F A S C I C U L I   M A T H E M A T I C I

Nr 38 2007

Yuji Liu

PERIODIC SOLUTIONS OF FIRST-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SUPPER-LINEAR NONLINEARITIES

Abstract. The existence results for periodic solutions concerning first-order functional differential equation

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), \cdots, x(t - \tau_m(t))),\ t \in R$$

are proved. The methods are based upon the coincidence degree theory by Mawhin. The results obtained are new. Examples, that can not be solved by known theorems, are given to illustrate the main results.

Key words: periodic solution, first-order functional differential equation, coincidence degree theory.

AMS Mathematics Subject Classification: 34B10, 34B15.

1. Introduction

In this paper, we investigate the existence of $\omega-$periodic solutions of first-order functional differential equation

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), \cdots, x(t - \tau_m(t))), \tag{1}$$

where $\tau_i (i = 1, \cdots, m)$ are $\omega-$periodic functions, f is a Carathedeodory function and $f(\bullet, x_1, \cdots, x_m)$ is a $\omega-$periodic function for all $(x_1, \cdots, x_m) \in R^m$.

The motivation for this paper is as follows: there were many papers concerning with the existence of periodic solutions of the first-order delay differential equations.

In [1], the authors studied the existence and asymptotic periodicity of delay differential equation

$$x'(t) + a(t)x(t) + b(t)x(t - \tau) = f(t), \tag{2}$$

where a, b and f are $\omega-$periodic, b has fixed sign, and there is a positive integer n such that $\tau = n\omega$, but they didn't discuss the existence of periodic solutions of above equation.

In [2], the author studied the existence of periodic solutions of the delay differential equation with piecewise constant variables

$$x'(t) \;=\; f(t, x([t]), x([t-1]), \cdots, x([t-k]), x(t)), \quad x \in R. \tag{3}$$

Under the assumption:

$$\begin{aligned}
&f(t, x_0, \cdots, x_{k+1}) > 0 \quad \text{for} \quad t \in R \quad \text{and} \quad x_i \ge D \;\; (i = 0, \cdots, k+1);\\
&f(t, x_0, \cdots, x_{k+1}) < 0 \quad \text{for} \quad t \in R \quad \text{and} \quad x_i \le -D \;\; (i = 0, \cdots, k+1);\\
&f(t, x_0, \cdots, x_{k+1}) > -M \quad \text{for} \quad (t, x_0, \cdots, x_{k+1}) \in R^{k+3},
\end{aligned}$$

where $D > 0$ is a constant, the authors proved that equation (3) has at least on periodic solution. Some other existence results were also obtained in this paper.

In [3, 4], the delay differential equation

$$x'(t) \;=\; a(t)x(t) + \lambda f(t, x(t - \tau(t))) \tag{4}$$

was studied by using Krasnoselskii's fixed point theorem, the authors established some existence results for positive periodic solutions of equation (4) at the case where f is sublinear or superlinear about x. When $f(t, x)$ is a linear function about x, (4) becomes the form of (2). In [11], Liu and Ge studied the following differential equation

$$x'(t) \;=\; -\delta(t)x(t) + f(t, x(t)).$$

It was showed that the equation has at least two positive periodic solutions under certain growth conditions imposed on f.

In [5], the delay differential system

$$x_i'(t) \;=\; f_i(t, x_1(\tau_{i,1}(t)), \cdots, x_n(\tau_{i,n}(t))), \quad i = 1, \cdots, n \tag{5}$$

was studied. One of the results in [5] as follows:

Theorem KP. *Let $\sigma_i \in \{-1, 1\}$, denote*

$$f_i^*(t, \rho_1, \cdots, \rho_n) = \max\{|f_i(t, x_1, \cdots, x_n)| : \; |x_1| \le \rho_1, \cdots, |x_n| \le \rho_n\},$$

for each $i \in \{1, \cdots, n\}$. The condition

$$f_i(t, x_1, \cdots, x_n)\mathrm{sgn}(\sigma_i x_i) \le p_i(t)|x_i| + \sum_{k=1}^{n} p_{i,k}(t)|x_k| + q(t) \tag{6}$$

hold on set $[0, \omega] \times R^n$ and for each $i = 1, \cdots, m$. Furthermore,

$$\int_0^{\tau_{i,i}^0(t)} |p_i(s)|ds \le p_{i,i}^*(t) \int_0^{\tau_{i,i}^0(t)} |f^*(s, |x_1|, \cdots, |x_n|)|ds \le \sum_{k=1}^{n} p_{i,k}^*(t)|x_k| + q^*,$$

$$(7)\quad |f_i(t,x_1,\cdots,x_i,\cdots,x_n)-f_i(t,x_1,\cdots,\overline{x_i},\cdots,x_n)| \le l_i(t)|x_i-\overline{x_i}|$$

hold on the set $I_i \times R^n$. Here $p_i : [0,\omega] \to R$, $p_{i,k}$, q, and $I_i : [0,\omega] \to [0,+\infty)$ $(i,k=1,\cdots,n)$ are summable functions, $p^*_{i,k} : [0,\omega] \to [0,+\infty)$ $(i,k=1,\cdots,n)$ are essentially bounded functions, and q^* is a nonnegative number. Moreover, let

$$\int_0^\omega p_i(s)ds < 0,\ \ i=1,\cdots,n$$

and there exist a constant nonnegative matrix $A=(a_{i,j})^n_{i,k=1}$ such that $r(A)<1$ and

$$\int_0^\omega g(\sigma_i p_i)(t,s)[p_{i,k}(s)+l_i(s)p^*_{i,k}(s)]ds \le a_{i,k},\ \ i,k=1,\cdots,n.$$

Then system (5) has at least one ω periodic solution.

