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LINEAR AND NONLINEAR BOUNDARY VALUE PROBLEMS FOR POLYHARMONIC EQUATIONS

Abstract. Boundary value problems for linear and nonlinear polyharmonic equations are studied in this paper. The theorems on uniqueness and existence of solutions for certain class of iterated elliptic equations of $2n$ order are proved.

Key words: polyharmonic type equation, boundary value problems, Green function, nonlinear integral equation.

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1. Introduction

Let $D \subset R^n$ denote the simple connected and bounded domain with boundary ∂D of class C^1. Let us consider the following polyharmonic equations

$$\Delta^n u(X) = f(X), \qquad X = (x_1, ..., x_n), \tag{1}$$

$$\Delta^n u(X) = f(X, \Delta^{n-1} u(X)), \quad X \in D \tag{1a}$$

and the following boundary conditions

$$\Delta^i u(X) = f_i(X) \quad \text{for} \quad X \in \partial D, \quad i = 0, 1, ..., n-1. \tag{2}$$

To the construction of solutions of the problems (1), (2) and (1a), (2) we shall apply the convenient Green function. For the solution of the problem (1a), (2) we shall apply the suitable nonlinear integral equation. We will prove the theorems on uniqueness and existence of solutions for above boundary value problems (1),(2) and (1a), (2).

In the paper [1], the similar boundary value problem for the nonlinear equation of the second and fourth order in the three dimensional ball have been considered.

2. Theorems on uniqueness for linear boundary value problem (1), (2)

Definition of class $(\boldsymbol{K})$. *Let*

$$(K) = \{u \in C^{2n}(D) \cap C^{2n-1}(\overline{D})\}.$$

We shall prove the following theorem:

Theorem 1. *If the functions u_1, u_2 are the solutions of the boundary value problem (1), (2) of the class (K),then*

$$u_1(X) \equiv u_2(X) \quad \textit{for} \quad X \in \overline{D}.$$

Proof. Let us consider the following identity

$$\Delta^n(u_1(X) - u_2(X)) \equiv 0 \quad \text{for} \quad X \in D.$$

We have

$$\Delta(\Delta^{n-1}(u_1(X) - u_2(X))) \equiv 0 \quad \text{for} \quad X \in D.$$

Hence, the function $\Delta^{n-1}(u_1(X) - u_2(X))$ is harmonic in the domain D and

$$\Delta^{n-1}(u_1(X) - u_2(X)) = 0 \quad \text{for} \quad X \in \partial D.$$

Thus

$$\Delta^{n-1}(u_1(X) - u_2(X)) \equiv 0 \quad \text{for} X \in \overline{D}.$$

Similarly, we can verify the identity

$$\Delta^k(u_1(X) - u_2(X)) = 0 \quad \text{for} \quad k = n-1, n-2, ..., 1, 0.$$

Hence, we obtain

$$u_1(X) \equiv u_2(X) \quad \text{for} \quad X \in \overline{D}.$$

■

3. Theorem on uniqueness for nonlinear boundary value problem (1a), (2)

Let N will be the inward normal to the boundary ∂D. The boundary value problem (1a), (2) is equivalent to the following system of problems:

$$\text{(1.1)} \quad \begin{cases} \Delta^{n-1}u(X) = W_0(X), \;\; \Delta W_0 = f(X, W_0(X)) \;\text{ for }\; X \in D, \\ W_0(X) = f_{n-1}(X) \;\text{ for }\; X \in \partial D, \end{cases}$$

$$\text{(1.2)} \qquad \begin{cases} \Delta^{n-2}u(X) = W_1(X), \;\; \Delta W_1(X) = W_0(X) \;\text{ for }\; X \in D, \\ W_1(X) = f_{n-2}(X) \;\text{ for }\; X \in \partial D, \end{cases}$$

$$\text{(1.3)} \qquad \begin{cases} \Delta^{n-3}u(X) = W_2(X), \;\; \Delta W_2(X) = W_1(X) \;\text{ for }\; X \in D, \\ W_2(X) = f_{n-3}(X) \;\text{ for }\; X \in \partial D, \end{cases}$$

. . . . ,

$$\text{(1.n)} \qquad \begin{cases} \Delta u(X) = W_{n-1}(X) \;\text{ for }\; X \in D, \\ u(X) = f_0(X) \;\text{ for }\; X \in \partial D. \end{cases}$$

