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SOME EXISTENCE AND UNIQUENESS COMMON FIXED POINT THEOREMS FOR SELFMAPPINGS IN UNIFORM SPACE

Abstract. In this paper, we establish some common fixed point theorems for selfmappings in uniform space by employing both the concepts of an $A-$distance and an $E-$distance introduced by Aamri and El Moutawakil [1]. We employ a contractive definition independent of those of Olatinwo [8] and Aamri and El Moutawakil [1]. Our results are also independent of those of Olatinwo [8] as well as independent of Theorems 3.1-3.3 of Aamri and El Moutawakil [1].

Key words: E-distance, contractive definition, uniform space.

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1. Introduction

Let (X, Φ) be a uniform space, where X is a nonempty set equipped with a nonempty family Φ of subsets of $X \times X$ satisfying certain properties. Φ is called the *uniform structure* of X and its elements are called *entourages or neighbourhoods or surroundings.* Interested readers can consult Bourbaki [4], Olatinwo [8] and Zeidler [14] for the definition of uniform space. The definition is also available on internet (by Wikipedia, the free encyclopedia).

The concept of a $W-$distance on metric space was introduced by Kada et al [6] to generalize some important results in nonconvex minimizations and in fixed point theory for both $W-$contractive and $W-$expansive maps. The theory of fixed point or common fixed point for contractive or expansive selfmappings in complete metric space has been well-developed. Interested readers can consult Berinde [2, 3], Jachymski [5], Kada et al [6], Kang [7], Rhoades [9], Rus [11], Rus et al [12], Wang et al [13] and Zeidler [14] for further study of fixed point or common fixed point theory.

Using the ideas of Kang [7], Montes and Charris [10] established some results on fixed and coincidence points of maps by means of appropriate

$W-$contractive or $W-$expansive assumptions in uniform space. Furthermore, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an $A-$distance and an $E-$distance.

In Aamri and El Moutawakil [1], the following contractive definition was employed:
Let $f,\ g : X \to X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \ \leq\ \psi(p(g(x), g(y))), \quad \forall\ x,\ y \in X, \tag{1}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying

(i) for each $t \in (0, +\infty),\ 0 < \psi(t)$,

(ii) $\lim_{n\to\infty} \psi^n(t) = 0,\ \forall\ t \in (0, +\infty)$. ψ satisfies also the condition $\psi(t) < t$, for each $t > 0$.

Olatinwo [8] employed the following contractive definition:
Let $f,\ g : X \to X$ be selfmappings of X. There exist $L \geq 0$ and a comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall\ x,\ y \in X$, we have

$$p(f(x), f(y)) \ \leq\ Lp(x, g(x)) + \psi(p(g(x), g(y))). \tag{2}$$

In this paper, we shall establish some common fixed point theorems by employing a contractive condition independent of (1) and (2).
We shall employ the concepts of an $A-$distance, an $E-$distance as well as the notion of comparison function in this work. Berinde [2, 3] extended the Banach's fixed point theorem using different contractive definitions involving the concept of the comparison functions. Rus [11] and Rus et al [12] also contain various generalizations and extensions of the Banach's fixed point theorem in which the contractive conditions involve some comparison functions.

Our results are generalizations of Theorems 3.1-3.3 of [1] and Theorems 3.1, 3.3 & 3.5 of Olatinwo [8].

2. Preliminaries

We shall require the following definitions and lemma in the sequel.
Let (X, Φ) be a uniform space.

Remark 1. When topological concepts are mentioned in the context of a uniform space (X, Φ), they always refer to the topological space $(X, \tau(\Phi))$.

Definition 1. *If $V \in \Phi$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be $V-$close. A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a Cauchy sequence for Φ if for any $V \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are $V-$close for $n,\ m \geq N$.*

Definition 2. *A function* $p : X \times X \to I\!R^+$ *is said to be an* $A-$*distance if for any* $V \in \Phi$*, there exists* $\delta > 0$ *such that if* $p(z, x) \leq \delta$ *and* $p(z, y) \leq \delta$ *for some* $z \in X$*, then* $(x, y) \in V$*.*

Definition 3. *A function* $p : X \times X \to I\!R^+$ *is said to be an* $E-$*distance if*

(p_1) p *is an* $A-$*distance,*

(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall\ x,\ y \in X.$

Definition 4. *A uniform space* (X, Φ) *is said to be Hausdorff if and only if the intersection of all* $V \in \Phi$ *reduces to the diagonal* $\{\ (x, x) \mid x \in X\}$*, i.e. if* $(x, y) \in V$ *for all* $V \in \Phi$ *implies* $x = y$*. This guarantees the uniqueness of limits of sequences.* $V \in \Phi$ *is said to be symmetrical if* $V = V^{-1} = \{\ (y, x) \mid (x, y) \in V\}$*.*

