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NONHOMOGENEOUS PARABOLIC PROBLEM IN THE SPACE R^∞

Abstract. A Cauchy problem for nonhomogeneous parabolic type partial differential equation in the infinite dimensional Cartesian space is studied in this paper.

Key words: parabolic type equation, nonhomogeneous Cauchy problem, infinite dimensional Cartesian space, parabolic type potentials.

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1. Introduction

The subject of the paper is the construction of the solution to the equation

$$\left(\sum_{i=1}^{\infty} D_{x_i}^2 - D_t\right) u(x,t) = \prod_{i=1}^{\infty} f_i(x_i,t), \quad x = (x_1, x_2, ...) \in R^\infty \tag{1}$$

in the domain

$$D \ = \ \{(x,t) : x_i \in (-\infty, \infty), \quad i = 1, 2, ..., \quad t \in (0,T)\},$$

satisfying the initial condition

$$u(x,0) = \prod_{i=1}^{\infty} h_i(x_i), \quad x_i \in (-\infty,\infty), \quad i = 1, 2, ... \tag{2}$$

To solve the problem (1)-(2) we shall apply the suitable kernels being fundamental solutions of the equations

$$(D_{x_i}^2 - D_t) U_i(x_i, t; y_i, s) = 0, \quad i = 1, 2, ...,$$

where

$$U_i(x_i, t; y_i, s) = A\ (t-s)^{-1/2} \exp(B(t,s)(x_i - y_i)^2$$
$$A = (2\sqrt{\pi})^{-1}, \qquad B(t,s) = (-4(t-s))^{-1}.$$

In [2] the similar Cauchy problem for homogeneous equation is treated.

2. Solution of the homogeneous problem in the space R^∞

Let us consider the following potentials:

$$V_i(x_i,\ t) = \int_{-\infty}^{\infty} h_i(y_i) U_i(x_i, t; y_i, 0) dy_i, \qquad i = 1, 2, ...,$$

$$V_\infty(x,\ t) = \prod_{i=1}^{\infty} V_i(x_i,\ t) = \int_{R^\infty} \prod_{i=1}^{\infty} h_i(y_i) U_i(x_i, t; y_i, 0) dy_i,$$

where $x = (x_1, x_2, ...) \in R^\infty$, $t \in (0, T)$.

Denote by (Π) the class of all functions $h_i : R \to R$, such that the infinite product $\prod_{i=1}^{\infty} h_i(y_i)$, $y_i \in (-\infty, \infty)$, $i = 1, 2, ...$, is convergent.

Denote by $C(R) \bigcap B(R)$ the class of all continuous and bounded functions $h_i : R \to R$.

By [2], we have

Theorem 1. *If* $h_i \in (\Pi)$, $i = 1, 2, ...,$ *and* $h_i \in C(R) \bigcap B(R)$, *then:*

1. $\left(\sum_{i=1}^{\infty} D^2_{x_i} - D_t \right) V_\infty(x, t) = 0, \quad x = (x_1, x_2, ...) \in R^\infty, \quad x_i \in (-\infty, \infty),$
$i = 1, 2, ..., \quad t \in (0, T).$

2. $V_\infty(x, 0) = \prod_{i=1}^{\infty} h_i(x_i), \quad x = (x_1, x_2, ...) \in R^\infty, \quad x_i \in (-\infty, \infty),$
$i = 1, 2, ... , \quad t \in (0, T).$

Conclusion 1. The function

$$u(x, t) = V_\infty(x, t) = \int_{R^\infty} \prod_{i=1}^{\infty} h_i(y_i) U_i(x_i, t; y_i, 0) dy_i,$$
$$x = (x_1, x_2, ...) \in R^\infty, \quad t \in (0, T),$$

is the solution of the Cauchy problem for homogeneous parabolic type equation in R^∞ with initial condition (2).