It follows from (6) and (7) that the growth conditions imposed on f_i are at most linear, we find the existence results have not established when f_i are supper linear even if $n=1$ in (5). Furthermore, the equations discussed in all above mentioned papers are delay differential equations. To the best of our knowledge, there is no paper concerning the existence of periodic solutions of mixed type differential equations or even of forward differential equations.

In this paper, the equation discussed will be mixed type differential equations. Some sufficient conditions, which allow the degrees of $x_0,\cdots,x_m$ in $f(t,x_0,\cdots,x_m)$ to be greater than 1 if f is polynomial, for the existence of periodic solutions of equation (1) will be established in section 2. Some examples will be given in this section to illustrate the main results. The proofs of the theorems will be present in section 3. Our methods and the results are different from known ones.

2. Main results and examples

In this section, we establish sufficient conditions for the existence of at least one $\omega-$periodic solution of equation (1). For convenience, we first introduce some notations and an abstract existence theorem by Gaines and Mawhin [8].

Let X and Y be Banach spaces, $L:\ \mathrm{dom}\ L \subset X \to Y$ be a Fredholm operator of index zero, $P:\ X \to X$, $Q:\ Y \to Y$ be projectors such that

$$\mathrm{Im}\ P=\mathrm{Ker}\ L,\ \mathrm{Ker}\ Q=\mathrm{Im}\ L,\ \ X=\mathrm{Ker}\ L\oplus\mathrm{Ker}\ P,\ \ Y=\mathrm{Im}\ L\oplus\mathrm{Im}\ Q.$$

It follows that

$$L|_{\mathrm{dom}\ L\cap\mathrm{Ker}\ P}:\ \mathrm{dom}\ L\cap\mathrm{Ker}\ P\to\mathrm{Im}\ L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, dom $L \cap \overline{\Omega} \neq \emptyset$, the map $N: X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N: \overline{\Omega} \to X$ is compact.

Theorem GM [8]. *Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:*
(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\text{dom}L \setminus \text{Ker}L) \cap \partial\Omega] \times (0,1)$;
(ii) $Nx \notin \text{Im}L$ for every $x \in \text{Ker}L \cap \partial\Omega$;
(iii) $\deg(\Lambda QN\big|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) \neq 0$, where $\Lambda: Y/\text{Im}L \to \text{Ker}L$ is an isomorphism.
Then the equation $Lx = Nx$ has at least one solution in $\text{dom}L \cap \overline{\Omega}$.

We use the classical Banach space C^0_ω, the set of all continuous $\omega-$periodic functions defined in R with the norm $||x|| = \max_{t\in[0,\omega]} |x(t)|$, let $X = C^0_\omega = Y$. Define the linear operator L and the nonlinear operator N by

$$L : X \cap \text{dom}L \to Y, \quad Lx(t) = x'(t) \quad \text{for} \quad x \in X \cap \text{dom}L,$$

$$N : X \to Y, \quad Nx(t) = f(t, x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t))), \quad \text{for} \quad x \in X,$$

respectively, where $\text{dom}L = C^1_\omega = \{x \in C^1(R) : x(t+\omega) = x(t),\ t \in R\}$.

Lemma 1. *The following results hold.*
(i) $KerL = \{x(t) \equiv c,\ t \in [0,1],\ c \in R\}$;
(ii) $ImL = \left\{ y \in Y, \int_0^\omega y(u)du = 0 \right\}$;
(iii) L is a Fredholm operator of index zero;
(iv) There are projectors $P: X \to X$ and $Q: Y \to Y$ such that $KerL = ImP$ and $KerQ = ImL$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap domL \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$;
(v) $x(t)$ is a $\omega-$periodic solution of equation (1) if and only if x is a solution of the operator equation $Lx = Nx$ in $domL$.

Proof. The proofs are simple and are omitted. ■

Theorem 1. *Suppose*
(A_1) there are continuous function g and h so that

$$f(t, x_0, \cdots, x_m) \ = \ g(t, x_0, \cdots, x_m) + h(t, x_0, \cdots, x_m),$$

with

$$g(t, x_0, \cdots, x_m)x_0 \ \leq \ 0,$$

and h satisfying that there are nonnegative continuous functions g_i, p_i and e such that

$$|h(t, x_0, \cdots, x_m)| \ \leq \sum_{i=0}^{m} g_i(t, x_i) + e(t), \quad i = 0, \cdots, m$$

and

$$\lim_{x\to\infty} \frac{g_i(t,x)}{|x|} \ = \ p_i(t), \ \ i = 0,\cdots,m, \ \ t \in R.$$

(A_2) *there is a constant $M > 0$ such that*

(i) $f(t, x_0, \cdots, x_m) > 0$ *for* $t \in R$ *and* $x_i > M(i = 0, \cdots, m)$;
(ii) $f(t, x_0, \cdots, x_m) < 0$ *for* $t \in R$ *and* $x_i < -M(i = 0, \cdots, m)$;

or

(A_3) *there is a constant $M > 0$ such that*

(i) $f(t, x_0, \cdots, x_m) < 0$ *for* $t \in R$ *and* $x_i > M(i = 0, \cdots, m)$;
(ii) $f(t, x_0, \cdots, x_m) > 0$ *for* $t \in R$ *and* $x_i < -M(i = 0, \cdots, m)$;