Definition 5. *Let* (X, Φ) *be a uniform space and* p *be an* $A-$*distance on* X*.*

(i) X *is said to be* $S-$*complete if for every* $p-$*Cauchy sequence* $\{x_n\}_{n=0}^{\infty}$*, there exists* $x \in X$ *with* $\lim\limits_{n\to\infty} p(x_n, x) = 0$*.*

(ii) X *is said to be* $p-$*Cauchy complete if for every* $p-$*Cauchy sequence* $\{x_n\}_{n=0}^{\infty}$*, there exists* $x \in X$ *with* $\lim\limits_{n\to\infty} x_n = x$ *with respect to* $\tau(\Phi)$*.*

(iii) $f : X \to X$ *is* $p-$*continuous if* $\lim\limits_{n\to\infty} p(x_n, x) = 0$ *implies* $\lim\limits_{n\to\infty} p(f(x_n), f(x)) = 0$*.*

(iv) $f : X \to X$ *is* $\tau(\Phi)-$*continuous if* $\lim\limits_{n\to\infty} x_n = x$ *with respect to* $\tau(\Phi)$ *implies* $\lim\limits_{n\to\infty} f(x_n) = f(x)$ *with respect to* $\tau(\Phi)$*.*

(v) X *is said to be* $p-$*bounded if* $\delta_p(X) = \sup\ \{\ \mathrm{p(x, y)} \mid \mathrm{x},\ \mathrm{y} \in \mathrm{X}\} < \infty$*.*

Definition 6. *Let* (X, Φ) *be a Hausdorff uniform space and* p *an* $A-$*distance on* X*. Two selfmappings* f *and* g *on* X *are said to be* $p-$*compatible if, for each sequence* $\{x_n\}_{n=0}^{\infty}$ *of* X *such that* $\lim\limits_{n\to\infty} p(f(x_n), u) = \lim\limits_{n\to\infty} p(g(x_n), u) = 0$ *for some* $u \in X$*, then we have* $\lim\limits_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$*.*

See [1] for Remark 1 as well as the Definitions 1-6. We shall also state the following definition of a comparison function which is mentioned in the contractive condition (2).

Definition 7. *A function* $\psi : I\!R^+ \to I\!R^+$ *is called a comparison function if:*

(i) ψ *is monotone increasing;*

(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0, \ \forall\ t \geq 0.$

Definition 7 is contained in Berinde [2, 3].

Remark 2. Every comparison function satisfies the condition $\psi(0) = 0$. Also, both conditions (i) and (ii) imply that $\psi(t) < t, \ \forall\ t > 0$. We state the

following example of a comparison function:
The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\psi(t) \;=\; \begin{cases} 0\,, & \text{if } \; t \in [0,1), \\ \frac{1}{4}\,t\,, & \text{if } \; t \geq 1, \end{cases}$$

is a comparison function.

In this paper, we shall employ the following contractive definition:
Let $f,\ g : X \to X$ be selfmappings of X. There exist a constant $k \in [0,1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, such that $\forall\ x,\ y \in X$, we have

$$p(f(x), f(y)) \;\leq\; \varphi(p(x, g(x))) + kp(g(x), g(y)). \tag{3}$$

Remark 3. The contractive condition (3) is independent of (1) and (2) since the right-hand side expressions of (1) and (2) cannot be obtained from that of (3) or vice-versa as shown below:
Thus, for instance, if in condition (3), $\varphi(u) = 0,\ \forall\, u \in \mathbb{R}^+$, then (3) reduces to

$$p(f(x), f(y)) \;\leq\; kp(g(x), g(y)),\ \ k \in [0,1),\ \ \forall\ x,\ y \in X, \tag{$\star$}$$

where $(\star)$ is a special case of (1), (2) & (3). Indeed, by putting $\psi(r) = kr$, $\forall\ r \in \mathbb{R}^+$, $k \in [0,1)$, then we get $(\star)$ from (1).
Also, if in condition (3), we have $\varphi(u) = Lu,\ L \geq 0,\ \forall\ u \in \mathbb{R}^+$, then (3) also reduces to

$$p(f(x), f(y)) \;\leq\; Lp(x, g(x)) + kp(g(x), g(y)),\ \ k \in [0,1),\ \forall\ \ x, y \in X, \tag{$\star\star$}$$

and $(\star\star)$ is a special case of (2) & (3). We note from this remark that it is not possible to obtain either condition (1) or (2) from (3). Rather, we can only both $(\star)$ and $(\star\star)$ from (3).