Definition of class (*F*). *The function $f(X, W(X)) \in (F)$ if the functions $D^j_W F(X, W)$, $j = 0, 1$, are continuous in the domain $Z = \{(X, W) : X \in \overline{D}, W \in [-r, r]\}$, where r is the positive number.*

Theorem 2. *If the functions u_1, u_2 are the solutions of the boundary value problems (1a), (2) belonging to the class (K), the function $f \in (F)$ and $D_W f(X, W) \geq 0$ for $(X, W) \in Z$, then $u_1(X) \equiv u_2(X)$ for $X \in \overline{D}$.*

Proof. Let the functions W^1_0, W^2_0 be the solutions of the problem (1.1) of the class $C^2(D) \cap C^1(\overline{D})$. Then

$$\text{(3)} \qquad \begin{aligned} \Delta(W^1_0(X) - W^2_0(X)) &= f(X, W^1_0) - f(X, W^2_0) \\ &= (W^1_0(X) - W^2_0(X)) D_N f(X, \overline{W}), \end{aligned}$$

where $\overline{W} = W^1_0 + t(W^2_0 - W^1_0)$, $t \in (0, 1)$.

Multiplying on either side the equation (3) by $W^1_0(X) - W^2_0(X)$ and further integrating on either side the domain D, we obtain $I = J$, where

$$I = \int_D (W^1_0(X) - W^2_0(X)) \Delta(W^1_0(X) - W^2_0(X)) dX,$$

$$J = \int_D (W^1_0(X) - W^2_0(X))^2 D_W f(X, \overline{W}) dX.$$

For J and I we have

$$\begin{aligned} I = &- \int_D (\mathrm{grad}(W^1_0(X) - W^2_0(X)))^2 dX \\ &+ \int_{\partial D} (W^1_0(X) - W^2_0(X)) D_N (W^1_0(X) - W^2_0(X)) dS \\ = &- \int_D (grad(W^1_0(X) - W^2_0(X)))^2 dX < 0 \end{aligned}$$

and $J \geq 0$. By $I = J$ and $I \leq 0$, $J \geq 0$ it follows that $grad(W_0^1(X) - W_0^2(X)) = 0$ for $X \in D$. Thus $W_0^1(X) - W_0^2(X) = const = 0$, since $W_0^1(X) - W_0^2(X) = 0$ for $X \in \partial D$. Let the functions W_1^1, W_1^2 of the class $C^4(D) \cap C^3(\overline{D})$ will be the solutions of the boundary value problem (1.2) for the $W = W_1^1$ and $W = W_1^2$, respectively.

Then

$$\Delta(W_1^1(X) - W_1^2(X)) = W_0^1(X) - W_0^2(X) \equiv 0 \quad \text{for} \quad X \in D$$

and

$$W_1^2(X) - W_1^1(X) \equiv 0 \quad \text{for} \quad X \in \partial D.$$

Thus, $W_1^1(X) = W_1^2(X)$ for $X \in \overline{D}$.

Analogically, for $W_{n-1}^i \in C^{2n-2}(D) \cap C^{2n-3}(\overline{D})$, we obtain

$$W_{n-1}^1(X) = W_{n-1}^2(X) \quad \text{for} \quad X \in \overline{D}$$

or

$$\Delta(u_1(X) - u_2(X)) \equiv 0 \ \text{ for } \ X \in \overline{D}.$$

Hence, by the homogeneous boundary condition for the function $u_1(X) - u_2(X)$, we obtain $u_1(X) - u_2(X) \equiv 0$ for $X \in \overline{D}$. ■

4. Theorem on existence of the solution of the boundary value problems (1), (2)

Let us consider the following functions

$$u_0(X) = A \int_{\partial D} f_0(Y) D_{N(Y)} G(X,Y) dS(Y),$$

$$u_1(X) = A \int_D G(X,Y) (\int_{\partial D} f_1(Z) D_{N(Z)} G(Y,Z) dS(Z)) dY,$$

$$u_2(X) = A \int_D G(X,Y) (\int_D G(Y,Z_1) \times (\int_{\partial D} f_2(Z_2) D_{N(Z_2)} G(Z_1,Z_2) dS(Z_2)) dZ_1) dY,$$

