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SEPARATION AXIOMS IN QUASI m-BITOPOLOGICAL SPACES

Abstract. By using the notion of m-spaces, we establish the unified theory for several variations of separation axioms quasi T_0, quasi T_1 and quasi T_2 in bitopological spaces.

Key words: quasi m-T_0, quasi m-T_1, quasi m-T_2, quasi-open, quasi m-structure, m_X-open, m-space, bitopological space.

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1. Introduction

The notion of quasi-open sets in bitopological spaces is introduced by Datta [7]. Some properties of quasi-open sets are studied in [11]. Quasi-semi-open sets in bitopological spaces are introduced and studied in [9], [12] and [20]. Thakur and Paik [24], [25] introduced and studied the notion of quasi-α-open sets in bitopological spaces. In these papers, the following separation axioms introduced and investigated: quasi T_i, quasi semi-T_i for $i = 0, 1, 2$. Recently, the present authors [21] have introduced the notions of minimal structures and m-spaces.

In this paper, by using the notion of minimal structures we obtain the unified definitions and characterizations of variations of separation axioms quasi T_0, quasi T_1 and quasi T_2 in bitopological spaces.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 1. *Let (X, τ) be a topological space. A subset A of X is said to be α-open [16] (resp. semi-open [10], preopen [14], β-open [1] or semi-preopen [3]) if $A \subset Int(Cl(Int(A)))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(A))$, $A \subset Cl(Int(Cl(A)))$).*

The family of all semi-open (resp. preopen, α-open, β-open, semi-preopen) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$).

Definition 2. *The complement of a semi-open (resp. preopen, α-open, β-open , semi-preopen) set is said to be semi-closed [6] (resp. preclosed [8], α-closed [15], β-closed [1], semi-preclosed [3]).*

Definition 3. *The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, semi-preclosed) sets of X containing A is called the semi-closure [6] (resp. preclosure [8], α-closure [15], β-closure [2], semi-preclosure [3]) of A and is denoted by $sCl(A)$ (resp. $pCl(A)$, $\alpha Cl(A)$, ${}_{\beta}Cl(A)$, $spCl(A)$).*

Definition 4. *The union of all semi-open (resp. preopen, α-open, β-open, semi-preopen) sets of X contained in A is called the semi-interior (resp. preinterior, α-interior, β-interior, semi-preinterior) of A and is denoted by $sInt(A)$ (resp. $pInt(A)$, $\alpha Int(A)$, ${}_{\beta}Int(A)$, $spInt(A)$).*

Throughout the present paper (X,τ) and (Y,σ) always denote topological spaces and (X,τ_1,τ_2) and (Y,σ_1,σ_2) denote bitopological spaces.

3. Minimal structures and m-continuity

Definition 5. *A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a minimal structure (or briefly m-structure) [21] on X if $\emptyset \in m_X$ and $X \in m_X$.*

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be m_X-closed (or briefly m-closed).

Remark 1. Let (X,τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ and $\mathrm{SPO}(X)$ are all m-structures on X.

Definition 6. *Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X-closure of A and the m_X-interior of A are defined in [13] as follows:*

(1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 2. Let (X,τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$), then we have

(1) $m_X\text{-Cl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$),

(2) $m_X\text{-Int}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_{\beta}\mathrm{Int}(A)$, $\mathrm{spInt}(A)$).

Lemma 1. (Popa and Noiri [21]) *Let (X, m_X) be an m-space and A a subset of X. Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 7. *A minimal structure m_X on a nonempty set X is said to have property ($\mathcal{B}$) [13] if the union of any family of subsets belonging to m_X belongs to m_X.*

Lemma 2. (Popa and Noiri [22]) *Let (X, m_X) be an m-space and m_X satisfy property ($\mathcal{B}$). Then for a subset A of X, the following properties hold:*

(1) $A \in m_X$ if and only if $m_X\text{-Int}(A) = A$,

(2) A is m_X-closed if and only if $m_X\text{-Cl}(A) = A$,

(3) $m_X\text{-Int}(A) \in m_X$ and $m_X\text{-Cl}(A)$ is m_X-closed.