Lemma 1. *Let (X, Φ) be a Hausdorff uniform space and p be an $A-$distance on X. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following hold:*

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$, $\forall\ n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$, $\forall\, n \in \mathbb{N}$, then $\{y_n\}_{n=0}^{\infty}$ converges to z.

(c) If $p(x_n, x_m) \leq \alpha_n\ \forall\ m > n$, then $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, Φ).

Remark 4. A sequence in X is $p-$Cauchy if it satisfies the usual metric condition.

3. The main results

The main results of this paper are the following:

Theorem 1. *Let (X,Φ) be a Hausdorff uniform space and p an $A-$distance on X. Suppose that X is $p-$bounded and $S-$complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), \quad n = 1, 2, ...,$$

with $x_0 \in X$. Let f and g be commuting $p-$continuous or $\tau(\Phi)-$continuous selfmappings of X such that

(i) $f(X) \subseteq g(X)$;

(ii) $p(f(x_i), f(x_i)) = 0,\ \forall\ x_i \in X,\ \ i = 0, 1, 2, ...$;

(iii) $f, g : X \to X$ satisfy the contractive condition (3).

Suppose also that $\varphi : I\!R^+ \to I\!R^+$ a monotone increasing function such that $\varphi(0) = 0$. Then, f and g have a common fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$, choose $x_1 \in X$ such that $f(x_1) = g(x_2)$, and in general, choose $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$. We recall that $x_n = f(x_{n-1})$, $n = 1, 2, ...$, so that by conditions (ii) and (iii) of the Theorem, we obtain

$$\begin{aligned}
p(f(x_n), f(x_{n+m})) &\le \varphi(p(x_n, g(x_n))) + kp(g(x_n), g(x_{n+m}))\\
&= \varphi(p(f(x_{n-1}), f(x_{n-1}))) + kp(f(x_{n-1}), f(x_{n+m-1}))\\
&= kp(f(x_{n-1}), f(x_{n+m-1}))\\
&\le k[\varphi(p(x_{n-1}, g(x_{n-1}))) + kp(g(x_{n-1}), g(x_{n+m-1}))]\\
&= k[\varphi(p(f(x_{n-2}), f(x_{n-2}))) + kp(f(x_{n-2}), f(x_{n+m-2})]\\
&= k^2(p(f(x_{n-2}), f(x_{n+m-2})))\\
&\le \ldots \ \le\ k^n(p(f(x_0), f(x_m))\ \le\ k^n\delta_p(X),
\end{aligned}$$

from which we have that

$$p(f(x_n), f(x_{n+m}))\ \le\ k^n\delta_p(X), \tag{4}$$

where $p(f(x_0), f(x_m)) \le \delta_p(X)$ and $\delta_p(X) = \sup\{p(x,y)|x,y \in X\} < \infty$. $\delta_p(X) < \infty$ since p is nonnegative with its range contained in $\mathbb{R}^+$. Therefore, using the fact that $k \in [0,1)$ in (4) yields $k^n\delta_p(X) \to 0$ as $n \to \infty$, from which it follows that

$$p(f(x_n), f(x_{n+m})) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, by applying Lemma 1(c), we have that $\{f(x_n)\}_{n=0}^{\infty}$ is a p-Cauchy sequence. Since X is $S-$complete, $\lim_{n\to\infty} p(f(x_n), u)) = 0$, for some $u \in X$, and therefore $\lim_{n\to\infty} p(g(x_n), u)) = 0$.

Since f and g are $p-$continuous, then $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0$. Also, since f and g are commuting, then $fg = gf$, so that we have $\lim\limits_{n\to\infty} p(f(g(x_n)), f(u)) = \lim\limits_{n\to\infty} p(f(g(x_n)), g(u)) = 0$, so that by Lemma 2.11(a), we obtain that $f(u) = g(u)$. Since $f(u) = g(u)$, $fg = gf$, we have $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u), f(f(u))) \neq 0$. Using (3), then we have that

$$\begin{aligned} p(f(u), f(f(u))) &\le \varphi(p(u, g(u))) + kp(g(u), g(f(u))) \\ &= \varphi(p(f(u), f(u))) + kp(f(u), f(f(u))) \\ &= kp(f(u), f(f(u))), \end{aligned}$$

from which

$$(1-k)p(f(u), f(f(u)) \le 0.$$

Since $k \in [0,1)$, then $1-k > 0$ and we have $p(f(u), f(f(u))) \le 0$, which is a contradiction since p is nonnegative. Therefore, $p(f(u), f(f(u))) = 0$. Since condition (ii) of the Theorem implies that $p(f(u), f(u)) = 0$, then $p(f(u), f(f(u))) = 0$ and $p(f(u), f(u)) = 0$, yield by Lemma 2.11(a) that $f(f(u)) = f(u)$. Thus, we have $g(f(u)) = f(f(u)) = f(u)$. Hence, $f(u)$ is a common fixed point of f and g.
The proof is similar when f and g are $\tau(\Phi)-$continuous as $S-$completeness implies $p-$Cauchy completeness. ■

Remark 5. Theorem 1 is independent of Theorem 3.1 of Aamri and El Moutawakil [1] as well as Theorem 3.1 of Olatinwo [8].