. . . . ,

$$u_{n-1}(X) = A \int_D G(X,Y) (\int_D G(Y,Z_1) ... \times (\int_D G(Z_{n-3}, Z_{n-2}) (\int_{\partial D} f_{n-1}(Z_{n-1}) \times D_{N(Z_{n-1})} G(Z_{n-2}, Z_{n-1}) dS(Z_{n-1})) dZ_{n-2}) ...) dZ_1) dY,$$

$$U_n(X) = A \int_D G(X,Y) (\int_D G(Y,Z_1) ... (\int_D G(Z_{n-3}, Z_{n-2}) \times (\int_D f(Z_{n-1}) G(Z_{n-2}, Z_{n-1}) dZ_{n-1}) ... dZ_1) dY,$$

where $A = (P_n)^{-1}$, P_n denote the measure of the unit n-dimensional sphere and $Z_i = (z_1^i, z_2^i, ..., z_n^i)$, $i = 1, ..., n-1$, is a point of R^n.

Lemma 1. *If the function f_0 belongs to the class $C(\partial D)$, then*

1°. *the function $u_0 \in C^{2n}(D)$ and satisfies the equation*

(4) $$\Delta^n u_0(X) = 0 \quad for \quad X \in D,$$

2°. *the function u_0 satisfies the boundary conditions*

(5) $$u_0(X) \to f_0(X_0) \quad when \quad X \to X_0 \in \partial D$$

(6) $$\Delta^i u_0(X) \to 0 \quad when \quad X \to X_0 \in \partial D, \quad i = 1, ..., n.$$

Proof. Ad. 1^o. Let us consider the integrals

$$u_0^i(X) = A \int_{\partial D} f_0(Y) D_{N(Y)} \Delta_X^i G(X,Y) dS(Y), \quad i = 0, 1, ..., n.$$

For each $X \in D$ the integrals u_0^i, $i = 0, 1, ..., n$ are locally uniformly convergent [2], p. 239. Thus, we obtain

(7) $$\Delta^i u_0(X) = 0, \quad i = 1, ..., n,$$

because $\Delta G(X,Y) = 0$ for each $(X,Y) \in D \times \partial D$.

Ad. 2^o. By [2], (p. 367) we get the condition(5). By (7), we obtain (6). ■

Lemma 2. *If the functions $f_i \in C(\partial D)$, $i = 1, ..., n-1$, then:*

1°. *the functions u_i, $i = 1, ..., n-1$, are of class $C^{2n}(D)$ and satisfy the equation (4) for $X \in D$,*

2°. *the functions u_i, $i = 1, ..., n-1$, satisfy the boundary conditions*

(8) $$\Delta^i u_i(X) \to f_i(X_0) \quad when \quad X \to X_0 \in \partial D, \quad i = 1, ..., n-1,$$

(9) $$\Delta^j u_i(X) \to 0 \quad when \quad X \to X_0 \in \partial D, \quad i, j = 1, ..., n-1, \quad i \neq j.$$

Proof. Ad. 1^o. We shall give the proof only for the integral u_2. The proof for the remaining integrals u_i, $i = 1, 3, 4, ..., n$, is similar. The integral

$$J(Y) = \int_D G(Y, Z_1)(\int_{\partial D} f_2(Z_2) D_{N(Z_2)} G(Z_1, Z_2) dS(Z_2)) dZ_1$$

is of class $C^1(\overline{D})$, because the integral

$$\int_{\partial D} f_2(Z_2) D_{N(Z_2)} G(Z_1, Z_2) dS(Z_2)$$

is continuous in $\overline{D}$. Hence, for the integral J we have

$$|J| \leq \sup_{\partial D} |f_2(Z_2)| \int_D G(Y, Z_1) dZ_1.$$

Using the spherical coordinates we can prove that the integral $J(Y)$ and, by [2], p. 327, the integrals

$$J^i(X) = \int_D D_{y_i} G(Y, Z_1)(\int_{\partial D} f_2(Z_2) D_{N(Z_2)} G(Z_1, Z_2) dS(Z_2)) dZ_1, \quad i = 1, ..., n.$$

are continuous in $\overline{D}$. By Poisson theorem we have

$$\Delta u_2(X) = A \int_D G(X, Z_1)(\int_{\partial D} f_2(Z_2) D_{N(Z_2)} G(Z_1, Z_2) dS(Z_2)) dZ_1.$$

Applying once more the Poisson theorem we obtain

$$\Delta^2 u_2(X) = A \int_{\partial D} f_2(Z_2) D_{N(Z_2)} G(X, Z_2) dS(Z_2). \tag{10}$$

