F A S C I C U L I   M A T H E M A T I C I

Nr 37 2007

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ON HIGHER ORDER VOLTERRA-FREDHOLM INTEGRODIFFERENTIAL EQUATION

Abstract: In the present paper we study the existence, uniqueness and other properties of soltions of a certain higher order Volterra-Fredholm integrodifferential equation. The well known Banach fixed point theorem coupled with Bilecki type norm and the new integral inequality with explicit estimate are used to establish the results.

Key words: higher order, Volterra-Fredholm, integrodifferential equation, Banach fixed point theorem, Bilecki type norm, integral inequality.

1. Introduction

Consider the initial value problem (IVP for short) for higher order Volterra-Fredholm integrodifferential equation of the form

$$x^{(n)}(t) = F\left(t, x(t), x'(t), ..., x^{(n-1)}(t), (Ax)(t), (Bx)(t)\right), \tag{1.1}$$

for $t \in I = [t_0, b], 0 \leq t_0 < b$, with the given initial conditions

$$x^{(k)}(t_0) = c_k, k = 0, 1, ..., n-1, \tag{1.2}$$

where

$$(Ax)(t) = \int_{t_0}^{t} k_1(t,\tau) h_1\left(\tau, x(\tau), x'(\tau), ..., x^{(n-1)}(\tau)\right) d\tau, \tag{1.3}$$

$$(Bx)(t) = \int_{t_0}^{b} k_2(t,\tau) h_2\left(\tau, x(\tau), x'(\tau), ..., x^{(n-1)}(\tau)\right) d\tau. \tag{1.4}$$

In (1.1)-(1.4), $F \in C\left(I \times R^{n+2}, R\right)$; for $i = 1, 2$ and $t_0 \leq \tau \leq t$, $k_i \in C\left(I^2, R\right)$, $h_i \in C\left(I \times R^n, R\right)$ are given functions and c_k are given real constants. The equation (1.1) considered in this paper is in the general sprit of the investigations in [4, 5, 7-9, 12, 14]. In particular, if we impose on F various meanings, it is apparent that the equation (1.1) has a great diversity.

The problems of existence, uniqueness and other properties of solutions of special forms of IVP (1)-(2) and its variants have been studied by many authors under a variety of hypotheses by using different techniques, see [4,5,7-9, 12-15] and some of the references cited therein. Our main objective here is to study the existence, uniqueness and other properties of solutions of IVP (1)-(2). The analysis used in the proofs is based on the applications of the well known Banach fixed point theorem (see [3,6]) coupled with Bilecki type norm (see [1]) and the new integral inequality with explicit estimate, recently established by the present author in (see [11], Theorem 1.5.3, part (c_2), p.47).

2. Existence and uniqueness

Let $E = R \times ... \times R$ (n times) be the product space. For continuous functions $u^{(j)}(t) : I \to R$ $(j = 0, 1, ..., n-1)$ we denote $|u(t)|_E = \sum_{j=0}^{n-1} \left|u^{(j)}(t)\right|$, for $\left(u(t), u'(t), ..., u^{(n-1)}(t)\right) \in E$, $t \in I$. Let G be a space of those functions $\left(u(t), u'(t), ..., u^{(n-1)}(t)\right) \in E$ which are continuous for $t \in I$ and fulfil the condition

$$|u(t)|_E = o(\exp(\lambda t)), t \in I, \tag{2.1}$$

where λ is a positive constant. In the space G we define the norm (see [1])

$$|u|_G = \sup_{t \in I} \left\{ |u(t)|_E \exp(-\lambda t) \right\}. \tag{2.2}$$

It is easy to see that G with norm defined by (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant $N_0 \geq 0$ such that

$$|u(t)|_E \leq N_0 \exp(\lambda t).$$

Using this fact in (6), we observe that

$$|u|_G \leq N_0. \tag{2.3}$$

It is easy to observe that the solution $x(t)$ of IVP (1.1)-(1.2) and its derivatives are equivalent to the integral equations

$$x^{(j)}(t) = \sum_{i=j}^{n-1} \frac{c_i (t - t_0)^{i-j}}{(i-j)!} \tag{2.4}$$

$$+\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}F\left(s,x(s),x'(s),...,x^{(n-1)}(s),(Ax)(s),(Bx)(s)\right)ds,$$

for $0\le j\le n-1$.

