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B.G. Pachpatte

NEW INEQUALITIES OF OSTROWSKI-GRÜSS TYPE

Abstract. The main aim of the present paper is to establish new inequalities of Ostrowski-Grüss type, involving two functions and their derivatives via certain integral identity.

Key words: inequalities, Ostrowski-Grüss type, integral identity, Korkine's identity, midpoint inequalities.

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1. Introduction

In 1938 A. Ostrowski proved the following inequality (see [5, p.468]).

Theorem A. *Let $f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable on (a,b), and whose derivative $f' : (a,b) \to R$ is bounded on (a,b), i.e. $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1}$$

for all $x \in [a,b]$.

Another well known inequality proved by G.Grüss in 1935 is given in the following theorem (see [6,p.296]).

Theorem B. *Let $f, g : [a,b] \to R$ be two integrable functions such that $\gamma \le f(x) \le \Gamma$, $\phi \le g(x) \le \Phi$ for all $x \in [a,b]$, $\gamma, \Gamma, \phi, \Phi \in R$ are constants. If*

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right),$$

then

$$|T(f,g)| \le \frac{1}{4}(\Gamma - \gamma)(\Phi - \phi), \tag{2}$$

In 1997 S.S. Dragomir and S. Wang [3] proved the following Ostrowski-Grüss type inequality.

Theorem C. *Let $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b). If f' is integrable and $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$; $\gamma, \Gamma \in R$ are constants, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma), \tag{3}$$

for all $x \in [a, b]$.

In the past few years, a number of authors have written about generalizations, extensions, improvements and variants of the above inequalities, see [1-12] and the references cited therein. Motivated by the recent result given in [1], in the present paper, we establish new inequalities similar to Ostrowski-Grüss type inequalities involving two functions and their derivatives. The analysis used in the proofs is based on the integral identity proved in [1] and in the special case we recapture the main result given in [1].

2. Statement of results

In what follows R and $'$ denote respectively the set of real numbers and the derivative of a function. For suitable functions $f, g : [a, b] \to R$, we use the following notations to simplify the details of presentation:

$$S(f, g) = f(x)g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(t)dt + f(x) \int_a^b g(t)dt \right] - \frac{1}{2} \left(x - \frac{a+b}{2} \right) [Fg(x) + Gf(x)],$$

$$H(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) - \frac{1}{2(b-a)} \int_a^b \left(x - \frac{a+b}{2} \right) [Fg(x) + Gf(x)] dx,$$

in which

$$F = \frac{f(b) - f(a)}{b-a}, \qquad G = \frac{g(b) - g(a)}{b-a}.$$

We are now in a position to state our results to be proved in this paper.

Theorem 1. *Let* $f, g : [a,b] \to R$ *be absolutely continuous functions whose derivatives* $f', g' \in L_2[a,b]$. *Then*

$$|S(f,g)| \le \frac{b-a}{4\sqrt{3}} \left[|g(x)| \left(\frac{1}{b-a} \|f'\|_2^2 - F^2 \right)^{\frac{1}{2}} + |f(x)| \left(\frac{1}{b-a} \|g'\|_2^2 - G^2 \right)^{\frac{1}{2}} \right], \tag{4}$$

for all $x \in [a,b]$ *and*

$$|H(f,g)| \le \frac{1}{4\sqrt{3}} \int_a^b \left[|g(x)| \left(\frac{1}{b-a} \|f'\|_2^2 - F^2 \right)^{\frac{1}{2}} + |f(x)| \left(\frac{1}{b-a} \|g'\|_2^2 - G^2 \right)^{\frac{1}{2}} \right] dx. \tag{5}$$

Under the additional assumptions on the derivatives of the functions, the following theorem holds.

Theorem 2. *Let the assumptions of Theorem 1 hold. If* $\gamma \le f'(x) \le \Gamma$, $\phi \le g'(x) \le \Phi$ *for* $x \in [a,b]$, *where* $\gamma, \Gamma, \phi, \Phi$ *are real constants. Then*

$$|S(f,g)| \le \frac{b-a}{8\sqrt{3}} \left[|g(x)| (\Gamma - \gamma) + |f(x)| (\Phi - \phi) \right], \tag{6}$$

for all $x \in [a,b]$ *and*

$$|H(f,g)| \le \frac{1}{8\sqrt{3}} \int_a^b \left[|g(x)| (\Gamma - \gamma) + |f(x)| (\Phi - \phi) \right] dx. \tag{7}$$

Remark 1. If we take $g(x) = 1$ and hence $g'(x) = 0$ in (4) and (6), then by simple computation we get the inequality established by Barnett, Dragomir and Sofo in [1, Theorem 2.1, p. 114].

