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NOTES ON SPACES WITH A WEAK-DEVELOPMENT

Abstract. Z. Li characterized spaces with a weak-development consisting of point-countable sfp-covers by pseudo-sequence-covering, quotient, and π-s-image of metric spaces. In this paper, we omit "pseudo-sequence-covering" in the above result, and prove that a space has a weak-development consisting of point-countable sfp-covers iff it is a quotient, and π-s-image of a metric space.

Key words: weak-(resp. sn-)development, sfp-(resp. fcs-, cs^*-) cover, quotient (resp. pseudo-sequence-covering, sequentially-quotient, π-) mapping.

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1. Introduction

Recently, Z. Li [7] obtained the following result.

Proposition 1. *A space has a weak-development consisting of point-countable sfp-covers iff it is a pseudo-sequence-covering, quotient, and π-s-image of a metric space.*

In this paper, we prove that a space has an sn-development consisting of point-countable cs^*-covers iff it is a sequentially-quotient, and π-s-image of a metric space. By this result, we prove that a space has a weak-development consisting of point-countable sfp-covers iff it is a quotient, and π-s-image of a metric space, which omits "pseudo-sequence-covering" in Proposition 1. Throughout this paper, all spaces mean regular and T_1 topological spaces, all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers. Let X be a space and $P \subset X$. We say that a sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of X and let $x \in X$. $\bigcup \mathcal{P}$, $st(x, \mathcal{P})$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$, the union $\bigcup\{P \in \mathcal{P} : x \in P\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$, respectively. If $f : X \longrightarrow Y$ is a mapping, $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. We shortly denote a point $b = (\beta_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space by (β_n).

Definition 1. *Let X be a space.*

(1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to x in X, then $\{x_n\}$ is eventually in P.

(2) Let $P \subset X$. P is called a sequentially open subset in X if P is a sequential neighborhood of x in X for each $x \in P$.

(3) X is called a sequential space if each sequentially open subset in X is open in X.

Remark 1. (1) P is a sequential neighborhood of x iff each sequence $\{x_n\}$ converging to x is frequently in P.

(2) The intersection of finitely many sequential neighborhoods of x is a sequential neighborhood of x.

Definition 2. *Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that the following conditions (a) and (b) are satisfied for each $x \in X$.*

(a) $\mathcal{P}_x$ is a network at x in X, i.e., $\mathcal{P}_x \subset (\mathcal{P})_x$ and for each neighborhood U of x in X, $P \subset U$ for some $P \in \mathcal{P}_x$;

(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \bigcap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is called a weak-base [1] for X if whenever $G \subset X$, G is open in X iff for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, where $\mathcal{P}_x$ is called a wn-base (i.e., weak neighborhood base) at x in X.

(2) $\mathcal{P}$ is called an sn-network [4] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$, where $\mathcal{P}_x$ is called an sn-network at x in X.

Remark 2. Each weak-base for a space is an *sn*-network and each *sn*-network for a sequential space is a weak-base [8].

Definition 3. *Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$.*

(1) $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a net-development of X if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for each $x \in X$.

(2) $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called an sn-development of X if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is an sn-network at x in X for each $x \in X$.

(3) $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a weak-development of X [7] if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a wn-base at x in X for each $x \in X$.

Remark 3. (1) By Remark 2, each weak-development of a space is an *sn*-development and each *sn*-development of a sequential space X is a weak-development.

(2) Each space with a weak-development is a sequential space [10].

Definition 4. *Let $\mathcal{P}$ be a cover of a space X.*

(1) $\mathcal{P}$ is called an sfp-cover [7] if for each sequence $\{x_n\}$ converging to x in X, there is a finite family $\{S_\alpha : \alpha \in \Gamma\}$ of closed subsets of S and a finite subfamily $\{P_\alpha : \alpha \in \Gamma\}$ of $\mathcal{P}$ such that $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ and $S_\alpha \subset P_\alpha$ for each $\alpha \in \Gamma$, where $S = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$.