3. The potential compatibility with considered problem and its properties

Now, let us consider the following potential:

$$W_\infty(x,\ t) = \int_0^t \int_{R^\infty} \prod_{i=1}^{\infty} f_i(y_i, s) U_i(x_i, t; y_i, s) dy_i ds,$$
$$x = (x_1, x_2, ...) \in R^\infty, \quad t \in (0, T).$$

Denote by (K) the class of all functions $f_i : D_1 \to R$, $D_1 = (-\infty, \infty) \times (0, T)$ such that:

1. $f_i \in C^{1,0}(D_1)$, $i = 1, 2, 3, ...$
2. $D^j_{y_i} f_i(-\infty, s) = D^j_{y_i} f_i(\infty, s) = 0$, $s \in (0, T)$, $j = 0, 1, 2$, $i = 1, 2, 3, ...$
3. $D^j_{y_i} \prod_{i=1}^{\infty} f_i(y_i, t)$ is convergent for $y_i \in (-\infty, \infty)$, $j = 0, 1, 2$,
4. $D_t \prod_{i=1}^{\infty} f_i(y_i, t)$ is convergent for $y_i \in (-\infty, \infty)$.
5. comp.supp $f_i \subset (-\infty, \infty) \times [0, T]$.

Theorem 2. *If $h_i \in (\Pi) \bigcap C(R) \bigcap B(R)$, $f_i \in (K)$, $i = 1, 2, ...$ and if $\prod_{i=1}^{\infty} y_i$ is convergent for $y_i \in (-\infty, \infty)$, then:*

1. $\left(\sum_{i=1}^{\infty} D^2_{x_i} - D_t \right) W_\infty(x, t) = \prod_{i=1}^{\infty} f_i(x_i, t)$, $x = (x_1, x_2, ...) \in R^{\infty}$, $x_i \in (-\infty, \infty)$, $i = 1, 2, ...,$ $t \in (0, T)$.
2. $W_\infty(x, 0) = 0$, $x = (x_1, x_2, ...) \in R^{\infty}$, $x_i \in (-\infty, \infty)$, $i = 1, 2, ...,$ $t \in (0, T)$.

Proof. Ad.1. We have

$$D^2_{x_i} W_\infty(x,\ t) = \int_0^t \int_{R^\infty} D^2_{x_i} \prod_{i=1}^{\infty} f_i(y_i, s) U_i(x_i, t; y_i, s) dy_i ds,$$

$$D_t W_\infty(x,\ t) = \lim_{s \to t} \int_{R^\infty} \prod_{i=1}^{\infty} f_i(y_i, s) U_i(x_i, t; y_i, s) dy_i$$
$$+ \int_0^t \int_{R^\infty} \prod_{i=1}^{\infty} f_i(y_i, s) \sum_{k=1}^{\infty} ((D_t U_k(x_i, t; y_i, s) \prod_{\substack{i=1 \\ i \neq k}}^{\infty} U_i(x_i, t; y, s) dy_i ds).$$

By Poisson theorem, [1], p. 336, we obtain

$$\left(\sum_{i=1}^{\infty} D^2_{x_i} - D_t \right) W_\infty(x, t) = \prod_{i=1}^{\infty} f_i(x_i, t)$$
$$+ \int_0^t \int_{R^\infty} \prod_{i=1}^{\infty} f_i(y_i, s) \sum_{k=1}^{\infty} ((D^2_{x_i} U_k(x_i, t; y_i, s) \prod_{\substack{i=1 \\ i \neq k}}^{\infty} U_i(x_i, t; y_i, s) dy_i ds)$$
$$- \int_0^t \int_{R^\infty} \prod_{i=1}^{\infty} f_i(y_i, s) \sum_{k=1}^{\infty} ((D_t U_k(x_i, t; y_i, s) \prod_{\substack{i=1 \\ i \neq k}}^{\infty} U_i(x_i, t; y_i, s) dy_i ds)$$

$$= \prod_{i=1}^{\infty} f_i(x_i,t) + \int_0^t \int_{R^\infty} \prod_{i=1}^{\infty} f_i(y_i,s) \sum_{k=1}^{\infty} ((D_{x_i}^2 - D_t)U_k(x_i,t;y_i,s) \times \prod_{\substack{i=1\\ i\neq k}}^{\infty} U_i(x_i,t;y_i,s)dy_i$$