Definition 8. *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be M-continuous [21] if for each $x \in X$ and each m_Y-open sets V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.*

Theorem 1. (Popa and Noiri [21]) *Let (X, m_X) be an m-space and m_X satisfy property ($\mathcal{B}$). For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is M-continuous;

(2) $f^{-1}(V)$ is m_X-open for every m_Y-open set V of Y;

(3) $f^{-1}(F)$ is m_X–closed for every m_Y-closed set F of Y.

Definition 9. *An m-space (X, m_X) is said to be*

(1) m-T_0 [17] if for any pair of distinct points x, y of X, there exists an m_X-open set containing x but not y or an m_X-open set containing y but not x,

(2) m-T_1 [17] if for any pair of distinct points x, y of X, there exists an m_X-open set containing x but not y and an m_X-open set containing y but not x,

(3) m-T_2 [21] if for any pair of distinct points x, y of X, there exist m_X-open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

4. Minimal structures and bitopological spaces

First, we shall recall some definitions of variations of quasi-open sets in bitopological spaces.

Definition 10. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be*

(1) quasi-open [7], [11] if $A = B \cup C$, where $B \in \tau_1$ and $C \in \tau_2$,

(2) quasi-semi-open [9], [12] if $A = B \cup C$, where $B \in \text{SO}(X, \tau_1)$ and $C \in \text{SO}(X, \tau_2)$,

(3) quasi-preopen [19] if $A = B \cup C$, *where* $B \in \mathrm{PO}(X, \tau_1)$ *and* $C \in \mathrm{PO}(X, \tau_2)$,

(4) quasi-semipreopen [26] if $A = B \cup C$, *where* $B \in \mathrm{SPO}(X, \tau_1)$ *and* $C \in \mathrm{SPO}(X, \tau_2)$,

(5) quasi-α*-open [24] if* $A = B \cup C$, *where* $B \in \alpha(X, \tau_1)$ *and* $C \in \alpha(X, \tau_2)$.

The family of all quasi-open (resp. quasi-semi-open, quasi-preopen, quasi-semipreopen, quasi-α-open) sets of (X, τ_1, τ_2) is denoted by $\mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$, $\mathrm{Q}\alpha(X)$).

Definition 11. *Let* (X, τ_1, τ_2) *be a bitopological space and* m_X^1 *(resp.* m_X^2*) an* m*-structure on the topological space* (X, τ_1) *(resp.* (X, τ_2)*). The family*

$$qm_X = \{A \subset X : A = B \cup C, \text{ where } B \in m_X^1 \text{ and } C \in m_X^2\}$$

is called a quasi m*-structure on* X*. Each member* $A \in qm_X$ *is said to be quasi-*m_X*-open (or briefly quasi-*m*-open). The complement of a quasi-*m_X*-open set is said to be quasi-*m_X*-closed (or briefly quasi-*m*-closed).*

Remark 3. Let (X, τ_1, τ_2) be a bitopological space.

(1) If m_X^1 and m_X^2 have property $(\mathcal{B})$, then qm_X is an m-structure with property $(\mathcal{B})$.

(2) If $(m_X^1, m_X^2) = (\tau_1, \tau_2)$ (resp. $(\mathrm{SO}(X, \tau_1), \mathrm{SO}(X, \tau_2))$, $(\mathrm{PO}(X, \tau_1), \mathrm{PO}(X, \tau_2))$, $(\mathrm{SPO}(X, \tau_1), \mathrm{SPO}(X, \tau_2))$, $(\alpha(X, \tau_1), \alpha(X, \tau_2))$), then $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$, $\mathrm{Q}\alpha(X)$).

(3) Since $\mathrm{SO}(X, \tau_i)$, $\mathrm{PO}(X, \tau_i)$, $\mathrm{SPO}(X, \tau_i)$ and $\alpha(X, \tau_i)$ have property $(\mathcal{B})$ for $i = 1, 2$, $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$ and $\mathrm{Q}\alpha(X)$ have property $(\mathcal{B})$.