Then equation (1) *has at least one solution provided*

$$\sum_{i=0}^{m} \int_0^{\omega} p_i(s)ds \ < \ 1/2. \tag{8}$$

Theorem 2. *Suppose* (A_2) *or* (A_3) *holds and*
(A_4) *there are continuous function g and h so that*

$$f(t, x_0, \cdots, x_m) = g(t, x_0, \cdots, x_m) + h(t, x_0, \cdots, x_m),$$

with

$$g(t, x_0, \cdots, x_m)x_0 \ \geq \ 0,$$

and h satisfying that there are nonnegative continuous functions g_i, p_i and e such that

$$|h(t, x_0, \cdots, x_m)| \ \leq \sum_{i=0}^{m} g_i(t, x_i) + e(t), \quad i = 0, \cdots, m$$

and

$$\lim_{x\to\infty} \frac{g_i(t,x)}{|x|} \ = \ p_i(t), \quad i = 0,\cdots,m, \ \ t \in R.$$

Then equation (1) *has at least one solution provided* (8) *holds.*

Theorem 3. *Suppose*
(A_5) *there are positive number α and nonnegative number α_i and L so that*

$$|f(t, x_0, \cdots, x_m)| \geq \alpha|x_0| - \sum_{i=1}^{m} \alpha_i|x_i| - L, \ \ (t, x_0, \cdots, x_m) \in R \times R^{m+1}$$

holds

(A_6) *there is a constant* $M > 0$ *so that*

$$cf(t, c, \cdots, c) \ > \ 0 \ \textit{for all} \ |c| > M$$

or

$$cf(t, c, \cdots, c) \ < \ 0 \ \textit{for all} \ |c| > M;$$

Then equation (1) *has at least one* ω *periodic solution if*

$$\frac{\sum_{i=1}^{m} \alpha_i}{\alpha} \ < \ 1. \tag{9}$$

Theorem 4. *Suppose* (A_3) *holds and*

(A_7) *there are continuous function* g *and* h *such that*

$$f(t, x_0, \cdots, x_m) = g(t, x_0, \cdots, x_m) + h(t, x_0, \cdots, x_m),$$

and there are positive numbers β *and* μ *such that*

$$g(t, x_0, \cdots, x_m)x_0 \ \geq \ \beta|x_0|^{\mu+1},$$

and there are continuous functions g_i, p_i *and* e *such that*

$$|h(t, x_0, \cdots, x_m)| \leq \sum_{i=0}^{m} g_i(t, x_i) + e(t), \quad i = 0, \cdots, m$$

and

$$\lim_{x\to\infty} \frac{g_i(t, x)}{|x|^\mu} \ = \ p_i(t), \quad i = 0, \cdots, m, \ \ t \in R$$

hold.

Furthermore, suppose $\tau_i'(t) < 1$ *for all* $t \in R$, *denote the inverse function of* $s = t - \tau_i(t)$ *by* $t = \mu_i(s)$, *let* $\lambda_i = \max_{t\in R} \frac{1}{|1-\tau_i'(\mu_i(t))|}$. *Then equation* (1) *has at least one solution provided*

$$\sum_{i=1}^{m} \lambda_i^{\mu+1}||p_i||_\infty + ||p_0||_\infty \ < \ \beta. \tag{10}$$

Theorem 5. *Suppose*

(A_8) *there are a continuous function* g *and* h *so that*

$$f(t, x_0, \cdots, x_m) \ = \ g(t, x_0, \cdots, x_m) + h(t, x_0, \cdots, x_m),$$

and there are positive numbers β *and* μ *such that*

$$g(t, x_0, \cdots, x_m)x_0 \ \leq -\beta|x_0|^{\mu+1},$$

and there are continuous functions g_i, p_i and e, and $M > 0$ so that

$$|h(t, x_0, \cdots, x_m)| \le \sum_{i=0}^{m} g_i(t, x_i) + e(t),\ i = 0, \cdots, m$$

and

$$\lim_{x\to\infty} \frac{g_i(t,x)}{|x|^\mu} = p_i(t), \quad i = 0, \cdots, m, \ \ t \in R$$

hold.

Furthermore, suppose $\tau_i'(t) < 1$ for all $t \in R$, denote the inverse function of $s = t - \tau_i(t)$ by $t = \mu_i(s)$, let $\lambda_i = \max_{t\in R} \frac{1}{|1-\tau_i'(\mu_i(t))}|$. Then equation (1) *has at least one solution provided*

$$\sum_{i=1}^{m} \lambda_i^{\mu+1} ||p_i||_\infty + ||p_0||_\infty < \beta. \tag{11}$$

Now, we present some examples to illustrate the main results.

Example 1. Consider the functional differential equation

$$x' = a_0 x(t) + \sum_{i=1}^{m} p_i x(t - \tau_i(t)) + p(t), \tag{12}$$

where $a_0 \in R$, $\tau_i(t) = \frac{1}{2}\sin t$, and $p_i \in R$, and p are continuous 2π-periodic functions. Let

$$f(t, x_0, \cdots, x_m) = a_0 x_0 + \sum_{i=1}^{m} p_i x_i + p(t).$$

It is easy to see

$$|f(t, x_0, \cdots, x_m)| \ge |a_0||x_0| - \sum_{i=1}^{m} |p_i|x_i| - \min_{t\in[0,\omega]} |p(t)|,$$

and from

$$cf(t, c, \cdots, c) = \left(a_0 + \sum_{i=1}^{m} p_i\right) c^2 + cp(t),$$

we see that $cf(t, c, \cdots, c) > 0$ for some positive constant $M > 0$ if $a_0 + \sum_{i=1}^m p_i > 0$ and $cf(t, c, \cdots, c) < 0$ for some positive constant $M > 0$ if $a_0 + \sum_{i=1}^m p_i < 0$. Hence (A_5) and (A_6) hold.