(2) $\mathcal{P}$ is called a fcs-cover [5] if for each sequence $\{x_n\}$ converging to x in X, there is a finite subfamily $\mathcal{P}'$ of $(\mathcal{P})_x$ such that $\{x_n\}$ is eventually in $\bigcup \mathcal{P}'$.

(3) $\mathcal{P}$ is called a cs^*-cover [4] if for each convergent sequence $\{x_n\}$ in X, $\{x_n\}$ is frequently in P for some $P \in \mathcal{P}$.

Definition 5. *Let $f : X \longrightarrow Y$ be a mapping.*

(1) f is called a pseudo-sequence-covering mapping [6] if for each sequence $\{y_n\}$ converging to y in Y, there is a compact subset K of X such that $f(K) = \{y_n : n \in \mathbb{N}\} \bigcup \{y\}$.

(2) f is called a sequentially-quotient mapping [2] if for each convergent sequence $\{y_n\}$ in Y, there is a convergent sequence $\{x_n\}$ in X such that $\{f(x_n)\}$ is a subsequence of $\{y_n\}$.

(3) f is called a quotient mapping [3] if U is open in Y iff $f^{-1}(U)$ is open in X.

(4) If X is a metric space with the metric d, f is called a π-mapping [9], if for each $y \in Y$ and for each neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$.

Remark 4. (1) Recall a mapping $f : X \longrightarrow Y$ is a compact mapping (resp. s-mapping), if $f^{-1}(y)$ is a compact (resp. separable) subset of X for each $y \in Y$. It is clear that each compact mapping from a metric space is a π-s-mapping.

(2) Each quotient mapping from a sequential space is a sequentially-quotient mapping [4, Remark 1.8].

(3) Each sequentially-quotient mapping onto a sequential space is a quotient mapping [4, Remark 1.8].

(4) Quotient mappings preserve sequential spaces [3, Exercises 2.4.G].

Lemma 1. *Let $\mathcal{P}$ be a cover of a space X. Then the following are equivalent.*

(1) $\mathcal{P}$ is an sfp-cover of X.

(2) $\mathcal{P}$ is an fcs-cover of X.

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{P}$ be an sfp-cover of X. Whenever $\{x_n\}$ is a sequence converging to x in X, put $S = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$. Then there is a finite family $\{S_\alpha : \alpha \in \Gamma\}$ of closed subsets of S and a finite subfamily $\{P_\alpha : \alpha \in \Gamma\}$ of $\mathcal{P}$ such that $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ and $S_\alpha \subset P_\alpha$ for each $\alpha \in \Gamma$. Put $\Gamma' = \{\alpha \in \Gamma : x \notin S_\alpha\}$ and $S' = \bigcup\{S_\alpha : \alpha \in \Gamma'\}$, then S' is closed in

S and $x \notin S'$. Thus there is $k \in \mathbb{N}$ such that $x_n \notin S'$ foe each $n > k$. It follows that $\{x_n\}$ is eventually in $\bigcup\{P_\alpha : \alpha \in \Gamma - \Gamma'\}$ and $\{P_\alpha : \alpha \in \Gamma - \Gamma'\}$ is a finite subfamily of $(\mathcal{P})_x$. So $\mathcal{P}$ is an fcs-cover of X.

(2) $\Longrightarrow$ (1). Let $\mathcal{P}$ be an fcs-cover of X. Whenever $\{x_n\}$ is a sequence converging to x in X, then there is a finite subfamily $\mathcal{P}' = \{P_\alpha : \alpha \in \Gamma_1\}$ of $(\mathcal{P})_x$ such that $\{x_n\}$ is eventually in $\bigcup \mathcal{P}'$. Put $S = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$, then $S - \bigcup \mathcal{P}' = \{x_\alpha : \alpha \in \Gamma_2\}$ is finite. For each $\alpha \in \Gamma_2$, there is $P_\alpha \in \mathcal{P}$ such that $x_\alpha \in P_\alpha$. Put $S_\alpha = P_\alpha \bigcap S$ for each $\alpha \in \Gamma_1$ and put $S_\alpha = \{x_\alpha\}$ for each $\alpha \in \Gamma_2$. Put $\Gamma = \Gamma_1 \bigcup \Gamma_2$. It is easy to see that $\{S_\alpha : \alpha \in \Gamma\}$ is a finite family of closed subsets of S and $\{P_\alpha : \alpha \in \Gamma\} \subset \mathcal{P}$. Moreover $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ and $S_\alpha \subset P_\alpha$ for each $\alpha \in \Gamma$. So $\mathcal{P}$ is an sfp-cover of X. ■