Definition 12. *Let* (X, τ_1, τ_2) *be a bitopological space. For a subset* A *of* X*, the quasi* m_X*-closure of* A *and the quasi* m_X*-interior of* A *are defined as follows:*

(1) $qm_X\text{-}Cl(A) = \cap\{F : A \subset F, X - F \in qm_X\}$,

(2) $qm_X\text{-}Int(A) = \cup\{U : U \subset A, U \in qm_X\}$,

qm_X*-Cl(A) and* qm_X*-Int(A) are simply denoted by qmCl(A) and qmInt(A), respectively.*

Remark 4. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. If $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$, $\mathrm{Q}\alpha(X)$), then we have

(1) $\mathrm{qmCl}(A) = \mathrm{qCl}(A)$ (resp. $\mathrm{qsCl}(A)$ [9], $\mathrm{qpCl}(A)$ [19], $\mathrm{qspCl}(A)$ [26], $\mathrm{q}\alpha\mathrm{Cl}(A)$ [24]),

(2) $\mathrm{qmInt}(A) = \mathrm{qInt}(A)$ (resp. $\mathrm{qsInt}(A)$, $\mathrm{qpInt}(A)$, $\mathrm{qspInt}(A)$, $\mathrm{q}\alpha\mathrm{Int}(A)$).

5. Quasi m-T_i-spaces

Definition 13. *A bitopological space (X,τ_1,τ_2) is said to be*

(1) quasi T_0 [20] (resp. quasi semi-T_0 [12], quasi α-T_0, quasi pre-T_0, quasi sp-T_0) if for each pair of distinct points in X, there exists a quasi-open (resp. quasi-semi-open, quasi-α-open, quasi-pre-open, quasi-semipre-open) set in (X,τ_1,τ_2) containing one of them and not containing the other,

(2) quasi T_1 [20] (resp. quasi semi-T_1 [12], quasi α-T_1, quasi pre-T_1, quasi sp-T_1) if for each pair of distinct points $x,y \in X$, there exist quasi-open (resp. quasi-semi-open, quasi-α-open, quasi-pre-open, quasi-semipre-open) sets U_x and U_y in (X,τ_1,τ_2) such that $x \in U_x, y \notin U_x, y \in U_y$ and $x \notin U_y$,

(3) quasi T_2 [20] (resp. quasi semi-T_2 [12], quasi α-T_2, quasi pre-T_2, quasi sp-T_2) if for each pair of distinct points $x,y \in X$, there exist disjoint quasi-open (resp. quasi-semi-open, quasi-α-open, quasi-pre-open, quasi-semi-pre-open) sets U_x and U_y in (X,τ_1,τ_2) such that $x \in U_x$ and $y \in U_y$.

Definition 14. *Let (X,τ_1,τ_2) be a bitopological space and qm_X a quasi m-structure on X. Then (X,τ_1,τ_2) is said to be quasi m-T_i if the m-space (X,qm_X) is m-T_i for $i=0,1,2$.*

Remark 5. Let (X,τ_1,τ_2) be a bitopological space. If $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$, $\mathrm{Q}\alpha(X)$, $\mathrm{QPO}(X)$, $\mathrm{QSPO}(X)$) and (X,qm_X) is m-T_i, then (X,τ_1,τ_2) is quasi T_i (resp. quasi semi-T_i, quasi α-T_i, quasi pre-T_i, quasi sp-T_i) for $i=0$, 1, 2.

We shall recall the definitions of Λ_m-sets, a topological space (X,Λ_m) and (Λ,m)-closed sets in order to obtain characterizations of quasi m-T_0 spaces and quasi m-T_1 spaces. Let (X,m) be an m-space and A a subset of X. A subset $\Lambda_m(A)$ is defined in [5] as follows: $\Lambda_m(A) = \cap\{U : A \subset U \in m\}$. The subset A is called a *Λ_m-set* [5] if $A = \Lambda_m(A)$. The family of all Λ_m-sets of (X,m_X) is denoted by $\Lambda_m(X)$ (or simply Λ_m). It follows from Theorem 3.1 of [5] that the pair (X,Λ_m) is an Alexandorff (topological) space. The subset A is said to be *(Λ,m)-closed* [5] if $A = U \cap F$, where U is a Λ_m-set and F is an m-closed set of (X,m). For a quasi m_X-structure qm_X, Λ_{qm}-sets, a topological space (X,Λ_{qm}) and (Λ,qm)-closed sets are similarly defined.