It follows from Theorem 2.3 that, for each $2\pi-$periodic function p, equation (12) has at least one 2π periodic solution if $a_0 + \sum_{i=1}^m p_i \neq 0$ and $\sum_{i=1}^m |p_i| < |a_0|$.

Example 2. Consider the functional differential equation

$$x' = a_0[x(t)]^{2k+1} + \sum_{i=1}^{m} p_i(t)[x(t-\tau_i(t))]^{2k+1} + p(t), \tag{13}$$

where $a_0 \in R$, $\tau_i(t) = \frac{1}{2}\sin t$, and p_i and p are continuous 2π-periodic functions, k an nonnegative integer.

It is easy to get

$$\lambda_i = \max_{t\in R} \frac{1}{|1-\tau_i'(\mu_i(t))|} = 2.$$

Let

$$\begin{aligned} f(t, x_0, \cdots, x_m) &= a_0 x_0^{2k+1} + \sum_{i=1}^{m} p_i(t) x_i^{2k+1} + p(t), \\ g(t, x_0, \cdots, x_m) &= a_0 x_0^{2k+1}, \\ h(t, x_0, \cdots, x_m) &= \sum_{i=1}^{m} p_i(t) x_i^{2k+1} + p(t). \end{aligned}$$

If $a_0 > 0$, it follows from Theorem 4 that, for each p, equation (13) has at least one 2π-periodic solution if

$$2^{2k+2} \sum_{i=1}^{m} ||p_i||_\infty < a_0. \tag{14}$$

If $a_0 < 0$, it follows from Theorem 4 that, for each p, equation (13) has at least one 2π-periodic solution if (14) holds.

Example 3. Consider the functional differential equation

$$x' = -(2 + x^2(t))[x(t)]^{2k+1} + \sum_{i=0}^{m} p_i x(t-\tau_i(t)) + p(t), \tag{15}$$

where $\tau_i(t) = \frac{1}{2}\sin t$, and $p_i \le 0$ and p is continuous 2π-periodic functions, k an nonnegative integer.

It follows from Theorem 1 that, for each p, that equation (15) has at least one solution if $\omega \sum_{i=0}^{m} |p_i| < 1/2$.

3. Proofs of theorems

In this section, we give the proofs of the main results.

Proof of Theorem 1. To apply Theorem GM, we should define an open bounded subset Ω of X such that (i), (ii) and (iii) of Theorem GM hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in \mathrm{dom}L/\mathrm{Ker}L,\ Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. It suffices to prove there is a constant $B > 0$ such that $||x|| \le B$.

For $x \in \Omega_1$, we get

$$x'(t) = \lambda f(t, x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t))). \tag{16}$$

The

$$\int_0^\omega f(t, x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t)))dt = 0. \tag{17}$$

It follows from (A_2) or (A_3) that there is $\xi \in [0,\omega]$ so that $|x(\xi)| \le M$.

Multiplying two sides of (16) by $x(t)$ and integrating from ξ to t, we get, for $\xi + \omega \ge t \ge \xi$, using (A_1),

$$\begin{aligned}
\frac{1}{2}[x(t)]^2 &= \frac{1}{2}[x(\xi)]^2 + \lambda \int_\xi^t f(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))x(s)ds \\
&\le \frac{1}{2}M^2 + \lambda \int_\xi^t f(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))x(s)ds \\
&= \frac{1}{2}M^2 + \lambda \int_\xi^t g(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))x(s)ds \\
&\quad + \lambda \int_\xi^t h(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))x(s)ds \\
&\le \frac{1}{2}M^2 + \lambda \int_\xi^t h(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))x(s)ds \\
&\le \frac{1}{2}M^2 + \int_0^\omega |h(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))||x(s)|ds \\
&\le \frac{1}{2}M^2 + \sum_{i=1}^m \int_0^\omega g_i(s, x(s-\tau_i(s))|x(s)|ds \\
&\quad + \int_0^\omega g_i(s, x(s))|x(s)|ds + \int_0^\omega |e(s)|x(s)|ds.
\end{aligned}$$

Choosing $\epsilon > 0$ so that

$$\sum_{i=0}^{m} \int_0^{\omega} (p_i(s) + \epsilon) ds \; < \; 1/2.$$

For such a ϵ, there is a $\delta > 0$ so that

$$g_i(t,x) \;\le\; (p_i(t) + \epsilon)|x|, \quad \text{for } \; |x| \ge \delta, \;\; t \in R. \tag{18}$$

Denote

$$\begin{aligned} \Delta_{i,1} &= \{t \in [0,\omega] : |x(t - \tau_i(t))| \le \delta\}, \\ \Delta_{i,2} &= \{t \in [0,\omega] : |x(t - \tau_i(t))| > \delta\}, \quad i = 1, \cdots, m, \end{aligned} \tag{19}$$

and

$$\Delta_{0,1} = \{t \in [0,\omega] : |x(t)| \le \delta\}, \qquad \Delta_{0,2} = \{t \in [0,\omega] : |x(t)| > \delta\}, \tag{20}$$

and

$$g_{i,\delta} = \max_{t \in [0,\omega], |x| \le \delta} g_i(t,x), \quad i = 0, 1, \cdots, m. \tag{21}$$