Theorem 1 is an existence result for the common fixed point of f and g, while the next two results guarantee the uniqueness of the common fixed point.

Theorem 2. *Let (X, Φ) be a Hausdorff uniform space and p an $E-$distance on X. Suppose that X is $p-$bounded and $S-$complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), \quad n = 1, 2, ...,$$

with $x_0 \in X$. Let f and g be commuting $p-$continuous or $\tau(\Phi)-$continuous selfmappings of X such that

(i) $f(X) \subseteq g(X)$;

(ii) $p(f(x_i), f(x_i)) = 0, \ \forall \ x_i \in X, \ i = 0, 1, 2, ...$;

(iii) $f, g : X \to X$ satisfy the contractive condition (3).

Suppose also that $\varphi : I\!R^+ \to I\!R^+$ *is a monotone increasing function such that* $\varphi(0) = 0$. *Then,* f *and* g *have a unique common fixed point.*

Proof. f and g have a common fixed point since an $E-$distance function p is an $A-$distance. Suppose that there exist $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$.

Let $p(u, v) \neq 0$. Then, we have

$$\begin{aligned} p(u,v) &= p(f(u), f(v)) \leq \varphi(p(u, g(u))) + kp(g(u), g(v)) \\ &= \varphi(p(u,u)) + kp(u,v) = kp(u,v), \end{aligned}$$

from which we have that $(1 - k)p(u, v) \leq 0$, leading to $p(u, v) \leq 0$, since $1 - k > 0$. This is a contradiction since p is nonnegative. Hence, we have $p(u, v) = 0$. In a similar manner, we also have that $p(v, u) = 0$. Using condition (p_2) of Definition 3, we have $p(u, u) \leq p(u, v) + p(v, u)$, from which it follows that $p(u, u) = 0$. Since $p(u, u) = 0$ and $p(u, v) = 0$, then by Lemma 1(a), we have that $u = v$. ■

Remark 6. Theorem 2 is independent of Theorem 3.2 as well as corollaries 3.1 & 3.2 of Aamri and El Moutawakil [1] and also independent of Theorem 3.3 of Olatinwo [8].

Theorem 3. *Let* (X, Φ) *be a Hausdorff uniform space and* p *an* $E-$*distance on* X. *Suppose that* X *is* $p-$*bounded and* $S-$*complete. Suppose that the sequence* $\{x_n\}_{n=0}^{\infty}$ *is defined by*

$$x_n = f(x_{n-1}), \quad n = 1, 2, ...,$$

with $x_0 \in X$. *Let* f *and* g *be* $p-$*compatible,* $p-$*continuous or* $\tau(\Phi)-$*continuous selfmappings of* X *such that*

(*i*) $f(X) \subseteq g(X)$;

(*ii*) $p(f(x_i), f(x_i)) = 0, \ \forall \ x_i \in X, \ i = 0, 1, 2, ...$;

(*iii*) $f, g : X \to X$ *satisfy the contractive condition (3).*

Suppose also that $\varphi : I\!R^+ \to I\!R^+$ *is a monotone increasing function such that* $\varphi(0) = 0$. *Then,* f *and* g *have a unique common fixed point.*

Proof. Just as in the proof of Theorem 1, we have for some $u \in X$, $\lim_{n\to\infty} p(f(x_n, u)) = \lim_{n\to\infty} p(g(x_n, u)) = 0$. Since f and g are $p-$continuous, we have $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0$, while the assumption that f and g are $p-$compatible implies $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$. Furthermore, by condition (p_2) of Definition 3, we have that

$$(5) \qquad p(f(g(x_n)), g(u)) \leq p(f(g(x_n)), g(f(x_n))) + p(g(f(x_n)), g(u)).$$

Taking limits in (4) and applying Lemma 1(a), then we have $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$. Since $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$, then by Lemma 1(a) we have $f(u) = g(u)$. The rest of the proof is as in Theorem 2. ■

Remark 7. Theorem 3 is independent of Theorem 3.3 of Aamri and El Moutawakil [1] as well as Theorem 3.5 of Olatinwo [8].

Remark 8. The results established in this paper can have applications in mathematical economics. Specifically, these results can find application in the study of demand and supply in relation to the determination of the market equilibrium point. Some applications in the areas of both engineering and science are also possible.

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