Theorem 2. (Noiri and Popa [17]) *An m-space (X,m_X) is m-T_0 if and only if m_X-$\mathrm{Cl}(\{x\}) \neq m_X$-$\mathrm{Cl}(\{y\})$ for any pair of distinct points $x,y \in X$.*

Theorem 3. (Cammaroto and Noiri [5]) *For an m-space (X,m_X), the following properties are equivalent:*

(1) (X,m) is m-T_0;

(2) The singleton $\{x\}$ is (Λ,m)-closed for each $x \in X$;

(3) (X,Λ_m) is T_0.

Corollary 1. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi* m*-structure on* X*. Then the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is quasi* m-T_0*;*

(2) $\mathrm{qmCl}(\{x\}) \neq \mathrm{qmCl}(\{y\})$ *for any pair of distinct points* $x, y \in X$*;*

(3) The singleton $\{x\}$ *is* (Λ, qm)*-closed for each* $x \in X$*;*

(4) (X, Λ_{qm}) *is* T_0*.*

Proof. This is an immediate consequence of Theorems 2 and 3. ■

Remark 6. In case $qm_X = \mathrm{QSO}(X)$, by Corollary 1 we obtain the following characterization due to Maheshwari, Chae and Thakur [12]: a bitopological space (X, τ_1, τ_2) is quasi semi-T_0 if and only if $\mathrm{qsCl}(\{x\}) \neq \mathrm{qsCl}(\{y\})$ for any pair of distinct points $x, y \in X$.

Theorem 4. (Noiri and Popa [17]) *Let* (X, m_X) *be an* m*-space and* m_X *have property* $(\mathcal{B})$*. Then* (X, m_X) *is* m-T_1 *if and only if for each points* $x \in X$*, the singleton* $\{x\}$ *is* m_X*-closed.*

Theorem 5. (Cammaroto and Noiri [5]) *Let* (X, m_X) *be an* m*-space and* m_X *have property* $(\mathcal{B})$*. Then for the* m*-space* (X, m_X)*, the following properties are equivalent:*

(1) (X, m_X) *is* m-T_1*;*

(2) The singleton $\{x\}$ *is a* Λ_m*-set for each* $x \in X$*;*

(2) (X, Λ_m) *is discrete.*

Corollary 2. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi* m*-structure on* X *having property* $(\mathcal{B})$*. Then for the space* (X, τ_1, τ_2)*, the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is quasi* m-T_1*;*

(2) The singleton $\{x\}$ *is quasi-*m_X*-closed for each point* $x \in X$*;*

(3) The singleton $\{x\}$ *is a quasi* Λ_m*-set for each* $x \in X$*;*

(4) (X, Λ_{qm}) *is discrete.*

Proof. This is an immediate consequence of Theorems 4 and 5. ■

Remark 7. In case $qm_X = \mathrm{QO}(X)$ (resp. $\mathrm{QSO}(X)$), by Corollary 2 we obtain the following characterization due to Maheshwari, Jain and Chae [11] (resp. Maheshwari, Chae and Thakur [12]): a bitopological space (X, τ_1, τ_2) is quasi T_1 (resp. quasi semi-T_1) if and only if the singleton $\{x\}$ is quasi-closed (resp. quasi-semi-closed) for each point $x \in X$.

Theorem 6. *Let* (X, m_X) *be an* m*-space and* m_X *have property* $(\mathcal{B})$*. Then, for the* m*-space* (X, m_X) *the following properties are equivalent:*

(1) (X, m_X) *is* m-T_2*;*

(2) For any distinct points $x, y \in X$*, there exists* $U \in m_X$ *containing* x *such that* $y \notin m_X\text{-Cl}(U)$*;*

(3) For each point $x \in X$*,* $\{x\} = \cap\{m_X\text{-Cl}(U) : x \in U \in m_X\}$*;*

(4) For each pair of distinct points $x, y \in X$*, there exists an* M*-continuous function* f *of* (X, m_X) *into an* m-T_2 m*-space* (Y, m_Y) *such that* $f(x) \neq f(y)$*.*

Proof. (1) ⇒ (2) For any distinct points $x, y \in X$, there exist $U, V \in m_X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$; hence $V \cap m_X\text{-Cl}(U) = \emptyset$ by Lemma 1. Therefore, we have $y \notin m_X\text{-Cl}(U)$.

(2) ⇒ (3): Let x be any point of X. Suppose that $y \in X - \{x\}$. By (2), there exists $U \in m_X$ such that $x \in U$ and $y \notin m_X\text{-Cl}(U)$. Thus, $y \notin \cap\{m_X\text{-Cl}(U) : x \in U \in m_X\}$. Therefore, we have $\{x\} = \cap\{m_X\text{-Cl}(U) : x \in U \in m_X\}$.