Our result on the existence of a unique solution of IVP (1)-(2) is embodied in the following theorem.

Theorem 1. *Assume that*

(*i*) *the functions* F, h_i $(i=1,2)$ *satisfy the conditions*

$$|F(t,u_0,u_1,...,u_{n-1},v_1,v_2)-F(t,w_0,w_1,...,w_{n-1},z_1,z_2)| \le p(t)\left[\sum_{j=0}^{n-1}|u_j-w_j|+|v_1-z_1|+|v_2-z_2|\right], \tag{2.5}$$

$$|h_i(t,u_0,u_1,...,u_{n-1})-h_i(t,w_0,w_1,...,w_{n-1})| \le q_i(t)\sum_{j=0}^{n-1}|u_j-w_j|, \tag{2.6}$$

where $p, q_i\in C(I,R_+)\,;R_+=[0,\infty)$,

(*ii*) *there exists a constant* α *such that* $0\le\alpha<1$ *and*

$$\sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}p(s)\left[\exp(\lambda s)+h_1^*(s)+h_2^*(s)\right]ds\le\alpha\exp(\lambda t), \tag{2.7}$$

for $t\in I$, *where*

$$h_1^*(t)=\int_{t_0}^{t}|k_1(t,\tau)|q_1(\tau)\exp(\lambda\tau)\,d\tau, \tag{2.8}$$

$$h_2^*(t)=\int_{t_0}^{b}|k_2(t,\tau)|q_2(\tau)\exp(\lambda\tau)\,d\tau,$$

and λ *is as given in (2.1),*

(*iii*) *there exists a nonnegative constant* P *such that*

$$h(t)+\sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}|F(s,0,0,...,0,(A0)(s),(B0)(s))|\,ds \le P\exp(\lambda t), \tag{2.9}$$

where

$$h(t) = \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{|c_i| (t - t_0)^{i-j}}{(i-j)!} \right], \tag{2.10}$$

and λ *is as given in (2.1).*

Then the IVP (1.1)-(1.2) has a unique solution $x(t)$ in G on I.

Proof. Let $x \in G$ and define the operator

$$(Tx)(t) = \sum_{i=0}^{n-1} \frac{(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} F\left(s, x(s), x'(s), ..., x^{(n-1)}(s), (Ax)(s), (Bx)(s)\right) ds. \tag{2.11}$$

Differentiating both sides of (2.11) with respect to t , it is easy to observe that

$$(Tx)^{(j)}(t) = \sum_{i=j}^{n-1} \frac{c_i (t-t_0)^{i-j}}{(i-j)!} + \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} F\left(s, x(s), x'(s), ..., x^{(n-1)}(s), (Ax)(s), (Bx)(s)\right) ds, \tag{2.12}$$

for $0 \leq j \leq n-1$. Evidently, $(Tx)^{(j)}(t)$ are continuous on I . Now, we shall show that T maps G into itself. We verify that (2.1) is fulfilled. From (2.12), (2.9), (2.5) we have

$$\begin{aligned} |(Tx)(t)|_E &= \sum_{j=0}^{n-1} \left|(Tx)^{(j)}(t)\right| \\ &\leq h(t) + \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} |F(s, 0, 0, ..., 0, (A0)(s), (B0)(s))| ds \\ &+ \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \left|F\left(s, x(s), x'(s), ..., x^{(n-1)}(s), (Ax)(s), (Bx)(s)\right)\right. \\ &\left. - F(s, 0, 0, ..., 0, (A0)(s), (B0)(s))\right| ds \\ &\leq P \exp(\lambda t) + \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} p(s) [|x(s)|_E \end{aligned} \tag{2.13}$$

$$+ \; |(Ax)(s) - (A0)(s)| + |(Bx)(s) - (B0)(s)|]\, ds$$

$$\leq P\exp(\lambda t) + \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} p(s)\left[\exp(\lambda s)|x|_G\right.$$