Then

$$\begin{aligned} \frac{1}{2}||x||_\infty^2 \;\le\; & \frac{1}{2}M^2 + \sum_{i=1}^{m} \int_{\Delta_{i,2}} g_i(s, x(s - \tau_i(s)))|x(s)| ds \\ & + \int_{\Delta_{0,2}} g_0(s, x(s))|x(s)| ds + \sum_{i=1}^{m} \int_{\Delta_{i,1}} g_i(s, x(s - \tau_i(s)))|x(s)| ds \\ & + \int_{\Delta_{0,1}} g_0(s, x(s))|x(s)| ds + \int_0^{\omega} |e(s)|x(s)| ds \\ \le\; & \sum_{i=0}^{m} g_{i,\delta} \int_0^{\omega} |x(s)| ds + \sum_{i=1}^{m} \int_0^{\omega} (p_i(s) + \epsilon)|x(s - \tau_i(s))||x(s)| ds \\ & + \int_0^{\omega} (p_0(s) + \epsilon)|x(s)|^2 ds + \int_0^{\omega} |e(s)|x(s)| ds + \frac{1}{2}M^2 \\ \le\; & \frac{1}{2}M^2 + \omega \sum_{i=0}^{m} g_{i,\delta}||x||_\infty + \sum_{i=1}^{m} \int_0^{\omega} (p_i(s) + \epsilon) ds ||x||_\infty^2 \\ & + \int_0^{\omega} (p_0(s) + \epsilon) ds ||x||_\infty^2 + \int_0^{\omega} |e(s)| ds ||x||_\infty. \end{aligned}$$

So we get

$$\left(\frac{1}{2} - \sum_{i=0}^{m} \int_0^{\omega} (p_i(s) + \epsilon) ds \right) ||x||_\infty^2 \le \frac{1}{2}M^2 + \int_0^{\omega} e(s) ds ||x||_\infty + \omega \sum_{i=0}^{m} g_{i,\delta}||x||_\infty.$$

It follows from (8) that there is a constant $A > 0$ so that $||x||_\infty \leq A$. Then Ω_1 is bounded.

Step 2. Let
$$\Omega_2 = \{x \in \text{Ker}L,\ Nx \in \text{Im}L\}.$$

We prove Ω_2 is bounded. Suppose $x \in \Omega_2$, then $x(t) = c \in R$, we prove $|c| \leq M$. In fact, if $c > M$, then (A_4) implies

$$\int_0^1 f(u, x(u), x(u-\tau_1(u)), \cdots, x(u-\tau_m u))du = \int_0^1 f(u, c, c, \cdots, c)du > 0.$$

Similarly, if $c < -M$, then we have

$$\int_0^1 f(u, c, c, \cdots, c)du \ < \ 0.$$

On the other hand, if $x \in \text{Ker}L$ and $Nx \in \text{Im}L$, we have $QNx = 0$, i.e.

$$\int_0^1 f(u, c, c, \cdots, c)du \ = \ 0.$$

This is a contradiction. So $|c| \leq M$. This shows that Ω_2 is bounded.

Step 3. Let

$$\Omega_3 = \{x \in \text{Ker}L,\ \lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge : \text{Ker}L \to \text{Im}Q$ is the linear isomorphism given by $\wedge(c) = c$ for all $c \in R$. Now we show that Ω_3 is bounded. Suppose $x_n(t) = c_n \in \Omega_3$ and $|c_n| \to +\infty$ as n tends to infinity. Then there exists $\lambda_n \in [0,1]$ such that

$$\lambda_n c_n + (1-\lambda_n)\int_0^1 f(u, c_n, \cdots, c_n)du \ = \ 0.$$

So
$$\lambda_n c_n \ = \ -(1-\lambda_n)\int_0^1 f(u, c_n, \cdots, c_n)du.$$

Since λ_n has a convergent subsequence, without loss of generality, suppose $\lambda_n \to \lambda_0$, Again, since $|c_n| \to +\infty$, there are two cases to be considered, i.e. there is a subsequence of c_n that tends to $+\infty$(without loss of generality suppose $c_n \to +\infty$) or there is a subsequence of c_n that tends to $-\infty$(without loss of generality suppose $c_n \to -\infty$). If $c_n \to +\infty$ as n tends to infinity. Then for sufficiently large n, we have $c_n > M$. Hence, using (A_4), we see

$$\lambda_n c_n^2 \ = \ -(1-\lambda_n)c_n\int_0^1 f(u, c_n, \cdots, c_n)du \ < \ 0,$$

a contradiction. If $c_n \to -\infty$, then for sufficiently large n, $c_n < -M$. Hence using (A_2), we see

$$\lambda_n c_n^2 \ = \ -(1-\lambda_n)c_n \int_0^1 f(u, c_n, \cdots, c_n)du \ < \ 0,$$

a contradiction. So Ω_3 is bounded.

In the following, we shall show that all conditions of Theorem GM are satisfied. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \cup_{i=1}^3 \overline{\Omega_i}$ centered at zero. By Lemma 1, L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have
(a) $Lx \neq \lambda Nx$ for $x \in (\text{dom}L/\text{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$;
(b) $Nx \notin \text{Im}L$ for $x \in \text{Ker}L \cap \partial\Omega$.

Step 4. We prove (c) $\deg(QN|_{\text{Ker}L}, \ \Omega \cap \text{Ker}L, 0) \neq 0$.