(3) ⇒ (1): For any pair of distinct points x, y in X, there exists $U \in m_X$ such that $x \in U$ and $y \notin m_X\text{-Cl}(U)$. Put $V = \text{X} - m_X\text{-Cl}(U)$. Since m_X has property ($\mathcal{B}$), by Lemma 2 $m_X\text{-Cl}(U)$ is m-closed and hence $y \in V$, $V \in m_X$ and $U \cap V = \emptyset$. Therefore, (X, m_X) is m-T_2.

(1) ⇒ (4): For any pair of distinct points x, y in X, it suffices to take the identity function on (X, m_X).

(4) ⇒ (1): Let x and y be any pair of distinct points of (X, m_X). By (4), there exists an M-continuous function of (X, m_X) into an m-T_2 m-space (Y, m_Y) such that $f(x) \neq f(y)$. Therefore, there exist disjoint m_Y-open sets V_x and V_y such that $f(x) \in V_x$ and $f(y) \in V_y$. Since f is M-continuous and m_X has property ($\mathcal{B}$), by Theorem 1 $f^{-1}(V_x)$ and $f^{-1}(V_y)$ are disjoint m_X-open sets containing x and y, respectively. This implies that (X, m_X) is m-T_2. ■

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *quasi* M*-continuous* if $f : (X, qm_X) \to (Y, qm_Y)$ is M-continuous, where qm_X and qm_Y are quasi m-structres on (X, τ_1, τ_2) and (Y, σ_1, σ_2), respectively.

Corollary 3. *Let* (X, τ_1, τ_2) *be a bitopological space and* qm_X *a quasi* m*-structure on* X *having property* ($\mathcal{B}$)*. Then, for the space* (X, τ_1, τ_2) *the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is quasi* m-T_2*;*

(2) For any distinct points $x, y \in X$*, there exists* $U \in qm_X$ *containing* x *such that* $y \notin \text{qmCl}(U)$*;*

(3) For each point $x \in X$*,* $\{x\} = \cap\{\text{qmCl}(U) : x \in U \in qm_X\}$*;*

(4) For each pair of distinct points $x, y \in X$*, there exists a quasi* M*-continuous function* f *of* (X, τ_1, τ_2) *into a quasi* m-T_2 *space* (Y, σ_1, σ_2) *such that* $f(x) \neq f(y)$*.*

Proof. This is an immediate consequence of Theorem 6. ■

Remark 8. In case $qm_X = \mathrm{QSO}(X)$, by Corollary 3 we obtain the results established in Theorem 6 of [20] and Theorem 24 of [12].

Theorem 7. *Let $f : (X, m_X) \to (Y, m_Y)$ be an injective M-continuous function and m_X have property ($\mathcal{B}$). If (Y, m_Y) is m-T_i, then (X, m_X) is m-T_i for $i = 0, 1, 2$.*

Proof. The proof of the case of m-T_0 is entirely analogous to that of m-T_1. The proof for m-T_2 is obvious from Theorem 6. Thus, we shall prove the case of m-T_1. Suppose that (Y, m_Y) is m-T_1. Let x, y be any pair of distinct points of X. Since f is injective, $f(x) \neq f(y)$ and there exist $V_x, V_y \in m_Y$ containing $f(x)$ and $f(y)$, resprctively, such that $f(y) \notin V_x$ and $f(x) \notin V_y$. Since f is M-continuous and m_X has property ($\mathcal{B}$), by Theorem 1 $f^{-1}(V_x)$ and $f^{-1}(V_y)$ are m_X-open sets containing x and y, respectively, such that $y \notin f^{-1}(V_x)$ and $x \notin f^{-1}(V_y)$. This implies that (X, m_X) is m-T_1. ■

Corollary 4. *Let qm_X and qm_Y be quasi m-structures on (X, τ_1, τ_2) and (Y, σ_1, σ_2), respectively, where qm_X has property ($\mathcal{B}$). If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a quasi M-continuous injection and (Y, σ_1, σ_2) is quasi m-T_i, then (X, τ_1, τ_2) is quasi m-T_i for $i = 0, 1, 2$.*

Proof. This follows immediately from Theorem 7. ■

Remark 9. If $qm_Y = \mathrm{QO}(Y)$, $qm_X = \mathrm{QSO}(X)$ and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a quasi M-continuous injection, then by Corollary 4 we obtain the result established in Theorem 2 of [20].