$$+ \; |(Ax)(s) - (A0)(s)| + |(Bx)(s) - (B0)(s)|]\, ds.$$

From (1.3), (2.6) and (2.8) we observe that

$$(2.14) \qquad |(Ax)(s) - (A0)(s)| \leq \int_{t_0}^{s} |k_1(s,\tau)|$$

$$\times \left| h_1\left(\tau, x(\tau), x'(\tau), ..., x^{(n-1)}(\tau)\right) - h_1(\tau, 0, 0, ..., 0)\right| d\tau$$

$$\leq \int_{t_0}^{s} |k_1(s,\tau)| q_1(\tau)\left[\exp(\lambda\tau)|x(\tau)|_E \exp(-\lambda\tau)\right] d\tau$$

$$\leq \int_{t_0}^{s} |k_1(s,\tau)| q_1(\tau)\left[\exp(\lambda\tau)|x|_G\right] d\tau = |x|_G\, h_1^*(s).$$

Similarly, from (1.4), (2.6) and (2.8) we observe that

$$(2.15) \qquad |(Bx)(s) - (B0)(s)| \leq |x|_G\, h_2^*(s).$$

From (2.13), (2.14), (2.15) and (2.3), (2.7) we observe that

$$(2.16) \qquad |(Tx)(t)|_E \leq P\exp(\lambda t)$$

$$+ \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} p(s)\left[\exp(\lambda s) + h_1^*(s) + h_2(s)\right]|x|_G\, ds$$

$$\leq P\exp(\lambda t) + N_0\alpha\exp(\lambda t) = [P + N_0\alpha]\exp(\lambda t).$$

From (2.16) it follows that $Tx \in G$. This proves that T maps G into itself.

Now, we shall show that the operator T is a contraction map. Let $x, y \in G$. From (2.12) and (2.5) we have

$$(2.17) \qquad |(Tx)(t) - (Ty)(t)|_E = \sum_{j=0}^{n-1} \left|(Tx)^{(j)}(t) - (Ty)^{(j)}(t)\right|$$

$$\leq \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \left| F\left(s, x(s), x'(s), ..., x^{(n-1)}(s), (Ax)(s), (Bx)(s)\right)\right.$$

$$-F\left(s,y(s),y'(s),...,y^{(n-1)}(s),(Ay)(s),(By)(s)\right)\Big|\,ds$$

$$\leq \sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}p(s)\left[|x(s)-y(s)|_E\right.$$

$$\left.+\,|(Ax)(s)-(Ay)(s)|+|(Bx)(s)-(By)(s)|\right]ds$$

$$\leq \sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}p(s)\left[\exp(\lambda s)|x-y|_G\right.$$

$$\left.+\,|(Ax)(s)-(Ay)(s)|+|(Bx)(s)-(By)(s)|\right]ds.$$

Following the proofs of (2.14) and (2.15) we obtain

$$|(Ax)(s)-(Ay)(s)| \leq |x-y|_G\, h_1^*(s), \tag{2.18}$$

and

$$|(Bx)(s)-(By)(s)| \leq |x-y|_G\, h_2^*(s). \tag{2.19}$$

Using (2.18), (2.19) in (2.17) and the condition (2.7) we get

$$|(Tx)(t)-(Ty)(t)|_E \tag{2.20}$$

$$\leq |x-y|_G\sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}p(s)\left[\exp(\lambda s)+h_1^*(s)+h_2^*(s)\right]ds$$

$$\leq |x-y|_G\,\alpha\exp(\lambda t).$$

Consequently, from (2.20) we have

$$|Tx-Ty|_G \leq \alpha\,|x-y|_G\,.$$

Since $\alpha < 1$, it follows from the Banach fixed point theorem (see [3, p.37]) that T has a unique fixed point in G . The fixed point of T is however a solution of IVP (1.1)-(1.2). The proof is complete. ■

Remark 1. We note that, the norm $|\,.\,|_G$ defined by (2.2) was first used by Bielecki [1] (see [2] for developments related to this topic), and has the role of improving the length of the interval on which the existence is assured.