Hence

$$\Delta^i u_2(X) = A \int_{\partial D} f_2(Z_2) D_{N(Z_2)} \Delta^i_X G(X, Z_2) dS(Z_2) \equiv 0, \quad i = 3, 4, ..., n. \tag{11}$$

Ad. 2^o. By (10), we have

$$\Delta^2 u_2(X) \to f_2(X_0) \text{ when } X \to X_0 \in \partial D.$$

By (11), we obtain

$$\Delta^i u_2(X) \to 0 \quad \text{when} \quad X \to X_0 \in \partial D, \quad i = 3, 4, ..., n-1.$$

■

Lemma 3. *If the function $f \in C^1(\overline{D})$, then:*

1^o *the function $U_n \in (K)$ and satisfies the equation (1) for $X \in D$,*

2^o *the function U_n satisfies the homogeneous boundary conditions*

$$\Delta^i U_n(X) \to 0 \ \ when \ \ X \to X_0 \in \partial D, \ \ i = 0, ..., n-1.$$

Proof. Ad. 1^o. Applying the Poisson theorem $(n-1)$-times we obtain

$$\Delta^{n-1} U_n(X) = A \int_D f(Y) G(X, Y) dY. \tag{12}$$

Applying once more the Poisson theorem for the formula (12) we obtain the assertion 1^o.

Ad. 2^o. We have

$$U_n(X) = A \int_D G(X, Y) U_{n-1}(Y) dY$$

and, by the boundary properties of the function G, we obtain

$$U_n(X) \to A \int_D G(X_0, Y) U_{n-1}(Y) dY = 0 \ \text{ when } \ X \to X_0 \in \partial D.$$

Similarly, we have

$$\Delta^i U_n(X) = A \int_D G(X, Y) U_{n-i-1}(Y) dY, \ \ i = 1, ..., n-1,$$

and

$$\Delta^i U_n(X) \to A \int_D G(X_0, Y) U_{n-i-1}(Y) dY = 0, \ \ i = 2, 3, ..., n-1.$$

■

By Lemmas 1, 2, 3, we obtain the following theorem:

Theorem 3. *If the functions $f_i \in C(\partial D)$, $i = 0, 1, ..., n-1$, the function $f \in C^1(\overline{D})$, then the function*

$$u(X) = \sum_{i=0}^{n-1} u_i(x) + U_n(X)$$

is the solution of the boundary value problem (1), (2).

5. Theorem on the existence and uniqueness of the solution of the problem (1),(2)

Theorem 4. *If the function $f \in C^1(\overline{D})$, then the function $U_n(X)$ is the unique solution of the problem (1), (2) satisfying the homogeneous boundary conditions.*

Proof. By Lemma 3, the function U_n satisfies the equation (1) for $X \in D$. In order to prove uniqueness it is sufficient to verify that the function $U_n \in (K)$. Indeed, by Lemma 3, function U_n satisfies the equation

$$\Delta^{n-1}U_n(X) = A\int_D f(Y)G(X,Y)dY \in C^1(D)$$

and $U_n(X) \in C^{2n-1}(\overline{D})$. Consequently, $U_n \in (K)$. ■

6. Solution of the boundary problem (1a), (2)

Let

$$S(X) = \sum_{i=0}^{n-1} u_i(X)$$

and

$$G_n(X,Y) = \int_D ... \int_D G(X,Z_1)G(Z_1,Z_2)...G(Z_{n-2},Y)dZ_1...dZ_{n-2}.$$

Let us consider the integral equation

$$V(X) = (TV)(X), \tag{13}$$

$$(TV)(X) = S(X) + (PV)(X),$$
$$(PV)(X) = A\int_D f(Y,V(Y))G_n(X,Y)dY$$

Definition of class (U). *Let $\|u\| = \sup_{\overline{D}} |u(x)|$ and let (U) denotes the class of all continuous functions for $X \in \overline{D}$ for which $\|u\| \leq r$, where r is positive number.*

Let the functions $R, Q \in (U)$ and let $d(R,Q) = \|R - Q\|$, where

$$q = A\sup_Z \left| \int_D D_W f(Y,W)G_n(X,Y)dY \right|.$$

Lemma 4. *If the functions $R, Q \in (U)$, the function $f \in (F), f(X,0) \equiv 0$ for $X \in D$, functions $f_i \in C(\partial D)$, $i = 0,1,...,n-1$, $q \in (0,1)$, $\|S\| \leq (1-q)r$, then*

1^o $(PV)(X\ )|_{V\equiv 0} = 0,$

2^o $d(TR,TQ) \leq qd(R,Q),$

3^o *for every* $u \in (U)$, $\|Tu\| \leq r$.