6. New forms of quasi-T_i spaces

There are many modifications of open sets in topological spaces. Recently, many researchers are interested in δ-preopen sets [23] and δ-semi-open sets [18]. First, we shall recall the definitions of the δ-closure and the θ-closure of a subset. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a *δ-cluster* (resp. *θ-cluster*) point of A if $\mathrm{Int}(\mathrm{Cl}(U)) \cap A \neq \emptyset$ (resp. $\mathrm{Cl}(U) \cap A \neq \emptyset$) for every open set U containing x. The set of all δ-cluster (resp. θ-cluster) points of A is called the *δ-closure* (resp. *θ-closure*) [27] of A and is denoted by $\mathrm{Cl}_\delta(A)$ (resp. $\mathrm{Cl}_\theta(A)$). It is shown in [27] that $A \subset \mathrm{Cl}(A) \subset \mathrm{Cl}_\delta(A) \subset \mathrm{Cl}_\theta(A)$ for every subset A of X. A subset A is said to be *δ-closed* (resp. *θ-closed*) if $\mathrm{Cl}_\delta(A) = A$ (resp. $\mathrm{Cl}_\theta(A) = A$). The complement of a δ-closed (resp. θ-closed) set is said to be *δ-open* (resp. *θ-open*). The δ-interior (resp. θ-interior) of A, $\mathrm{Int}_\delta(A)$ (resp. $\mathrm{Int}_\theta(A)$), is defined as the union of δ-open (resp. θ-open) sets contained in A.

Definition 15. *A subset A of a topological space (X, τ) is said to be*

(1) δ-semiopen [18] (resp. θ-semiopen [4]) if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$ *(resp.* $A \subset \mathrm{Cl}(\mathrm{Int}_\theta(A))$*),*

(2) δ-preopen [23]) (resp. θ-preopen) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ *(resp.* $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$*),*

(3) δ-semipreopen (resp. θ-semipreopen) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ *(resp.* $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$*).*

The family of all δ-semiopen (resp. δ-preopen, δ-semipreopen, θ-semiopen, θ-preopen, θ-semipreopen) sets of (X, τ) is denoted by $\delta\mathrm{SO}(X, \tau)$ (resp. $\delta\mathrm{PO}(X, \tau)$, $\delta\mathrm{SPO}(X, \tau)$, $\theta\mathrm{SO}(X, \tau)$, $\theta\mathrm{PO}(X, \tau)$, $\theta\mathrm{SPO}(X, \tau)$).

Definition 16. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be*

(1) quasi-δ-semiopen if $A = B \cup C$, where $B \in \delta\mathrm{SO}(X, \tau_1)$ and $C \in \delta\mathrm{SO}(X, \tau_2)$,

(2) quasi-δ-preopen if $A = B \cup C$, where $B \in \delta\mathrm{PO}(X, \tau_1)$ and $C \in \delta\mathrm{PO}(X, \tau_2)$,

(3) quasi-δ-semipreopen if $A = B \cup C$, where $B \in \delta\mathrm{SPO}(X, \tau_1)$ and $C \in \delta\mathrm{SPO}(X, \tau_2)$.

(4) quasi-θ-semiopen if $A = B \cup C$, where $B \in \theta\mathrm{SO}(X, \tau_1)$ and $C \in \theta\mathrm{SO}(X, \tau_2)$,

(5) quasi-θ-preopen if $A = B \cup C$, where $B \in \theta\mathrm{PO}(X, \tau_1)$ and $C \in \theta\mathrm{PO}(X, \tau_2)$,

(6) quasi-θ-semipreopen if $A = B \cup C$, where $B \in \theta\mathrm{SPO}(X, \tau_1)$ and $C \in \theta\mathrm{SPO}(X, \tau_2)$.