In fact, let $H(x,\lambda) = \lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \text{Ker}L$, thus by the homotopy property of degree,

$$\begin{aligned}\deg(QN|\text{Ker}L, \Omega \cap \text{Ker}L, 0) &= \deg(H(\cdot,0), \Omega \cap \text{Ker}L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \text{Ker}L, 0) = \deg(I, \Omega \cap \text{Ker}L, 0) \neq 0.\end{aligned}$$

Thus by Theorem GM, $Lx = Nx$ has at least one solution in dom $L \cap \overline{\Omega}$, which is a ω periodic solution of equation (1). The proof is complete. ■

Proof of Theorem 2. To apply Theorem GM, we should define an open bounded subset Ω of X such that (i), (ii) and (iii) of Theorem GM hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in \text{dom}L/\text{Ker}L, \ Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. It suffices to prove there is a constant $B > 0$ such that $||x|| \leq B$.

For $x \in \Omega_1$, we get (16). Similarly, we have that there is $\xi \in [0,\omega]$ such that $|x(\xi)| \leq M$. Multiplying two sides of (16) by $x(t)$ and integrating from t to $\xi + \omega$, we get, for $\xi + \omega \geq t \geq \xi$, using (A_4),

$$\begin{aligned}\frac{1}{2}[x(t)]^2 &= \frac{1}{2}[x(+\omega+\xi)]^2 - \lambda \int_t^{\omega+\xi} f(s, x(s), x(s-\tau_1(s)), \cdots, \\ &\qquad\qquad x(s-\tau_m(s)))x(s)ds\\ &\leq \frac{1}{2}M^2 - \lambda \int_t^{\omega+\xi} f(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))x(s)ds\\ &= \frac{1}{2}M^2 - \lambda \int_t^{\omega+\xi} g(s, x(s), x(s-\tau_1(s)), \cdots, x(s-\tau_m(s)))x(s)ds\end{aligned}$$

$$
\begin{aligned}
&-\lambda\int_t^{\omega+\xi} h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s)))x(s)ds\\
\le\ & \frac{1}{2}M^2-\lambda\int_t^{\omega+\xi} h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s)))x(s)ds\\
\le\ & \frac{1}{2}M^2+\int_0^{\omega} |h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s)))||x(s)|ds\\
\le\ & \frac{1}{2}M^2+\sum_{i=1}^m\int_0^{\omega} g_i(s,x(s-\tau_i(s))|x(s)|ds+\int_0^{\omega} g_i(s,x(s))|x(s)|ds\\
&+\int_0^{\omega}|e(s)|x(s)|ds.
\end{aligned}
$$

The remaider of the proof of this step and the other steps are similar to those of the proof of Theorem 1 and are omitted. ■

Proof of Theorem 3. The method is exactly similar to that of Theorem 1. Let Ω_1 be defined as that in the proof of Theorem 1. For $x \in \Omega_1$, we get (16). Since $x(0) = x(\omega)$, there is $\xi \in [0,\omega]$ so that $x'(\xi) = \lambda f(\xi, x(\xi), x(\xi-\tau_1(\xi)),\cdots,x(\xi-\tau(\xi))) = 0$ and ξ is a maximum or minimum point of $x(t)$. It follows from (H_5) that

$$
||x||_\infty = |x(\xi)| \le \frac{L}{\alpha}+\frac{\sum\limits_{i=1}^m \alpha_i|x(\xi-\tau_i(\xi))|}{\alpha} \le \frac{L}{\alpha}+\frac{\sum\limits_{i=1}^m \alpha_i}{\alpha}||x||_\infty.
$$

Since (9) holds, we know that there is a constant $A > 0$ so that $||x||_\infty \le A$.

The remainder of the proof of this Theorem is similar to that of Theorem 1 and is omitted. ■

Proof of Theorem 4. To apply Theorem GM, we should define an open bounded subset Ω of X such that (i), (ii) and (iii) of Theorem GM hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$
\Omega_1 = \{x \in \mathrm{dom}L \setminus \mathrm{Ker}L,\ Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.
$$

We prove Ω_1 is bounded. Similar to that of the proof of Theorem 1, if $x \in \Omega_1$, we get (16) and (17). It suffices to prove there is a constant $B > 0$ such that $||x|| \le B$. We divide this step into two sub-steps.

Sub-step 1.1. Prove there is constant $\overline{M} > 0$ such that $\int_0^\omega |x(s)|^{\mu+1}ds \le \overline{M}$.

Multiplying two sides of (16) by $x(t)$ and integrating from 0 to ω, using (A_7), we get

$$\begin{aligned} 0 &= \lambda \int_0^\omega f(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds \\ &= \lambda \left(\int_0^\omega h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds \right. \\ &\quad \left. + \int_0^\omega g(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds \right). \end{aligned}$$

Thus, from the second part of (A_7),

$$\begin{aligned} \beta \int_0^\omega |x(s)|^{\mu+1}\, ds &\le \int_0^\omega g(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds \\ &= -\int_0^\omega h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds \\ &\le \int_0^\omega |h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))||x(s)|ds \\ &\le \sum_{i=1}^m \int_0^\omega g_i(s,x(s-\tau_i(s)))|x(s)|ds + \int_0^\omega g_i(s,x(s))|x(s)|ds \\ &\quad + \int_0^\omega |e(s)||x(s)|ds. \end{aligned}$$

Choosing $\epsilon > 0$ so that

$$\sum_{i=1}^m \lambda_i^{\mu+1}(||p_i||_\infty + \epsilon) + (||p_0||_\infty + \epsilon) \ < \ \beta.$$