3. Properties of solutions

We need the following new inequality, recently established in [11, p.47] to study various properties of solutions of IVP (1.1)-(1.2). We shall state and prove it in the following lemma for completeness. For a detailed account on such inequalities, see [10, 11].

Lemma (Pachpatte [11]). Let $u(t), f(t), g(t), h(t) \in C(I, R_+)$ and suppose

$$u(t) \leq c + \int_{t_0}^{t} f(s) \left[u(s) + \int_{t_0}^{s} g(\sigma) u(\sigma) d\sigma + \int_{t_0}^{b} h(\sigma) u(\sigma) d\sigma \right] ds, \tag{3.1}$$

for $t \in I$, where $c \geq 0$ is a constant. If

$$d = \int_{t_0}^{b} h(\sigma) \exp\left(\int_{t_0}^{\sigma} [f(\tau) + g(\tau)] d\tau \right) d\sigma < 1, \tag{3.2}$$

then

$$u(t) \leq \frac{c}{1-d} \exp\left(\int_{t_0}^{t} [f(s) + g(s)] ds \right), \tag{3.3}$$

for $t \in I$.

Proof. Define a function $z(t)$ by the right side of (3.1). Then $z(t_0) = c$, $u(t) \leq z(t)$ and

$$\begin{aligned} z'(t) &= f(t) \left[u(t) + \int_{t_0}^{t} g(\sigma) u(\sigma) d\sigma + \int_{t_0}^{b} h(\sigma) u(\sigma) d\sigma \right] \\ &\leq f(t) \left[z(t) + \int_{t_0}^{t} g(\sigma) z(\sigma) d\sigma + \int_{t_0}^{b} h(\sigma) z(\sigma) d\sigma \right], \end{aligned}$$

for $t \in I$. Define a function $v(t)$ by

$$v(t) = z(t) + \int_{t_0}^{t} g(\sigma) z(\sigma) d\sigma + \int_{t_0}^{b} h(\sigma) z(\sigma) d\sigma,$$

then we observe that $z(t) \leq v(t)$, $z'(t) \leq f(t) v(t)$,

$$v(t_0) = c + \int_{t_0}^{b} h(\sigma) z(\sigma) d\sigma, \tag{3.4}$$

and

$$v'(t) = z'(t) + g(t)\,z(t) \le f(t)\,v(t) + g(t)\,z(t) \le [f(t) + g(t)]\,v(t),$$

which implies

$$v(t) \le v(t_0)\exp\left(\int_{t_0}^{t} [f(s) + g(s)]\,ds\right), \tag{3.5}$$

for $t \in I$. Using the fact that $z(t) \le v(t)$ and (3.5) on the right side of (3.4) and in view of (3.2), it is easy to observe that

$$v(t_0) \le \frac{c}{1-d}. \tag{3.6}$$

Using (3.6) in (3.5) and the fact that $u(t) \le z(t)$ we get the desired inequality in (3.3). ■

Remark 2. If we take (i) $g(t) = 0$, (ii) $h(t) = 0$, then we get the new inequalities which can be used conveniently in certain situations.

The following theorem deals with the estimate on the solution of IVP (1.1)-(1.2).

Theorem 2. *Assume that the functions F, k_i, h_i $(i = 1, 2)$ satisfy the conditions*

$$|F(t, u_0, u_1, ..., u_{n-1}, v_1, v_2)| \le p(t)\left(\sum_{j=0}^{n-1} |u_j| + |v_1| + |v_2|\right), \tag{3.7}$$

$$|k_i(t, s)| \le M_i, \tag{3.8}$$

$$|h_i(t, u_0, u_1, ..., u_{n-1})| \le q_i(t)\sum_{j=0}^{n-1} |u_j|, \tag{3.9}$$

where $M_i \ge 0$ are constants and $p, q_i \in C(I, R_+)$. Let

$$d' = \int_{t_0}^{b} M_2 q_2(\sigma)\exp\left(\int_{t_0}^{\sigma} [Np(s) + M_1 q_1(s)]\,ds\right) d\sigma < 1, \tag{3.10}$$