The family of all quasi-δ-semiopen (resp. quasi-δ-preopen, quasi δ-semipreopen, quasi-θ-semiopen, quasi-θ-preopen, quasi θ-semipreopen) sets of (X, τ_1, τ_2) is denoted by $\mathrm{Q}\delta\mathrm{SO}(X)$ (resp. $\mathrm{Q}\delta\mathrm{PO}(X)$, $\mathrm{Q}\delta\mathrm{PSO}(X)$, $\mathrm{Q}\theta\mathrm{SO}(X)$, $\mathrm{Q}\theta\mathrm{PO}(X)$, $\mathrm{Q}\theta\mathrm{PSO}(X)$).

Remark 10. Let (X, τ_1, τ_2) be a bitopological space. Since $\delta\mathrm{SO}(X, \tau_i)$, $\delta\mathrm{PO}(X, \tau_i)$, $\delta\mathrm{SPO}(X, \tau_i)$, $\theta\mathrm{SO}(X, \tau_i)$, $\theta\mathrm{PO}(X, \tau_i)$ and $\theta\mathrm{SPO}(X, \tau_i)$ are all m-structures with property $(\mathcal{B})$ for $i = \mathrm{i}, 2$, $\mathrm{Q}\delta\mathrm{SO}(X)$, $\mathrm{Q}\delta\mathrm{PO}(X)$, $\mathrm{Q}\delta\mathrm{PSO}(X)$, $\mathrm{Q}\theta\mathrm{SO}(X)$, $\mathrm{Q}\theta\mathrm{PO}(X)$ and $\mathrm{Q}\theta\mathrm{PSO}(X)$ are all quasi m-structures on X with property $(\mathcal{B})$.

For a bitopological space (X, τ_1, τ_2), we can define new types of quasi T_i. For example, in case $qm_X = \mathrm{Q}\delta\mathrm{SO}(X)$, $\mathrm{Q}\delta\mathrm{PO}(X)$, $\mathrm{Q}\delta\mathrm{PSO}(X)$, $\mathrm{Q}\theta\mathrm{SO}(X)$, $\mathrm{Q}\theta\mathrm{PO}(X)$ or $\mathrm{Q}\theta\mathrm{PSO}(X)$, we can define new types of quasi T_i as follows:

Definition 17. *A bitopological space (X, τ_1, τ_2) is said to be quasi δ-semi-T_0 (resp. quasi δ-pre-T_0, quasi δ-sp-T_0, quasi θ-semi-T_0, quasi θ-pre-T_0, quasi θ-sp-T_0) if for each pair of distinct points in X, there exists a quasi-δ-semiopen (resp. quasi-δ-preopen, quasi-δ-semipreopen, quasi-θ-semiopen, quasi-θ-preopen,* *quasi-θ-semipreopen) set in (X, τ_1, τ_2) containing one of them and not containing the other.*

Definition 18. *A bitopological space (X, τ_1, τ_2) is said to be quasi δ-semi-T_1 (resp. quasi δ-pre-T_1, quasi δ-sp-T_1, quasi θ-semi-T_1, quasi θ-pre-T_1, quasi θ-sp-T_1) if for each pair of distinct points $x, y \in X$, there exist quasi-δ-semiopen (resp. quasi-δ-preopen, quasi-δ-semipreopen, quasi-θ-semiopen, quasi-θ-preopen, quasi-θ-semipreopen) sets U_x and U_y in (X, τ_1, τ_2) such that $x \in U_x, y \notin U_x, y \in U_y$ and $x \notin U_y$.*

Definition 19. *A bitopological space (X, τ_1, τ_2) is said to be quasi δ-semi-T_2 (resp. quasi δ-pre-T_2, quasi δ-sp-T_2, quasi θ-semi-T_2, quasi θ-pre-T_2, quasi θ-sp-T_2) if for each pair of distinct points $x, y \in X$, there exist disjoint quasi-δ-semiopen (resp. quasi-δ-preopen, quasi-δ-semipreopen, quasi-θ-semiopen, quasi-θ-preopen, quasi-θ-semipreopen) sets U_x and U_y in (X, τ_1, τ_2) such that $x \in U_x$ and $y \in U_y$.*

Conclusion. We can apply the results established in Section 5 to bitopological spaces as follows:
(*1*) bitopological spaces defined in Definition 13,
(*2*) bitopological spaces defined in Definitions 17, 18 and 19,
(*3*) bitopological spaces with any quasi m-structure having property $(\mathcal{B})$.

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