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ON APPROXIMATION AND INTERPOLATION ERRORS OF ANALYTIC FUNCTIONS*

Abstract. Kasana and Kumar [5] obtained the $(p, q)-$growth parameters in terms of Chebyshev and interpolation errors for entire functions on a compact set E of positive transfinite diameter. Rizvi and Nautiyal [9] studied the order and type in terms of these errors for the functions which are not entire. But these results do not give any specific information about the growth of non-entire functions if maximum modules is increasing so rapidly that the order of function is infinite. In this paper an attempt has been made to extend the results contained in [9] for functions having rapidly increasing maximum modulus.

Key words: Largest equipotential curve, regularity domain, approximation and interpolation errors.

AMS Mathematics Subject Classification: 40D10, 30E05.

1. Introduction

Let E be a compact set in complex plane and $\xi^{(n)} = \{\xi_{n0}, \xi_{n1}, \ldots, \xi_{nn}\}$ be a system of $(n+1)$ points of the set E and define

$$V(\xi^{(n)}) = \prod_{0\le j<k\le n} |\xi_{nj} - \xi_{nk}|$$

and

$$\Delta^j\left(\xi^{(n)}\right) = \prod_{k=0,k\neq j}^{n} |\xi_{nj} - \xi_{nk}|, \quad j = 0, 1, \ldots, n$$

Again, let $\eta^{(n)} = \eta_{no}, \eta_{n1}, \ldots, \eta_{nn}$ be the system of $(n+1)$ points in E ([11],[16]) such that

$$V_n \equiv V(\xi^{(n)}) = \sup_{\xi^{(n)}\subset E} V(\xi^{(n)})$$

* This work dedicated to my teacher Late Prof. H.S. Kasana Senior Associate ICTP, Trieste, Italy.

and
$$\Delta^0(\eta^{(n)}) \leq \Delta^j(\eta^{(n)}) \ \text{ for } \ j = 1, 2, \ldots, n.$$

Such a system always exists and is called the n^{th} extremal system of E. The polynomials

$$L^{(j)}(z, \eta^{(n)}) = \prod_{k=0, k \neq j}^{n} \left(\frac{z - \eta_{nk}}{\eta_{nj} - \eta_{nk}} \right), \quad j = 0, 1, \ldots, n,$$

are called the Lagrange extremal polynomials and the limit $d \equiv d(E) = \lim_{n\to\infty} V_n^{2/n(n+1)}$ is called the transfinite diameter of E.

Let $C(E)$ denote the algebra of analytic functions on E. Let us define the approximation errors as follows:

$$E_{n,1}(f, E) \equiv E_{n,1}(f) = \inf_{g \in \pi_n} \|f - g\|,$$

where $\|.\|$ is the sup norm and $\pi_n(z)$ denotes the set of all polynomials of degree $\leq n$

Further, we also define

$$E_{n,2}(f; E) \equiv E_{n,2}(f) = \|L_n - L_{n-1}\|, n \geq 2$$

$$E_{n,3}(f; E) \equiv E_{n,3}(f) = \|L_n - f\|,$$

where $n \in N$ and
$$L_n(z) = \sum_{j=0}^{N} L^{(j)}(z, \eta^{(n)}) f(\eta_{nj})$$

is the Lagrange interpolation polynomial of degree n.

Kasana and Kumar [5] have studied the growth parameters in terms of Chebyshev and interpolation errors $E_{n,j}(f)$, $j = 1, 2, 3$ for entire functions of index-pair (p, q). Rice [8] and Winiarski [16] have obtained these results for $(p, q) = (2, 1)$. Also when $E = [-1, 1]$, Bernstein [2], Juneja [3], Reddy [7], Shah [10] and Varga [14] have studied the rate of decay of these errors for entire functions. All these results do not give any information about the rates of decay of these errors when $f \epsilon C(E)$ is not entire. However, Rizvi and Nautiyal [9] studied the rates of decay of $E_{n,j}$ when f is not entire. But the results contained in [9] do not give any specific information about the growth of $f(z)$ if maximum modulus of $f(z)$ is increasing so rapidly that the order of $f(z)$ is infinite. In the present paper we have extended the results of Rizvi and Nautiyal [9] for the functions having rapidly increasing maximum modulus. Our results give the generalizations of the results in [9] obtained for $q = 2$. Although, Jozef Siciak [12] obtained some results on approximation and interpolation by transcendental polynomials in several variables for any function holomorphic in a neighborhood of compact set E. But our results are different from those of Jozef Siciak [12].