For such a ϵ, there is a $\delta > 0$ so that

$$g_i(t,x) \le (p_i(t)+\epsilon)|x|^m, \quad \text{for } \ |x| \ge \delta, \ \ t \in R. \tag{22}$$

Denote

$$\begin{aligned} \Delta_{i,1} &= \{t \in [0,\omega] : |x(t-\tau_i(t))| \le \delta\}, \\ \Delta_{i,2} &= \{t \in [0,\omega] : |x(t-\tau_i(t))| > \delta\}, \quad i = 1,\cdots,m, \end{aligned} \tag{23}$$

and

$$\Delta_{0,1} = \{t \in [0,\omega] : |x(t)| \le \delta\}, \quad \Delta_{0,2} = \{t \in [0,\omega] : |x(t)| > \delta\}, \tag{24}$$

and

$$g_{i,\delta} \ = \max_{t\in[0,\omega],|x|\le\delta} g_i(t,x), \quad i = 0,1,\cdots,m. \tag{25}$$

$$\beta\int_0^\omega |x(s)|^{\mu+1}\,ds \le \sum_{i=1}^m \int_0^\omega (p_i(s)+\epsilon)|x(s-\tau_i(s))|^\mu |x(s)|ds$$
$$+\int_0^\omega (p_0(s)+\epsilon)|x(s)|^{\mu+1}ds + \int_0^\omega |e(s)||x(s)|ds + \sum_{i=0}^m g_{i,\delta}\int_0^\omega |x(s)|ds$$
$$\le \sum_{i=1}^m (||p_i||_\infty+\epsilon)\int_0^\omega |x(s-\tau_i(s))|^\mu |x(s)|ds + (||p_0||_\infty+\epsilon)\int_0^\omega |x(s)|^{\mu+1}ds$$
$$+||e||_\infty \int_0^\omega |x(s)|ds + \sum_{i=0}^m g_{i,\delta}\int_0^\omega |x(s)|ds$$
$$\le \sum_{i=1}^m (||p_i||_\infty+\epsilon)\Big(\int_0^\omega |x(s-\tau_i(s))|^{\mu+1}|ds\Big)^{\mu/(\mu+1)}\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$
$$+(||p_0||_\infty+\epsilon)\int_0^\omega |x(s)|^{\mu+1}ds + \omega^{\mu/(\mu+1)}||e||_\infty\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$
$$+\omega^{\mu/(\mu+1)}\sum_{i=0}^m g_{i,\delta}\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}.$$

It is easy to see that

$$\int_0^\omega |x(s-\tau_i(s))|^{\mu+1}|ds = \int_0^\omega \left(\frac{|x(s)|}{1-\tau_i'(\mu_i(s))}\right)^{\mu+1} ds \le \lambda_i^{\mu+1}\int_0^\omega |x(s)|^{\mu+1}ds.$$

Hence

$$\beta\int_0^1 |x(s)|^{\mu+1}ds \le \sum_{i=1}^m (||p_i||_\infty+\epsilon)\left(\int_0^\omega\left(\frac{|x(s)|}{|1-\tau_i'(\mu_i(s))|}\right)^{\mu+1}ds\right)^{\mu/(\mu+1)}$$
$$\times\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)} + (||p_0||_\infty+\epsilon)\int_0^\omega |x(s)|^{\mu+1}ds$$
$$+\omega^{\mu/(\mu+1)}||e||_\infty\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$
$$+\omega^{\mu/(\mu+1)}\sum_{i=0}^m g_{i,\delta}\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$
$$\le \sum_{i=1}^m \lambda_i^{\mu+1}(||p_i||_\infty+\epsilon)\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{\mu/(\mu+1)}$$
$$\times\left(\int_0^\omega ||x(s)|^{\mu+1}ds\right)^{1/(\mu+1)} + (||p_0||_\infty+\epsilon)\int_0^\omega |x(s)|^{\mu+1}ds$$
$$+\omega^{\mu/(\mu+1)}||e||_\infty\left(\int_0^\omega ||x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$

$$+\,\omega^{\mu/(\mu+1)}\sum_{i=0}^{m} g_{i,\delta}\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}.$$

Then

$$\left(\beta-\sum_{i=1}^{m}\lambda_i^{\mu+1}(||p_i||_\infty+\epsilon)-(||p_0||_\infty+\epsilon)\right)\int_0^1 |x(s)|^{\mu+1}ds$$
$$\le \omega^{\mu/(\mu+1)}||e||_\infty\left(\int_0^\omega ||x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$
$$+\,\omega^{\mu/(\mu+1)}\sum_{i=0}^{m} g_{i,\delta}\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}.$$

Hence there is a constant $\overline{M}>0$ such that

$$\int_0^\omega |x(s)|^{m+1}ds \;\le\; \overline{M}.$$

Sub-step 1.2. Prove there is $B>0$ such that $||x||\le B$.
From Sub-step 1.1, we have that there is $\xi\in[0,\omega]$ so that $x(\xi)|\le \overline{M}^{1/(\mu+1)}$. For each $t\in R$, let $k_0=\max\{k\in Z \text{ so that } k\omega+\xi\ge t\}$ and $t_0=k_0\omega+\xi$. Multiplying two side of (14) by $x(t)$, integrating it from t to t_0, we get