Lemma 2. *Let $\mathcal{P}$ be a point-countable cover of a space X. Then the following are equivalent.*

(1) $\mathcal{P}$ is an sfp-cover of X.

(2) $\mathcal{P}$ is an fcs-cover of X.

(3) $\mathcal{P}$ is a cs^*-cover of X.

Proof. (1) $\Longleftrightarrow$ (2). It holds from Lemma 1.

(2) $\Longrightarrow$ (3). It is clear from Definition 4.

(3) $\Longrightarrow$ (2). Let $\mathcal{P}$ be a point-countable cs^*-cover of X. Whenever $\{x_n\}$ is a sequence converging to x in X, put $S = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$. $\mathcal{P}$ is point-countable, put $(\mathcal{P})_x = \{P_n : n \in \mathbb{N}\}$. We claim that $\{x_n\}$ is eventually in $\bigcup_{n \le k} P_n$ for some $k \in \mathbb{N}$. In fact, if not, then there is $x_{n_k} \in S - \bigcup_{n \le k} P_n$ for each $k \in \mathbb{N}$, . We may assume $n_1 < n_2 < \cdots < n_{k-1} < n_k < n_{k+1} < \cdots$. Put $S' = \{x_{n_k} : k \in \mathbb{N}\}$, then S' is a sequence converging to x. Since $\mathcal{P}$ is a cs^*-cover, there is $m \in \mathbb{N}$ such that S' is frequently in P_m. This contradicts the construction of S'. ■

Lemma 3. *Let $f : X \longrightarrow Y$ be a mapping, and let $\{y_n\}$ be a sequence converging to y in Y. If $\{B_k\}$ is a decreasing network at some point $x \in f^{-1}(y)$, and $\{y_n\}$ is frequently in $f(B_k)$ for each $k \in \mathbb{N}$, then there is a sequence $\{x_k\}$ converging to x in X such that $\{f(x_k)\}$ is a subsequence of $\{y_n\}$.*

Proof. Since $\{y_n\}$ is frequently in $f(B_1)$, there is $n_1 \in \mathbb{N}$ such that $y_{n_1} \in f(B_1)$. Choose $x_1 \in f^{-1}(y_{n_1}) \bigcap B_1$. We construct a sequence $\{x_k\}$ by induction as follows. Assume x_k has been chosen for $k \in \mathbb{N}$. Since $\{y_n\}$ is frequently in $f(B_{k+1})$, there is $n_{k+1} \in \mathbb{N}$ and $n_{k+1} > n_k$ such that $y_{n_{k+1}} \in f(B_{k+1})$, so we may choose $x_{k+1} \in f^{-1}(y_{n_{k+1}}) \bigcap B_{k+1}$. By induction, we construct a sequence $\{x_k\}$ such that $\{f(x_k)\} = \{y_{n_k}\}$ is a subsequence of

$\{y_n\}$. Note that $x_k \in B_k$ for each $k \in \mathbb{N}$, and $\{B_k\}$ is a decreasing network at x. So $\{x_k\}$ converges to x. ■

Theorem 1. *Let X be a space. Then the following are equivalent.*
(1) X is a sequentially-quotient, and π-s-image of a metric space.
(2) X has a net-development consisting of point-countable cs^*-covers.
(3) X has a net-development consisting of point-countable fcs-covers.
(4) X has a net-development consisting of point-countable sfp-covers.

Proof. (2)$\Longleftrightarrow$(3) $\Longleftrightarrow$ (4) from Lemma 2.