2. Definitions and auxiliary results

We first introduce the concept of domain of regularity for a function $f \in C(E)$.

Let $E_r = \{z : |\phi(z)| = r\}, r > 1$, where the univalent function $\omega = \phi(z)$ maps the complement of E onto $|\omega| > 1$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Then E_r is an analytic Jordan curve. Let D_r be the bounded domain with the boundary E_r. Then $E \subset D_{r'}$ for each r $(1 < r < \infty)$ and $E_r \subset D_{r'}$ for $r' > r$. Since through an arbitrary point $z_0 \notin E$ there passes one and only one curve E_r $(1 < r \leq \infty)$, it follows that for each $f \in C(E)$ there exists a unique $R \equiv R(f)$ $(1 < R \leq \infty)$ such that f can be extended analytically to D_r for each $r \leq R$ but for no $r > R$. We call D_R the 'domain of regularity' for f and denote the class of those $f \in C(E)$ which have domain of regularity D_R by $C(E, R)$.

The concept of index q, the q-order and q-type were introduced by Bajpai et al. [1] in order to obtain a measure of growth of the maximum modulus, when it is rapidly increasing. Thus we define the growth parameters for a function $f \in C(E, R)$ as follows:

A function $f \in C(E, R), 1 < R < \infty$, will be said to be of q-order $\rho(q)$ $(\rho(q) < \infty$, $\rho(q-1) = \infty$, $q = 2, 3, \ldots)$ if

$$\rho(q) = \lim \sup_{r \to R^-} \frac{\log^{[q]} \bar{M}(r)}{\log(R/(R-r))}$$

where

$$\bar{M}(r) \equiv \bar{M}(r, f) = \max_{z \in E_r} |f(z)|$$

and

$$\log^{[0]} \bar{M}(r) = \bar{M}(r), \qquad \log^{[q]} \bar{M}(r) = \log \log^{[q-1]} \bar{M}(r)$$

In case $0 < \rho(q) < \infty$, the q-type $T(q)(0 \leq T(q) < \infty)$ of f is defined as

$$T(q) = \lim \sup_{r \to R^-} \frac{\log^{[q-1]} \bar{M}(r)}{(R/(R-r))^{\rho(q)}}$$

Now, it will be justified to give the definition of q-order and q-type of a function analytic in the disk $|z| < R$, $0 < R < \infty$. Let ρ_0 and T_0 are ρ and T for the case of E the finite disk.

$f(z)$ is analytic in $|z| < R$, then the q-order $\rho_o(q)$ of $f(z)$ is defined as

$$\rho_0(q) = \lim\sup_{r\to R^-} \frac{\log^{[q]} \bar{M}(r)}{\log(R/(R-r))}, \quad 0 \le \rho_0 \le \infty,$$

where $M(r) \equiv M(r,f) = \max_{z\in E_r} |f(z)|$. If $0 \le \rho_0(q) \le \infty$, then the type $T_0(q)$ of f is defined as

$$T_0(q) = \lim\sup_{r\to R^-} \frac{\log^{[q-1]} M(r)}{(R/(R-r))^{\rho(q)}}.$$

Now we prove

Lemma 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$ $(0 < R < \infty)$ and have q -order $\rho_0(q)$ $(\rho_0(q) > 0, q > 2)$. Then*

$$\rho(q) + A(q) = \lim\sup_{n\to\infty} \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+ |a_n| R^n}, \quad 0 \le \rho_0 \le \infty,$$

where $A(q) = 1$ if $q = 2$, $A(q) = 0$ if $q \ge 3$ and for $x > 0$, we put $\log^+ x = \max(\log x, 0)$.

Lemma 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$ $(0 < R < \infty)$. Then f is of q-order $\rho_0(q)(0 < \rho_0(q) < \infty)$ and q-type $T_0(q)$ if and only if*

$$V(q) = T_0(q)B(q),$$

where $B(q) = \frac{(\rho_0+1)^{\rho_0+1)}}{\rho_0^{\rho_0}}$ for $q = 2$ and $B(q) = 1$ if $q = 3, 4, \ldots$ and

$$V(q) = \lim\sup_{n\to\infty} (\log^{[q-2]} n)(\log^+ |a_n| R^n)^{\rho(q)+A(q)}.$$

The above lemmas can be easily proved c.f.([4], Theorems 1 and 5).