where

$$N = \sum_{j=0}^{n-1} \frac{1}{(n-j-1)!}(b - t_0)^{n-j-1}. \tag{3.11}$$

If $x(t)$ is any solution of IVP (1.1)-(1.2), then

$$\sum_{j=0}^{n-1}\left|x^{(j)}(t)\right| \leq \frac{M}{1-d'}\exp\left(\int_{t_0}^{t}\left[Np(s)+M_1q_1(s)\right]ds\right), \tag{3.12}$$

for $t \in I$, where

$$M=\sum_{j=0}^{n-1}\left[\sum_{i=j}^{n-1}\frac{|c_i|(b-t_0)^{i-j}}{(i-j)!}\right]. \tag{3.13}$$

Proof. The solution $x(t)$ of IVP (1.1)-(1.2) and its derivatives satisfy the equivalent integral equations in (2.4). Let $u(t) = \sum_{j=0}^{n-1}|x^{(j)}(t)|$, $t \in I$. Then from (2.4) and the hypotheses, we observe that

$$\begin{aligned} u(t) &\leq \sum_{j=0}^{n-1}\left[\sum_{i=j}^{n-1}\frac{|c_i|(t-t_0)^{i-j}}{(i-j)!}\right]+\sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!} \\ &\quad\times|F(s,x(s),x'(s),...,x^{(n-1)}(s),(Ax)(s),(Bx)(s))|ds \\ &\leq \sum_{j=0}^{n-1}\left[\sum_{i=j}^{n-1}\frac{|c_i|(b-t_0)^{i-j}}{(i-j)!}\right] \\ &+\sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(b-t_0)^{n-j-1}}{(n-j-1)!}p(s)\left[\sum_{j=0}^{n-1}\left|x^{(j)}(s)\right|+|(Ax)(s)|+|(Bx)(s)|\right]ds \\ &\leq M+\int_{t_0}^{t}Np(s)\left[u(s)+\int_{t_0}^{s}M_1q_1(\tau)u(\tau)\,d\tau+\int_{t_0}^{b}M_2q_2(\tau)u(\tau)\,d\tau\right]ds. \end{aligned} \tag{3.14}$$

Now, in view of hypotheses (3.10), an application of Lemma to (3.14) yields (3.12). The proof is complete. ■

Remark 3. We note that the inequality (3.12) gives the bounds in terms of known functions, which majorizes the solution $x(t)$ of IVP (1.1)-(1.2) as well as its derivatives $x^{(j)}(t)$ $(j = 1, ..., n-1)$ for $t \in I$. If the right side in (3.12) is finite, then the solution $x(t)$ as well as its derivatives $x^{(j)}(t)$ are bounded in I .

Next, by applying Lemma we establish the uniqueness of solutions of IVP (1.1)-(1.2).

Theorem 3. *Assume that the functions $F, h_i\,(i=1,2)$ satisfy the conditions (2.5), (2.6) and the functions $k_i\,(i=1,2)$ satisfy (3.8). Let d' be as in (3.10). Then the IVP (1.1)-(1.2) has at most one solution on I .*

Proof. Let $x(t)$ and $y(t)$ be two solutions of IVP (1.1)-(1.2) on I and $v(t)=\sum_{j=0}^{n-1}\left|x^{(j)}(t)-y^{(j)}(t)\right|$. Then it is easy to observe that

$$v(t) \le \sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!} \tag{3.15}$$
$$\times\left|F\left(s,x(s),x'(s),...,x^{(n-1)}(s),(Ax)(s),(Bx)(s)\right)\right.$$
$$\left.-\;F\left(s,y(s),y'(s),...,y^{(n-1)}(s),(Ay)(s),(By)(s)\right)\right|ds$$
$$\le \sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}p(s)\left[\sum_{j=0}^{n-1}\left|x^{(j)}(s)-y^{(j)}(s)\right|\right.$$
$$\left.+\left|(Ax)(s)-(Ay)(s)\right|+\left|(Bx)(s)-(By)(s)\right|\right]ds$$
$$\le \int_{t_0}^{t}Np(s)\left[v(s)+\int_{t_0}^{s}M_1q_1(\tau)v(\tau)\,d\tau+\int_{t_0}^{b}M_2q_2(\tau)v(\tau)\,d\tau\right]ds.$$

A suitable application of Lemma (with $c=0$) to (3.15) yields $v(t)\le 0$ and consequently $x(t)=y(t)$, i.e. there is at most one solution of IVP (1.1)-(1.2) on I. ∎

The following theorem shows the dependency of solutions of (1.1) on given initial data.