(1) $\Longrightarrow$ (2). Let (M,d) be a metric space, and let $f : M \longrightarrow X$ be a sequentially-quotient, and π-s-mapping. We write $B(a,n) = \{b \in M : d(a,b) < 1/n\}$ for each $a \in M$ and each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, put $\mathcal{B}_n = \{B(a,n) : a \in M\}$, and let $\mathcal{A}_n$ be a locally-finite open refinement of $\mathcal{B}_n$. Put $\mathcal{F}_n = \{\bigcap_{i\leq n} A_i : A_i \in \mathcal{A}_i\}$, then $\mathcal{F}_n$ is a locally-finite open refinement of $\mathcal{B}_n$. Put $\mathcal{P}_n = f(\mathcal{F}_n)$, then $\mathcal{P}_n$ refines $f(\mathcal{B}_n)$.

Claim 1. $\mathcal{P}_n$ is a point-countable cover of X for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $x \in X$. As f is an s-mapping, $f^{-1}(x)$ is a Lindelöf subset of M, so $\{F \in \mathcal{F}_n : F \bigcap f^{-1}(x) \neq \emptyset\}$ is countable. Thus x only belongs to countable elements of $\mathcal{P}_n$, This proves that $\mathcal{P}_n$ is a point-countable cover of X.

Claim 2. $\mathcal{P}_n$ is a cs^*-cover of X for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and S be a sequence converging to x in X. Since f is sequentially-quotient, there is a sequence L in M converging to $a \in f^{-1}(x) \subset M$ such that $f(L)$ is a subsequence of S. Choose $F \in \mathcal{F}_n$ such that $a \in F$. Then L is eventually in F, so $f(L)$ is eventually in $f(F) \in \mathcal{P}_n$, thus S is frequently in $f(F) \in \mathcal{P}_n$. This proves that $\mathcal{P}_n$ is a cs^*-cover of X.

Claim 3. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a net-development of X,

For each $n \in \mathbb{N}$, $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$, so $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. Let $x \in X$, it suffices to prove that $\{st(x,\mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X. Let $x \in U$ with U open in X. Since f is a π-mapping, there is $m \in \mathbb{N}$ such that $d(f^{-1}(x), M - f^{-1}(U)) > 1/m$. Pick $k \in \mathbb{N}$ such that $k > 2m$. Then $st(x, f(\mathcal{B}_k)) \subset U$. In fact, let $x \in f(B(a,k)) \in f(\mathcal{B}_k)$, where $a \in M$. Then $f^{-1}(x) \bigcap B(a,k) \neq \emptyset$. If $B(a,k) \not\subset f^{-1}(U)$, choose $b \in f^{-1}(x) \bigcap B(a,k)$ and $c \in B(a,k) - f^{-1}(U)$, then $d(b,c) \leq d(b,a) + d(a,c) < 1/k + 1/k = 2/k$, thus $d(f^{-1}(x), M - f^{-1}(U)) \leq 2/k < 1/m$. This is a contradiction. So $B(a,k) \subset f^{-1}(U)$, hence $f(B(a,k)) \subset ff^{-1}(U) = U$, thus $st(x, f(\mathcal{B}_k)) \subset U$. Note that $st(x,\mathcal{P}_k) \subset st(x, f(\mathcal{B}_k))$ because $\mathcal{P}_k$ refines $f(\mathcal{B}_k)$. So $st(x,\mathcal{P}_k) \subset U$. This proves that $\{st(x,\mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x.

By the above, X has a net-development $\{\mathcal{P}_n : n \in \mathbb{N}\}$ consisting of point-countable cs^*-covers.

(2) $\Longrightarrow$ (1). Let X have a net-development $\{\mathcal{P}_n : n \in \mathbb{N}\}$ consisting of point-countable cs^*-covers.

For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\beta : \beta \in \Lambda_n\}$, and endow Λ_n a discrete topology. Put $M = \{b = (\beta_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \{P_{\beta_n} : n \in \mathbb{N}\}$ *is a network at some* $x_b \in X\}$. Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} \Lambda_n$, is a metric space with metric d described as follows:

Let $b = (\beta_n), c = (\gamma_n) \in M$. If $b = c$, then $d(b,c) = 0$. If $b \neq c$, then $d(b,c) = 1/min\{n \in \mathbb{N} : \beta_n \neq \gamma_n\}$.