Let $w = \psi(z)$ be the univalent function which maps the complement of E onto $|w| > d$ such that $\psi(\infty) = \infty$ and $\psi'(\infty) = 1$. Set $\bar{E}_r = \{z : |\psi(z)| = r\}, r > d, d > 1, r > 1$ and denote by $\bar{D}_r$ the domain interior of $\bar{E}_r$. $\bar{E}_r$ is the largest equipotential curve of E. For $r = d = 1, \bar{E}_r = E$. We have taken the case $r > d$. Then we have

Lemma 3. *Let f(z) be analytic in $\bar{D}_{r_0}, r_0 > d$. Then for every positive integer n there exists a polynomial $Q_n \in \pi_n$ such that*

$$|f(z) - Q_n(z)| \le A\bar{M}(r)\left(\frac{d}{r}\right)^n, \quad z \in E.$$

for all $r(< r_0)$ sufficiently close to r_0. Here A is a constant depending on the set E and r_0 but is independent of n and r and

$$\bar{M}(r) \equiv \bar{M}(r,f) = \max_{z\in E_r} |f(z)|.$$

Proof. It can be shown [13, p. 138] that, inside $\bar{D}_{r_0}$,

$$f(z) = \sum_{k=0}^{\infty} C_k g_k(z), \tag{1}$$

where $\{g_k\}_{k=0}^{\infty}$ is the sequence of Faber polynomials for E and the series on the right hand side of (1) converges uniformly on compact subsets of $\bar{D}_{r_0}$ Further, the $C_k's$ satisfy

$$|C_k| \le \bar{M}(r)/r^k \quad \text{for} \quad d < r < r_0 \quad \text{and} \quad k = 0, 1, 2, \dots \tag{2}$$

Also from [13, p. 137], we have

$$|g_k(z)| \le 2d^k \quad \text{for} \quad z \in E \quad \text{and} \quad k = 0, 1, 2, 3, \dots \tag{3}$$

Taking $Q_n = \sum_{k=0}^{n} C_k g_k(z)$ with (2) and (3), we get

$$\begin{aligned} |f(z) - Q_n(z)| &\le \sum_{k=n+1}^{\infty} |C_k|\,|g_k(z)| \\ &\le 2\bar{M}(r) \sum_{k=n+1}^{\infty} (d/r)^k, d < r < r_0 \\ &= 2\bar{M}(r) \left(\frac{d}{d-r}\right) \left(\frac{d}{r}\right)^n \end{aligned}$$

The lemma easily follows by taking $r \ge ((d + r')/2)$ if $r' < \infty$, and $r \ge 2d$ if $r' = \infty$. ∎

Lemma 4. *Let $f \in C(E, R)$, $R > 1$. Then*

$$E_{n,1}(f) \le A\bar{M}(r, f)(r)^n, \quad z \in E, \quad n = 0, 1, 2, \dots$$

for all $r(< R)$ sufficiently close to R. Here A is a constant depending on E and R but independent of n and r.

Proof. By Lemma 3, for every positive integer n, there exists a polynomial Q_n of degree at most n such that

$$|f(z) - Q_n(z)| \le 2d^k A\bar{M}(r, f)(r)^n, \quad z \in E, \tag{4}$$

for all r $(< R)$ sufficiently close to R. In view of definition of $E_{n,1}(f)$ and constant A we get the required result. ∎

Lemma 5. *Let $f \in C(E)$, then*

$$E_{n,1}(f) \;\leq\; E_{n,3}(f) \;\leq\; (n+2)E_{n,1}(f),$$

$$E_{n,2}(f) \;\leq\; 2(n+2)E_{n-1,1}(f).$$

The lemma follows from Winiarski [16].

3. Main results

In this section we investigate the growth parameters of a function $f \in C(E,R)$, $1 < R < \infty$, in terms of $E_{n,j}(f)$.

Theorem 1. *Let $f \in C(E)$, then $f \in C(E,R), R > 1$, if and only, if*

$$\limsup_{n\to\infty} (E_{n,j}(f))^{1/n} \;=\; 1/R, \quad j = 1,2,3.$$

Proof. First let $f \in C(E,R)$. Then, by Lemma 4, we have

$$\limsup_{n\to\infty} (E_{n,1}(f))^{1/n} \;\leq\; 1/r$$

for all r $(< R)$ sufficiently close to R and so

$$\limsup_{n\to\infty} (E_{n,1}(f))^{1/n} \;\leq\; 1/R.$$

Also it is known [15, Chapter XII] that there exist polynomials $P_n \equiv P_n(f) \in \; \pi_n$ such that

$$E_{n,1}(f) \;=\; \|f - p_n\|, \quad n = 0,1,2,\ldots$$

and

$$f(z) \;=\; P_0(z) + \sum_{n=0}^{\infty}(P_{n+1}(z) - P_n(z)) \tag{5}$$

holds in D_R and the series on the right hand side of (5) converges uniformly on compact subsets of D_R. Now

$$\|P_{n+1}(z) - P_n(z))\| \;\leq\; 2E_{n,1}(f)$$

and

$$|P_{n+1}(z) - P_n(z))| \leq 2E_{n,1}(f)r^{n+1} \quad \text{for} \quad z \in E_r, \quad r > 1. \tag{6}$$