3. Proofs of Theorems 1 and 2

Define a function

$$p(x,t) = \begin{cases} t - a, & \text{if } t \in [a,x], \\ t - b, & \text{if } t \in (x,b]. \end{cases}$$

By using the well known Korkine's identity (see [6]) for mappings $g, h : [a, b] \to R$:

$$T(g,h) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (g(t)-g(s))(h(t)-h(s))dtds,$$

which can be easily proved by direct computation, we obtain

$$\frac{1}{b-a}\int_a^b p(x,t)f'(t)dt - \left(\frac{1}{b-a}\int_a^b p(x,t)dt\right)\left(\frac{1}{b-a}\int_a^b f'(t)dt\right) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(f'(t)-f'(s))dtds. \tag{8}$$

By simple computation we obtain

$$\frac{1}{b-a}\int_a^b p(x,t)f'(t)dt = f(x) - \frac{1}{b-a}\int_a^b f(t)dt,$$

$$\frac{1}{b-a}\int_a^b p(x,t)dt = x - \frac{a+b}{2},$$

and

$$\frac{1}{b-a}\int_a^b f'(t)dt = \frac{f(b)-f(a)}{b-a} = F.$$

Using these facts in (8) we get (see [1, p. 115]):

$$f(x) - \frac{1}{b-a}\int_a^b f(t)dt - F\left(x - \frac{a+b}{2}\right) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(f'(t)-f'(s))dtds, \tag{9}$$

for all $x \in [a, b]$. Similarly, we get

$$g(x) - \frac{1}{b-a}\int_a^b g(t)dt - G\left(x - \frac{a+b}{2}\right) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(g'(t)-g'(s))dtds. \tag{10}$$

Multiplying both sides of (9) and (10) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting we get

$$S(f,g)=\frac{1}{2}\left[g(x)\frac{1}{2(b-a)^2}\int_a^b\int_a^b(p(x,t)-p(x,s))(f'(t)-f'(s))dtds\right.$$
$$\left.+f(x)\frac{1}{2(b-a)^2}\int_a^b\int_a^b(p(x,t)-p(x,s))(g'(t)-g'(s))dtds\right]. \tag{11}$$

From (11) and using the properties of modulus we get

$$|S(f,g)|\le\frac{1}{2}\left[|g(x)|\frac{1}{2(b-a)^2}\int_a^b\int_a^b|p(x,t)-p(x,s)|\left|f'(t)-f'(s)\right|dtds\right.$$
$$\left.+|f(x)|\frac{1}{2(b-a)^2}\int_a^b\int_a^b|p(x,t)-p(x,s)|\left|g'(t)-g'(s)\right|dtds\right]. \tag{12}$$

By using the Cauchy-Schwarz inequality for double integrals we observe that

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b|p(x,t)-p(x,s)|\left|f'(t)-f'(s)\right|dtds \tag{13}$$
$$\le\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b(p(x,t)-p(x,s))^2dtds\right)^{\frac{1}{2}}$$
$$\times\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b(f'(t)-f'(s))^2dtds\right)^{\frac{1}{2}}.$$

It is easy to observe that

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b(p(x,t)-p(x,s))^2dtds \tag{14}$$
$$=\frac{1}{b-a}\int_a^b p^2(x,t)dt-\left(\frac{1}{b-a}\int_a^b p(x,t)dt\right)^2$$
$$=\frac{1}{b-a}\left[\int_a^x(t-a)^2dt+\int_x^b(b-t)^2dt\right]-\left(x-\frac{a+b}{2}\right)^2$$
$$=\frac{1}{b-a}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]-\left(x-\frac{a+b}{2}\right)^2=\frac{(b-a)^2}{12},$$

and

$$\text{(15)}\quad \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2 dtds$$
$$= \frac{1}{b-a}\left\|f'\right\|_2^2 - \left(\frac{f(b)-f(a)}{b-a}\right)^2 = \frac{1}{b-a}\left\|f'\right\|_2^2 - F^2.$$

Using (14) and (15) in (13) we get

$$\text{(16)}\quad \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left|p(x,t)-p(x,s)\right|\left|f'(t)-f'(s)\right| dtds$$
$$\leq \frac{b-a}{2\sqrt{3}}\left(\frac{1}{b-a}\left\|f'\right\|_2^2 - F^2\right)^{\frac{1}{2}}.$$

Similarly, we get

$$\text{(17)}\quad \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left|p(x,t)-p(x,s)\right|\left|g'(t)-g'(s)\right| dtds$$
$$\leq \frac{b-a}{2\sqrt{3}}\left(\frac{1}{b-a}\left\|g'\right\|_2^2 - G^2\right)^{\frac{1}{2}}.$$

Using (16) and (17) in (12) we get the desired inequality in (4).