Theorem 4. *Let $x(t)$ and $y(t)$ be solutions of (1.1) with initial data*

$$x^{(k)}(t_0)=c_k,\; k=0,1,...,n-1, \tag{3.16}$$

and

$$y^{(k)}(t_0)=d_k,\; k=0,1,...,n-1, \tag{3.17}$$

respecively, where c_k, d_k are real constants. Suppose that the functions F, k_i, h_i $(i=1,2)$ be as in Theorem 3. Let d' be as in (3.10). Then

$$\sum_{j=0}^{n-1}\left|x^{(j)}(t)-y^{(j)}(t)\right| \le \frac{\bar{M}}{1-d'}\exp\left(\int_{t_0}^{t}\left[Np(s)+M_1q_1(s)\right]ds\right), \tag{3.18}$$

for $t \in I$ *, where*

$$\bar{M} = \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{(b-t_0)^{i-j}}{(i-j)!} \right] |c_i - d_i| \,, \tag{3.19}$$

and N *is given by (3.11).*

Proof. Let $v(t) = \sum_{j=0}^{n-1} \left| x^{(j)}(t) - y^{(j)}(t) \right|$, $t \in I$. Using the facts that $x(t)$ and $y(t)$ are the solutions of IVP (1.1)-(3.16) and (1.1)-(3.17), respectively, and the hypotheses we have

$$\begin{aligned} v(t) \le{} & \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{(t-t_0)^{i-j}}{(i-j)!} |c_i - d_i| \right] + \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \\ & \times \Big| F\left(s, x(s), x'(s), ..., x^{(n-1)}(s), (Ax)(s), (Bx)(s)\right) \\ & - F\left(s, y(s), y'(s), ..., y^{(n-1)}(s), (Ay)(s), (By)(s)\right) \Big| \, ds \\ \le{} & \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{(b-t_0)^{i-j}}{(i-j)!} |c_i - d_i| \right] \\ & + \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(b-t_0)^{n-j-1}}{(n-j-1)!} p(s) \left[\sum_{j=0}^{n-1} \left| x^{(j)}(s) - y^{(j)}(s) \right| \right. \\ & \left. + |(Ax)(s) - (Ay)(s)| + |(Bx)(s) - (By)(s)| \right] ds \\ \le{} & \bar{M} + \int_{t_0}^{t} N p(s) \left[v(s) + \int_{t_0}^{s} M_1 q_1(\tau) v(\tau) \, d\tau + \int_{t_0}^{b} M_2 q_2(\tau) v(\tau) \, d\tau \right] ds. \end{aligned} \tag{3.20}$$

Now, an application of Lemma to (3.20) yields (3.18), which shows the dependency of solutions of IVP (1.1)-(3.16) and IVP (1.1)-(3.17) on given initial data. ■

We next consider the following initial value problems

$$x^{(n)}(t) = H\left(t, x(t), x'(t), ..., x^{(n-1)}(t), (Ax)(t), (Bx)(t), \mu\right), \tag{3.21}$$

$$x^{(k)}(t_0) = c_k, k = 0, 1, ..., n-1, \tag{3.22}$$

and

$$x^{(n)}(t) = H\left(t, x(t), x'(t), ..., x^{(n-1)}(t), (Ax)(t), (Bx)(t), \mu_0\right), \tag{3.23}$$

$$x^{(k)}(t_0) = c_k, k = 0, 1, ..., n-1, \tag{3.24}$$

where $H \in C\left(I \times R^{n+3}, R\right)$, $(Ax)(t)$, $(Bx)(t)$ are as in (1.3), (1.4), c_k are real constants and μ, μ_0 are real parameters.