In view of (6) we see that if $\limsup_{n\to\infty}(E_{n,1}(f))^{1/n} < 1/R$, then the series on the right hand side of (5) converges uniformly on compact subsets of $D_{R'}$

for some $R' > R$, which is a contradiction. Hence $\limsup(E_{n,1}(f))^{1/n} = 1/R$ as $n \to \infty$. This proves the necessary part of the theorem for $j = 1$. Theorem 1 of [13, p.17] with Lemma 5 proves also the necessary part for $j = 2, 3$. The sufficiency part can also be proved similarly. ■

Theorem 2. *Let $f \in C(E, R)$, $1 < R < \infty$, be of order $\rho(q)$. Then*

$$(7) \qquad \rho(q) + A(q) \;=\; \limsup_{n\to\infty} \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+ E_{n,j}(f) R^n}, \qquad j = 1, 2, 3.$$

Proof. In view of Lemma 5 and Theorem 1 of [13, p. 17] it is sufficient to prove the theorem for the case $j = 1$. Thus for $j = 1$, let

$$\liminf_{n\to\infty} \frac{\log n - \log^+ \log^+ E_{n,1}(f) R^n}{\log^{q-1} n} \;=\; \alpha.$$

Obviously $0 \le \alpha \le \infty$. First suppose that $0 < \alpha < \infty$. Then, by the definition of α, there exists a sequence $\{n_k\}$ of positive integers tending to infinity such that

$$(8) \qquad \log E_{n_{k,1}}(f) R^{n_k} > n_k (\log^{[q-2]} n_k)^{(-\alpha+\epsilon)} \quad \text{for} \quad k = 1, 2, 3, \ldots$$

Using Lemma 4 and (8) we obtain

$$(9) \qquad \log \bar{M}(r) \;\ge\; n_k (\log^{[q-2]} n_k)^{(-\alpha+\epsilon)} + n_k \log(r/R) - \log A$$

for the sequence $\{n_k\}$ and all $r(< R)$ sufficiently close to R. Let $\{r_k\}$ be a sequence defined by

$$n_k \;=\; \exp^{[q-2]} \{e \log{(R/r_k)}\}, \quad k = 1, 2, 3, \ldots, \quad \text{then} \quad r_k \to R \quad \text{as} \quad k \to \infty.$$

Thus, using (9), for all sufficiently large values of k, we get

$$\begin{aligned} \log \bar{M}(r_k) \;&\ge\; (1 - e) n_k (\log^{[q-2]} n_k)^{-(\alpha+\epsilon)} [1 + 0(1)] \\ &=\; e(e-1) \left\{ \exp^{[q-2]} (e \log(R/r_k))^{-1/(\alpha+\epsilon)} \right\} \log(R/r_k)^{-1} [1 + 0(1)]. \end{aligned}$$

Since $\log(R/(R - r_k))\; - \log\log(R/r_k)$ *as* $k \to \infty$, after a simple calculation the above inequality gives

$$(10) \qquad \rho(q) + A(q) \;\ge\; 1/\alpha.$$

Now by (5), we have that

$$f(z) \;=\; P_0(z) + \sum_{n=0}^{\infty} (P_{n+1}(z) - P_n(z)) \;\text{ holds in }\; D_r.$$

Thus in view of (6), we get

$$|f(z)| \leq P_0(z) + \sum_{n=0}^{\infty} |(P_{n+1}(z) - P_n(z))| \leq K + 2\sum_{n=0}^{\infty} E_{n,1}(f) r^{n+1} \tag{11}$$

for $z \in E_r, 1 < r < R$. (11) gives

$$\bar{M}(r, f) \ \leq \ K + 2M(r, h), \tag{12}$$

where $h(z) = \sum_{n-\infty}^{\infty} E_{n,1}(f) z^{n+1}$. By Theorem 1, $h(z)$ is analytic in $|z| < R$. Using (12) and Lemma 1 for h(z), we get

$$\rho(q) + A(q) \ \leq \ 1/\alpha. \tag{13}$$

Combining (10) and (13), the proof is completed for $0 < \alpha < \infty$. Also both inequalities are trivially true if $\alpha = 0$ or $\alpha = \infty$. Hence the proof is completed. ■