Integrating both sides of (11) with respect to x over $[a,b]$ and dividing throughout by $(b-a)$ we get

$$\text{(18)}\quad H(f,g) = \frac{1}{2(b-a)}\int_a^b \left[\frac{g(x)}{2(b-a)^2}\right.$$
$$\times \int_a^b\int_a^b (p(x,t)-p(x,s))(f'(t)-f'(s))dtds$$
$$\left. + \frac{f(x)}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(g'(t)-g'(s))dtds\right] dx.$$

From (18) and using the properties of modulus we have

$$
\begin{aligned}
(19)\quad |H(f,g)| \le \frac{1}{2(b-a)} \int_a^b \Bigg[& \frac{|g(x)|}{2(b-a)^2} \\
& \times \int_a^b \int_a^b |p(x,t)-p(x,s)| \left| f'(t)-f'(s) \right| dtds \\
& + \frac{|f(x)|}{2(b-a)^2} \int_a^b \int_a^b |p(x,t)-p(x,s)| \left| g'(t)-g'(s) \right| dtds \Bigg] dx.
\end{aligned}
$$

Using (16) and (17) in (19) we get the required inequality in (5). This completes the proof of Theorem 1.

From the hypotheses of Theorem 2 and using the Grüss inequality in Theorem B, it is easy to observe that

$$0 \;\le\; \frac{1}{b-a} \int_a^b (f'(t))^2 dt - \left(\frac{1}{b-a} \int_a^b f'(t)dt \right)^2 \;\le\; \frac{1}{4}(\Gamma-\gamma)^2,$$

i.e.

$$0 \;\le\; \frac{1}{b-a} \left\| f' \right\|_2^2 - F^2 \;\le\; \frac{1}{4}(\Gamma-\gamma)^2. \tag{20}$$

Similarly, we obtain

$$0 \;\le\; \frac{1}{b-a} \left\| g' \right\|_2^2 - G^2 \;\le\; \frac{1}{4}(\Phi-\phi)^2. \tag{21}$$

Using (20), (21) in (4) and (5) we get the required inequalities in (6) and (7). This completes the proof of Theorem 2.

Remark 2. If we take $x = \frac{a+b}{2}$ in (4) and (6), then we get the corresponding midpoint inequalities. For similar results, see [1, 4, 11, 12]. We believe that the Grüss type inequalities established in (5) and (7) are new to the literature.

References

[1] Barnett N.S., Dragomir S.S., Sofo A., Better bounds for an inequality of the Ostrowski type with applications, *RGMIA Res.Rep.Coll.,* 3(1)(2000), 113-122.

[2] Cheng X.L., Improvement of some Ostrowski-Grüss type inequalities, *Computers Math. Applic.,* 42(1/2)(2001), 109-114.

[3] Dragomir S.S., Wang S., An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.,* 33(11)(1997), 16-20.

[4] Matić M., Pečarić J., Ujević N., Improvement and further generalization of inequalities of Ostrowski-Grüss type, *Computers Math. Applic.,* 39(2000), 161-175.

[5] Mitrinović D.S., Pečarić J.E., Fink A.M., *Inequalities Involving Functions and Their Integrals and Derivatives,* Kluwer Academic Publisher, Dordrecht, 1991.

[6] Mitrinović D.S., Pečarić J.E., Fink A.M., *Classical and new Inequalities in Analysis,* Kluwer Academic Publisher, Dordrecht, 1993.

[7] Pachpatte B.G., On Trapezoid and Grüss like integral inequalities, *Tamkang J. Math.,* 34(4)(2003), 365-369.

[8] Pachpatte B.G., On Grüss like integral inequalities via Pompeiu's mean value theorem, *J. Inequal. Pure and Appl. Math.,* 6(3) Art.82, 2005.

[9] Pachpatte B.G., A note on Ostrowski like inequalities, *J. Inequal. Pure and Appl. Math.,* 6(4) Art.114, 2005.

[10] Pachpatte B.G., On Ostrowski-Grüss-Čebšev type inequalities for functions whose modulus of derivatives are convex, *J. Inequal. Pure and Appl. Math.,* 6(4) Art.128, 2005.

[11] Pearce C.E.M., Pečarić J., Ujević N., Varošanec S., Generalizations of some inequalities of Ostrowski-Grüss type, *Math. Inequal. Appl.,* 3(1)(2000), 25 -34.

[12] Ujević N., Better bounds for an inequality of the Ostrowski type with applications, *Computers Math.Applic.,* 46(2003), 421-427.

B.G. Pachpatte
57 Shri Niketan Colony, Near Abhinay Talkies
Aurangabad 431 001 (Maharashtra) India
e-mail: bgpachpatte@gmail.com

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