, Tx_{2n})]} + \beta d(Tx_{2n}, Sz), \\ &\leq \alpha[D(Fx_{2n}, Sz) + D(Gz, Tx_{2n})] + \beta d(Tx_{2n}, Sz), \\ &\leq \alpha[d(y_{2n+1}, Sz) + d(Sz, y_{2n})] + \beta d(y_{2n}, Sz). \end{aligned}$$

On letting $n \to \infty$ the above inequality yields $d(z, Sz) \leq (2\alpha + \beta)d(z, Sz)$, contradiction giving there by $Sz = z$.

Now to show $Tz = z$, by (3.3), we have

$$\begin{aligned} d(Tz, Tx_{2n}) &\leq H(Fz, Gx_{2n-1}) \\ &\leq \alpha \frac{[D(Fz, Sx_{2n-1})]^2 + [D(Gx_{2n-1}, Tz)]^2}{D(Fz, Sx_{2n-1}) + D(Gx_{2n-1}, Tz)} + \beta d(Tz, Sx_{2n-1}), \\ &\leq \alpha[D(Fz, Sx_{2n-1}) + D(Gx_{2n-1}, Tz)] + \beta d(Tz, Sx_{2n-1}), \\ &\leq \alpha[d(Tz, y_{2n-1}) + d(y_{2n}, Tz)] + \beta d(Tz, y_{2n-1}). \end{aligned}$$

On letting $n \to \infty$ the above inequality yields $d(Tz, z) \leq (2\alpha + \beta)d(Tz, z)$, which is a contradiction. So we have $Tz = z$. Combining the results we have $z = Tz = Sz \in Fz \cap Gz$. This completes the proof. ■

Remark 3.1. Theorem 3.1 is extension, improvement and generalization of Theorem A and Theorem B.

Remark 3.2. (i) For $\alpha = 0$, we get an extension of the well known Banach fixed point theorem.

(ii) For $\beta = 0$, we get a new result.

If we put $S = T$ in Theorem 3.1, we have the following:

Corollary 3.2. *Let (X, d) be a complete metric space. Let $T : X \to X$ and $F, G : X \to CB(X)$ satisfying the following conditions:*

$$F(X) \subseteq T(X), \qquad G(X) \subseteq T(X), \tag{3.4}$$

$$\begin{aligned} &\text{the pairs } \{F, T\} \text{ and } \{G, S\} \text{ are } R-\text{weakly commuting} \\ &\text{of type } A_T \text{ at coincidence points in } X, \end{aligned} \tag{3.5}$$

$$H(Fx, Gy) \le \alpha \frac{[D(Fx, Sy)]^2 + [D(Gy, Tx)]^2}{D(Fx,\ Sy) + D(Gy, Tx)} + \beta d(Tx, Sy), \tag{3.6}$$

$x \ne y$, $Fx \ne Fy$, $Gx \ne Gy$ for all $x, y \in X$, $\alpha, \beta \ge 0$, $2\alpha + \beta < 1$, whenever $D(Fx, Sy) + D(Gy, Tx) \ne 0$ and $H(Fx, Gy) = 0$, whenever $D(Fx, Sy) + D(Gy, Tx) = 0$. Then there exists a point z in X such that $z = Tz = Sz \in Fz \cap Gz$.

Then there exists a point z in X such that $z = Tz \in Fz \cap Gz$.

Remark 3.3. Corollary 3.2 improves Theorem A and Theorem B in the sense that compatibility of pairs $\{F, T\}$ and $\{G, T\}$ are replaced by a weaker condition that is R-weak commutativity of type A_T and the continuity of any mapping is not required.

Remark 3.4. The condition in the hypothesis of Theorem 3.1 and Corollary 3.2 $x \ne y$, $Fx \ne Fy$, $Gx \ne Gy$ is necessary since the Theorem 3.1 and Corollary 3.2 fail for F and G taken as constant mappings.

References

[1] ASAD A.J., AHMAD Z., Common fixed point of multivalued mappings with weak commutativity conditions, *Radovi, Math.,* 9(1999), 119–124.

[2] FISHER B., Mappings satisfying rational inequality, *Nanta Math.,* 12(1979), 195–199.

[3] JUNGCK G., Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.,* 4(1986), 771–779.

[4] KAMARAN T., Fixed points of asymptotically regular noncompatible maps, *Demonstratio Math.,* XXXVIII(2)(2005), 485–494.

[5] Kaneko H., A common fixed point of weakly commuting multivalued mappings, *Math. Japon.,* 33(5)(1988), 741–744.
[6] Kaneko H., Sessa S., Fixed point theorem for compatible multivalued and single-valued mappings, *Internat. J. Math. Math. Sci.,* 12(2)(1989), 257–262.
[7] Kuratowski K., Topology, *Demon. Math.,* 1, Academic press (1966).
[8] Kyzyska S., Kubiaczyk I., Fixed point theorems for upper semicontinuous and weakly-weakly upper semicontinuous multivalued mappings, *Math. Japonica,* 47(2)(1988), 237–240.
[9] Nadler S.B., Jr. Multivalued contraction mappings, *Pacfic J. Math.,* 30(1969), 475–488.
[10] Pant R.P., Common fixed points of non-commuting mappings, *J. Math. Anal. Appl.,* 188(1994), 436-440.
[11] Pant R.P., Common fixed point theorems for contractive maps, *J. Math. Anal. Appl.,* 226(1998), 251–258.
[12] Pant R.P., Common fixed points of Lipschitz type mappings pair, *J. Math. Anal. Appl.,* 240(1999), 280-283.
[13] Pant R.P., Discontinuity and fixed points, *J. Math. Anal. Appl.,* 240(1999), 284–289.
[14] Pathak H.K., Cho Y.J., Kang, Remarks on R-weakly commuting mappings and common fixed point theorems, *Bull. Korean Math. Soc.,* 34(1997), 247–257.
[15] Sessa S., On a weak commutativity condition of mappings in fixed point consideration, *Publ. Inst. Math.,* 32(1982), 149–153.
[16] S. Sessa S., Khan M.S., Some remarks in best approximation theory, *Math. J. Toyoma Univ.,* 17(1994), 151–165.
[17] Shahzad N., Kamran T., Coincidence points and R-weakly commuting maps, *Archivum Mathematicum (Brno),* 37(2001), 179–183.
[18] Singh S.L., Mishra S.L., Coincidence and fixed points of non-self hybrid contractions, *J. Math. Anal. Appl.,* 256(2001), 486–497.

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