Theorem 3. *Let $f \in C(E, R)$, $1 < R < \infty$, and have $q-order$ $\rho(q)(0 < \rho(q) < \infty)$, $q-type$ $T(q)$, then*

$$G(q) \ = \ T(q) B_0(q), \tag{14}$$

where $B_0(2) = \frac{(\rho(2)+1)^{\rho(2)+1}}{(\rho(2))^{\rho(2)}} d^{\rho(2)}$, $A(2) = 1$ *and* $B_0(q) = 1$, $A(q) = 0$ *if* $q = 3, 4, \ldots$

$$G(q) \ = \ \limsup_{n\to\infty} (\log^{[q-2]} n) \left(\frac{\log^+ E_{n,j}(f) R^n}{n}\right)^{\rho(q)+A(q)}. \tag{15}$$

Proof. Let $G(q) < \infty$. For given $\epsilon > 0$, by (15) we have

$$\left(\log^{[q-2]} n\right) \left(\frac{\log^+ E_{n,j}(f) R^n}{n}\right)^{\rho(q)+A(q)} < G(q) + \epsilon, \ \text{ for all } \ n > n_0 \equiv n_0(\epsilon),$$

or

$$\log^{[q-1]} n + (\rho(q) + A(q)) \left[\log^+ \log^+ E_{n,j}(f) R^n - \log n\right] < \log(G(q) + \epsilon),$$

or

$$\rho(q) + A(q) > \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+ E_{n,j}(f) R^n} - \frac{\log(G(q) + \epsilon)}{\log n - \log^+ \log^+ E_{n,j}(f) R^n}.$$

Let $0 < T(q) < \infty$. For given $\epsilon > 0$, by definition, we have

$$\log \bar{M}(r) < \exp^{[q-2]} \left\{(T(q) + \epsilon)(R/(R - r))^{\rho(q)}\right\} \tag{16}$$

for all r such that $0 < r_0 = r_0(\epsilon) < r < R$.

In view of Theorem 1 of [13, p. 1] and Lemma 5 it is sufficient to prove the theorem for $j = 1$. Thus using Lemma 4, (16) gives

$$\log^+ E_{n,1}(f)R^n \leq \exp^{[q-2]}\left\{(T(q)+\epsilon)(R/(R-r))^{\rho(q)}\right\} + n\log(R/r) + \log A. \tag{17}$$

The maximum value of right hand side of (17) is uniquely determined by the value of r given by

$$\prod_{i=0}^{q-2} \exp^{[i]}\left\{(T(q)+\epsilon)(R/(R-r))^{\rho(q)}\right\} = \frac{n(R-r)}{R\rho(q)}. \tag{18}$$

For $q = 2$, using (18) in (17), we get

$$\log^+ E_{n,1}(f)R^n \leq \frac{(T(q)+\epsilon)^{1/(\rho(q)+1)} n^{\rho(q)/(1+\rho(q))}}{(\rho(q))^{\rho(q)/(1+\rho(q)+1)}}(1+\rho(q)+o(1))$$

for all sufficiently large value of n. On proceeding to limits, the above inequality gives (14) for $q = 2$

Next, for $q = 3, 4, \ldots$ (18) gives

$$\frac{R}{R-r} \simeq \left(\frac{\log^{[q-2]} n}{T(q)+\epsilon}\right)^{1/\rho(q)} \quad \text{as} \quad n \to \infty.$$

Thus for $n > n_0$, (17) gives

$$\log^+ E_{n,1}(f)R^n \ < \ n + n\log(R/r) + logA,$$

or

$$\log^{[q-2]} n^{(1+o(1))} \left(\frac{\log^+ E_{n,1}(f)R^n}{n}\right)^{\rho(q)} \ < \ (T(q)+\epsilon)(1+o(1)).$$

Proceeding to limits as $n \to \infty$, the above inequality gives

$$T(q) \ \geq \ G(q) \quad \text{for} \quad q \geq 3.$$

The reverse inequality follows from (12) by applying Lemma 2 to the function $h(z)$. If $G(q)$ is infinite then $T(q) = \infty$ and f is of growth $(\rho(q), \infty)$. Hence the theorem is completed. ■

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