$$= \prod_{i=1}^{\infty} f_i(x_i,t) \text{ for } (x,t) \in D.$$

Ad 2. We have

$$\begin{aligned} W_\infty(x,\ 0) &= \lim_{\substack{t\to 0\\ n\to\infty}} \int_0^t \int_{R^n} \prod_{i=1}^{n} f_i(y_i,s)U_i(x_i,t;y_i,s)dy_i ds \\ &= \left| \lim_{\substack{t\to 0\\ n\to\infty}} \int_0^t \int_{R^n} \prod_{i=1}^{n} f_i(y_i,s)(t-s)^{-\frac{1}{2}} \exp\left(-\frac{(x_i-y_i)^2}{4(t-s)}\right) dy_i ds \right| \\ &\le \sup_{(-\infty,\infty)\times[0,T]} |f_i(y_i,s)| \left| \lim_{\substack{t\to 0\\ n\to\infty}} \int_0^t \int_{R^n} (t-s)^{-\frac{1}{2}} dy_i ds \right| \\ &= \sup_{(-\infty,\infty)\times[0,T]} |f_i(y_i,s)| \lim_{\substack{t\to 0\\ n\to\infty}} 2\cdot t^{n/2} \prod_{i=1}^{n} y_i \\ &\le \sup_{(-\infty,\infty)\times[0,T]} |f_i(y_i,s)| \lim_{\substack{t\to 0\\ n\to\infty}} 2\cdot t\cdot \mathit{const} \to 0 \text{ as } t\to 0. \end{aligned}$$

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Conclusion 2. The function

$$\begin{aligned} u(x,t) &= V_\infty(x,\ t) + W_\infty(x,\ t) = \\ &= \int_{R^\infty} \prod_{i=1}^{\infty} h_i(y_i)U_i(x_i,t;y_i,0)dy_i \\ &\quad + \int_0^t \int_{R^\infty} \prod_{i=1}^{\infty} f_i(y_i,s)U_i(x_i,t;y_i,s)dy_i ds, \\ &\qquad x=(x_1,x_2,...) \in R^\infty,\ \ t\in(0,T) \end{aligned}$$

is the solution of the Cauchy problem (1)-(2).

4. Examples of the convergence of the infinite products

Example 1. By [2], p. 97, we have that the following infinite products: $\prod_{i=1}^{\infty}\left(1+\frac{x_i}{i^2}\right)$ and $x_i\left(1-\frac{x_i^2}{1^2\pi^2}\right)\left(1-\frac{x_i^2}{3^2\pi^2}\right)\left(1-\frac{x_i^2}{5^2\pi^2}\right)...$ are convergent for every $x_i \in R$.

Example 2. By [2], p. 97, we have that the following infinite product: $\prod_{i=1}^{\infty} h_i(x_i)$, where $h_i(x_i) = \left(1 - \frac{1}{(i+1)^2}\right) x_i^2$ for $x_i \in (0,1)$ and $h_i(x_i) = 0$ for $x_i \in R\backslash(0,1)$, is convergent for every $x_i \in (0,1)$ and $\prod_{i=1}^{\infty} h_i(x_i) = 0$ for $x_i \in R\backslash(0,1)$.

Example 3. By [2] and [3] , we have that the following infinite product: $\prod_{i=1}^{\infty} f_i(x_i,t)$, where $f_i(x_i,t) = \left(1 - \frac{1}{(i+1)^2}\right) x_i^2 t^{1/2}$ for $(x_i,t) \in (0,1) \times (0,1)$ and $f_i(x_i,t) = 0$ for $(x_i,t) \in R^2\backslash(0,1) \times (0,1)$, is convergent for every $(x_i,t) \in (0,1) \times (0,1)$ and $\prod_{i=1}^{\infty} f_i(x_i,t) = 0$ for $(x_i,t) \in R^2\backslash(0,1) \times (0,1)$.

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