The following theorem shows the dependency of solutions of IVP (3.21)-(3.22) and IVP (3.23)-(3.24) on parameters.

Theorem 5. *Assume that the functions k_i, h_i $(i = 1, 2)$ satisfy the conditions (3.8), (2.6), respectively, and the function H satisfies the conditions*

$$\begin{aligned} &|H(t, u_0, u_1, ..., u_{n-1}, v_1, v_2, \mu) - H(t, w_0, w_1, ..., w_{n-1}, z_1, z_2, \mu)| \\ &\leq p(t)\left[\sum_{j=0}^{n-1} |u_j - w_j| + |v_1 - z_1| + |v_2 - z_2|\right], \end{aligned} \tag{3.25}$$

$$\begin{aligned} &|H(t, u_0, u_1, ..., u_{n-1}, v_1, v_2, \mu) - H(t, u_0, u_1, ..., u_{n-1}, v_1, v_2, \mu_0)| \\ &\leq r(t)|\mu - \mu_0|, \end{aligned} \tag{3.26}$$

where $p, r \in C(I, R_+)$ and

$$\int_{t_0}^{t} Nr(s)\, ds \leq Q, \tag{3.27}$$

where Q is a nonnegative constant and N is given by (3.11). Let d' be as in (3.10). Let $x_1(t)$ and $x_2(t)$ be the solutions of IVP (3.21)-(3.22) and IVP (3.23)-(3.24). Then

$$\begin{aligned} \sum_{j=0}^{n-1} \left|x_1^{(j)}(t) - x_2^{(j)}(t)\right| &\leq \frac{Q|\mu - \mu_0|}{1 - d'} \\ &\times \exp\left(\int_{t_0}^{t} [Np(s) + M_1 q_1(s)]\, ds\right), \end{aligned} \tag{3.28}$$

for $t \in I$.

Proof. Let $z(t) = \sum_{j=0}^{n-1} \left| x_1^{(j)}(t) - x_2^{(j)}(t) \right|$, $t \in I$. From the hypotheses it is easy to observe that

$$
\begin{aligned}
(3.29) \qquad z(t) \leq & \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \\
& \times \left| H\left(s, x_1(s), x_1'(s), ..., x_1^{(n-1)}(s), (Ax_1)(s), (Bx_1)(s), \mu\right) \right. \\
& \left. - H\left(s, x_2(s), x_2'(s), ..., x_2^{(n-1)}(s), (Ax_2)(s), (Bx_2)(s), \mu\right) \right| ds \\
& + \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \\
& \times \left| H\left(s, x_2(s), x_2'(s), ..., x_2^{(n-1)}(s), (Ax_2)(s), (Bx_2)(s), \mu\right) \right. \\
& \left. - H\left(s, x_2(s), x_2'(s), ..., x_2^{(n-1)}(s), (Ax_2)(s), (Bx_2)(s), \mu_0\right) \right| ds \\
& \leq \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(b-t_0)^{n-j-1}}{(n-j-1)!} p(s) \left[\sum_{j=0}^{n-1} \left| x_1^{(j)}(s) - x_2^{(j)}(s) \right| \right. \\
& \left. + \left|(Ax_1)(s) - (Ax_2)(s)\right| + \left|(Bx_1)(s) - (Bx_2)(s)\right| \right] ds \\
& + \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(b-t_0)^{n-j-1}}{(n-j-1)!} r(s) \left|\mu - \mu_0\right| ds \\
\leq Q\left|\mu - \mu_0\right| + & \int_{t_0}^{t} N p(s) \left[z(s) + \int_{t_0}^{s} M_1 q_1(\tau) z(\tau) d\tau + \int_{t_0}^{b} M_2 q_2(\tau) z(\tau) d\tau \right] ds.
\end{aligned}
$$

Now an application of Lemma to (3.29) yields (3.28), which shows the dependency of solutions of IVP (3.21)-(3.22) and IVP (3.23)-(3.24) on parameters μ, μ_0 . ■

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Received on 08.11.2005 and, in revised form, on 03.01.2006.