$$\frac{1}{2}|x(t)|^2 \;=\; \frac{1}{2}|x(\xi)|^2-\lambda\int_t^{t_0} f(s,x(s),x(s-\tau_1(s)),\cdots,$$
$$x(s-\tau_m(s)))x(s)ds$$
$$=\;\overline{M}^{1/(\mu+1)}-\lambda\left(\int_t^{t_0} g(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds\right.$$
$$\left.+\int_t^{t_0} h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds\right)$$
$$\le \overline{M}^{1/(\mu+1)}-\lambda\left(\beta\int_t^{t_0}|x(s)|^{\mu+1}ds\right.$$
$$\left.+\int_t^{t_0} h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds\right)$$
$$\le\;\overline{M}^{1/(\mu+1)}-\lambda\int_t^{t_0} h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))x(s)ds$$
$$\le \overline{M}^{1/(\mu+1)}+\int_0^\omega |h(s,x(s),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))))||x(s)|ds$$
$$\le \overline{M}^{1/(\mu+1)}+\sum_{i=1}^{m}\int_0^\omega g_i(s,x(s-\tau_i(s))|x(s)|ds+\int_0^\omega g_0(s,x(s))|x(s)|ds.$$

Similarly to Sub-step 1.1, we can get

$$\frac{1}{2}|x(t)|^2 \le \overline{M}^{1/(\mu+1)} + \sum_{i=1}^{m} \lambda_i^{\mu+1}(||p_i||_\infty + \epsilon)\left(\int_0^\omega |x(s)|^{m+1}ds\right)^{\mu/(\mu+1)}$$
$$\times \left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)} + (||p_0||_\infty + \epsilon)$$
$$\times \int_0^\omega |x(s)|^{\mu+1}ds + \omega^{\mu/(\mu+1)}||e||_\infty \left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$
$$+ \omega^{\mu/(\mu+1)} \sum_{i=0}^{m} g_{i,\delta}\left(\int_0^\omega |x(s)|^{\mu+1}ds\right)^{1/(\mu+1)}$$
$$\le \overline{M}^{1/(\mu+1)} + \sum_{i=1}^{m} \lambda_i^{\mu+1}(||p_i||_\infty + \epsilon)\overline{M} + (||p_0||_\infty + \epsilon)\overline{M}$$
$$+ \omega^{\mu/(\mu+1)}||e||_\infty \overline{M}^{1/(\mu+1)} + \omega^{\mu/(\mu+1)} \sum_{i=0}^{m} g_{i,\delta}\overline{M}^{1/(\mu+1)}.$$

So there is $\overline{M_1} > 0$ such that $|x(t)| \le \overline{M_1}$. It follows that $||x|| \le \overline{M}_1$. Hence Ω_1 is bounded. This completes the step 1.

Step 2. Let

$$\Omega_2 = \{x \in \mathrm{Ker}L,\ Nx \in \mathrm{Im}L\}.$$

Similar to that of the proof of Step 2 of Theorem 1, we can prove Ω_2 is bounded.

Step 3. Let

$$\Omega_3 = \{x \in \mathrm{Ker}L,\ \lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge : \mathrm{Ker}L \to \mathrm{Im}Q$ is the linear isomorphism given by $\wedge(c) = c$ for all $c \in R$. Similar to that of the proof of Step 3 of Theorem 1, we can show that Ω_3 is bounded.

The remainder step, Step 4, is similar to that of the proof of Step 4 of Theorem 1 and is omitted.

Thus by Theorem GM, $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$, which is a periodic solution of equation (1). The proof is complete. ■

Proof of Theorem 5. It is similar to that of Theorem 4 and is omitted. ■

References

[1] FARKAS M.,GRAEF J., QIAN C., Asymptotic periodicity of delay differential equations, *J. Math. Anal. Appl.,* 226(1998), 150-165.

[2] WANG G., Existence theorems of periodic solutions for a delay nonlinear differential equation with piecewise constant argument, *J. Math. Anal. Appl.,* 298(2004), 298-307.

[3] ZHANG G., CHENG S., Existence of positive periodic solutions of non-autonomous functional differential equations, *Electronic J. of Differential Equations,* 59(2001), 1-8.

[4] JIANG D., WEI J., ZHANG B., Positive periodic solutions of functional differential equations and population models, *Electronic J. of Differential Equations,* 71(2002), 1-13.

[5] KIGARADZE I.T., PUZA B., On periodic solutions of systems of differential equations with deviating arguments, *Nonlinear Analysis,* 42(2000), 229-242.

[6] KUANG Y., *Delay Differential Equations with Applications in Population Dynamics,* Academic Press, New York, 1993.

[7] YANG X., Upper and lower solutions for periodic solutions, *Appl. Math. Comput.,* 137(2003), 413-422.

[8] GAINES R.E., MAWHIN J.L., *Coincidence Degree and Nonlinear Differential Equations,* Lecture Notes in Math. 568, Springer, Berlin, 1977.

[9] LOUD W.S., *Periodic solutions of nonlinear differential equations of Duffine types, in differential equations,* New York, Benjami (1967), 199-224.

[10] REISSIG R., Contractive mapping and periodically perturbed non-conservertive systems, *Lincei Rend. Sco. Fiz. Mat. Enat.,* 58(1975), 696-702.

[11] LIU Y., GE W., Positive periodic solutions of nonlinear differential equations, *Appl. Math. a Journal of Chinese Universities,* 18(4)(2003), 373-382.

YUJI LIU
DEPARTMENT OF MATHEMATICS
GUANGDONG UNIVERSITY OF BUSINESS STUDIES
GUANGZHOU 510320, P.R.CHINA
e-mail: liuyuji888@sohu.com

The author is supported by the Natural Science Foundation of Hunan Province (06JJ50008).

Received on 27.02.2006 and, in revised form, on 09.02.2007.