Define $f : M \longrightarrow X$ by $f(b) = x_b$ for each $b = (\beta_n) \in M$, where $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x_b. It is not difficult to prove that f is continuous and onto.

Claim 1. f is a π-mapping.

Let $x \in U$ with U open in X. Since $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a net-development of X, there is $m \in \mathbb{N}$ such that $st(x, \mathcal{P}_m) \subset U$. If $b = (\beta_n) \in M$ such that $d(f^{-1}(x), b) < 1/m$, then there is $c = (\gamma_n) \in f^{-1}(x)$ such that $d(b,c) < 1/m$, thus $\beta_k = \gamma_k$ if $k \leq m$. Notice that $x \in P_{\gamma_m} \in \mathcal{P}_m$ and $\beta_m = \gamma_m$. So $f(b) \in P_{\beta_m} = P_{\gamma_m} \subset st(x, \mathcal{P}_m) \subset U$, hence $b \in f^{-1}(U)$. Thus $d(f^{-1}(x), b) \geq 1/m$ if $b \in M - f^{-1}(U)$, and so $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/m > 0$. This proves that f is a π-mapping.

Claim 2. f is an s-mapping.

Let $x \in X$. For each $n \in \mathbb{N}$, put $D_n = \{\beta \in \Lambda_n : x \in P_\beta\}$, then D_n is countable, and so $\prod_{n\in\mathbb{N}} D_n$, which is a product of countably many separable spaces, is separable. It suffices to prove that $f^{-1}(x) = \prod_{n\in\mathbb{N}} D_n$. If $b = (\beta_n) \in f^{-1}(x)$, then $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x in X. For each $n \in \mathbb{N}$, $x \in P_{\beta_n}$ and $\beta_n \in \Lambda_n$, i.e., $\beta_n \in D_n$. So $b \in \prod_{n\in\mathbb{N}} D_n$, thus $f^{-1}(x) \subset \prod_{n\in\mathbb{N}} D_n$. Conversely, if $b = (\beta_n) \in \prod_{n\in\mathbb{N}} D_n$, then $x \in P_{\beta_n} \in \mathcal{P}_n$. It is easy to see that $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x in X. so $f(b) = x$, i.e., $b \in f^{-1}(x)$. Thus $\prod_{n\in\mathbb{N}} D_n \subset f^{-1}(x)$. This proves that $f^{-1}(x) = \prod_{n\in\mathbb{N}} D_n$.

Claim 3. f is a sequentially-quotient mapping.

Let $x \in X$ and let S be a sequence converging to x in X. $\mathcal{P}_1$ is a cs^*-cover of X, so there is a subsequence S_1 of S such that S_1 is eventually in P_{β_1} for some $\beta_1 \in \Lambda_1$. For $m \in \mathbb{N}$, assume that we have obtained a subsequence S_m of S such that S_m is eventually in P_{β_m} for some $\beta_m \in \Lambda_m$. $\mathcal{P}_{m+1}$ is a cs^*-cover of X, so there is a subsequence S_{m+1} of S_m such that S_{m+1} is eventually in $P_{\beta_{m+1}}$ for some $\beta_{m+1} \in \Lambda_{m+1}$. By induction, for each $n > 1$, we may choose $\beta_n \in \Lambda_n$ and a subsequence S_n of S_{n-1} such that S_n is eventually in $P_{\beta_n} \in \mathcal{P}_n$. Put $b = (\beta_n) \in \prod_{n\in\mathbb{N}} \Lambda_n$. It is clear that $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x in X, so $b \in M$ and $f(b) = x$. For each $n \in \mathbb{N}$, put $B_n = \{(\gamma_k) \in M : \gamma_k = \beta_k \ for \ k \leq n\}$. Then $\{B_n\}$ is a decreasing neighborhood base at b in M. We claim that $f(B_n) = \bigcap_{k\leq n} P_{\beta_k}$ for each $n \in \mathbb{N}$. In fact, let $c = (\gamma_k) \in B_n$, then $f(c) \in \bigcap_{k\in\mathbb{N}} P_{\gamma_k} \subset \bigcap_{k\leq n} P_{\gamma_k} = \bigcap_{k\leq n} P_{\beta_k}$, so $f(B_n) \subset \bigcap_{k\leq n} P_{\beta_k}$. On the other hand, let $y \in \bigcap_{k\leq n} P_{\beta_k}$, then there is