Proof. The assertion 1^o is evident.

Ad. 2^o We have

$$\begin{aligned} d(TR,TQ) &= d(PR,PQ) \\ &= A \sup_Z \left| \int_D [f(Y,R(Y) - f(Y,Q(Y))]G_n(X,Y)dY \right| \\ &= A \sup_Z \left| \int_D (R(Y) - Q(Y))D_W f(Y,\overline{W})G_n(X,Y)dY \right|, \end{aligned}$$

where

$$\overline{W} = R + t(Q-R), \quad t \in (0,1), \quad \overline{W} \in (U).$$

Thus

$$d(TR,TR) \ \leq \ qd(R,Q).$$

Ad. 3^o By 1^o, we have

$$\begin{aligned} \|Tu\| &= \|Tu - TO + TO\| \leq \|Tu - TO\| + \|TO\| \\ &= \|S + Pu - S - P(0)\| \leq \|Pu - P(0)\| = d(u,0) = \|u\| \leq r. \end{aligned}$$

From Lemma 4, it follows that the mapping T is the contracting mapping for the function of class (U) and the operator T transforms each function $u \in (U)$ into the function belonging to the class (U). Consequently, by the Banach theorem there exists the unique solution $V \in (U)$ of the equation (13). ■

Theorem 5. *If the function $f \in (F), f(X,0) \equiv 0$ for $X \in \overline{D}$, the functions $f_i \in C(\partial D)$, $i = 0,1,...,n-1$, $q \in (0,1)$, $\|S\| \leq (1-q)r$, then the function V is the solution of the integral equation (13) satisfying the conditions:*

1^o *the function V satisfies (1a) for $X \in D$,*

2^o *the function V satisfies the boundary conditions (2) for $X \in \partial D$.*

Proof. Ad. 1^o By Lemmas 1, 2, the function $S(X)$ satisfies the equation

$$\Delta^n S(X) = 0 \text{ for } X \in D.$$

Applying n-times the Poisson theorem we obtain

$$\Delta^n (PV)(X) = f(X, V(X)) \text{ for } X \in D$$

and finally, by (13), we obtain

$$\Delta^n V(X) = \Delta^n S(X) + \Delta^n (PV)(X) = f(X, V(X)) \text{for } X \in D.$$

Ad. 2^o By Lemma 2, the function S satisfies the boundary conditions (2) and, by Lemma 3, we have

$$\Delta^k (PU)(X) = 0 \text{ for } X \in \partial D, \ \ k = 0, 1, ..., n-1.$$

■

7. Theorem on existence and uniqueness for the equation (1a) and homogeneous boundary data

Theorem 6. *If the function $f \in (F)$, $f(X,0) \equiv 0$ for $X \in \overline{D}$, $D_W f(X,W) \geq 0$ for $(X,W) \in Z$, $q \in (0,1)$, then the function U is the solution of the*

$$U(X) = (PU)(X) \text{ for } X \in D \tag{14}$$

belonging to the class (U) and satisfies the conditions:

1^o *the function $U(X)$ satisfies the equation (1a) for $X \in D$,*

2^o *the function U satisfies the homogeneous boundary conditions*

$$\Delta^i U(X) = 0 \text{ for } X \in \partial D, \ \ i = 0, ..., n-1, \tag{2a}$$

3^o *the function U is the unique solution of the problem (1a), (2a).*

Proof. Ad. 1^o Similarly as for the equation (13) we construct it solution U belonging to the class (U). Similarly, as in Lemma 3, we can verify that the function U satisfies the conditions (1a), (2a). ■

8. Example of a physical application of biharmonic boundary value problems in the theory of elasticity

In the theory of elasticity the following boundary problem is considered:

$$\Delta^2 u(X) = f(X, u(X)), \quad \text{for } X \in D, \tag{15}$$

$$\Delta^i u(X) = f_i(X), \quad \text{for } X \in D, \quad i = 0, 1, \tag{16}$$

where D is a disc or a three-dimensional ball and $f = 0$.

If $f \neq 0$, then solution of above boundary value problem (14), (15) is not known. By foregoing results we obtain the solution of problem (14), (15) applying the Green function for the Laplace equation for the domain D and for the Dirichlet boundary condition.

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