$c' = (\gamma'_k) \in M$ such that $f(c') = y$. For each $k \in \mathbb{N}$, put $\gamma_k = \beta_k$ if $k \leq n$, and $\gamma_k = \gamma'_k$ if $k > n$. It is easy to see that $\{P_{\gamma_n} : n \in \mathbb{N}\}$ is a network at y in X. Put $c = (\gamma_k)$, then $c \in B_n$ and $f(c) = y$. This show that $y \in f(B_n)$. So $\bigcap_{k \leq n} P_{\beta_k} \subset f(B_n)$, thus $f(B_n) = \bigcap_{k \leq n} P_{\beta_k}$. For each $n \in \mathbb{N}$, by the construction of S_n, S_n is eventually in P_{β_k} for each $k \leq n$, and so S_n is eventually in $\bigcap_{k \leq n} P_{\beta_k} = f(B_n)$. Thus S is frequently in $f(B_n)$ for each $n \in \mathbb{N}$. By Lemma 3, there is a sequence $\{b_n\}$ converging to b such that $\{f(b_n)\}$ is a subsequence of S. So f is sequentially-quotient map.

By the above, X is a sequentially-quotient, and π-s-image of a metric space. ■

Lemma 4. *Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a net-development of space X. If $\mathcal{P}_n$ is a cs*-cover of X for each $n \in \mathbb{N}$, then $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is an sn-development of X.*

Proof. It suffices to prove that $st(x, \mathcal{P}_n)$ is a sequential neighborhood of x in X for each $x \in X$ and each $n \in \mathbb{N}$. Whenever S is a sequence converging to x in X, $\mathcal{P}_n$ is a cs^*-cover of X, so S is frequently in P for some $P \in \mathcal{P}_n$. Note that $P \subset st(x, \mathcal{P}_n)$, S is frequently in $st(x, \mathcal{P}_n)$. By Remark 1(1), $st(x, \mathcal{P}_n)$ is a sequential neighborhood of x in X. ■

The following corollary is obtained immediately from Theorem 1 and Lemma 4.

Corollary 1. *Let X be a space. Then the following are equivalent.*
(1) X is a sequentially-quotient, and π-s-image of a metric space.
(2) X has a sn-development consisting of point-countable cs*-covers.
(3) X has a sn-development consisting of point-countable fcs-covers.
(4) X has a sn-development consisting of point-countable sfp-covers.

Theorem 2. *Let X be a space. Then the following are equivalent.*
(1) X is a quotient, and π-s-image of a metric space.
(2) X has a weak-development consisting of point-countable cs*-covers.
(3) X has a weak-development consisting of point-countable fcs-covers.
(4) X has a weak-development consisting of point-countable sfp-covers.

Proof. (2)$\Longleftrightarrow$(3) $\Longleftrightarrow$ (4) from Lemma 2.

(1) $\Longrightarrow$ (2). Let $f : M \longrightarrow X$ be a quotient, and π-s-mapping, where M is a metric space. Then f is sequentially-quotient from Remark 4(2). So X has a sn-development $\{\mathcal{P}_n : n \in \mathbb{N}\}$ consisting of point-countable cs^*-covers from Corollary 1. By Remark 4(4), X is a sequential space. So $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is weak-development of X from Remark 3(1).

(2) $\Longrightarrow$ (1). Let X have a weak-development consisting of point-countable cs^*-covers. Then X has an sn-development consisting of point-countable

cs^*-covers from Remark 3(1). So X is a sequentially-quotient, and π-s-image of a separable metric space from Corollary 1. By Remark 3(2) and Remark 4(3), X is a quotient, and π-s-image of a metric space. ∎

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