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COMMON FIXED POINT RESULTS WITH APPLICATIONS IN CONVEX METRIC SPACE

Abstract. Sufficient conditions for the existence of a common fixed point for uniformly C_q- commuting mappings satisfying a generalized contractive conditions in the framework of a convex metric space are obtained. As an application, related results on best approximation are derived. Our results generalize various known results in the literature.

Key words: Convex metric space, common fixed point, uniformly C_q- commuting mapping, asymptotically $S-$ nonexpansive mapping, best approximation.

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1. Introduction and preliminaries

Metric fixed point theory is a branch of fixed point theory which finds its primary applications in functional analysis. The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric conditions on mappings and/or underlying spaces play a crucial role in metric fixed point problems. Although it has a purely metric flavor, it is also a major branch of nonlinear functional analysis with close ties to Banach space geometry, see for example [10] and references mentioned therein. Several results concerning the existence and approximation of a fixed point of a mapping rely on convexity hypotheses and geometric properties of the Banach spaces. Takahashi [21] introduced the notion of a convexity on metric spaces. Afterwards, Beg and Abbas ([4], [5]), Ciric [8], Ding [9], Guay, Singh and Whitfield [11] and other authors have studied fixed point theorems in convex metric spaces (see also [6], [19]). On the other hand, Shahzad [18] introduced a class of noncommuting mappings called $R-$ subweakly commuting mappings, and applied it to $S-$ nonexpansive mappings in normed spaces. In this paper, common fixed points, for C_q- commuting maps which are more general than weakly compatible maps, are obtained in the setting of a convex metric space. Applying uniformly C_q-commuting mappings to asymptotically $S-$nonexpansive mappings, common fixed point theorems are proved. As an application, invariant approximation results for these mappings are also derived.

For the sake of convenience, we gather some basic definitions and set out the terminology needed in the sequel.

Definition 1. *Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X, if, for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space X together with a convex structure W is called a convex metric space. Obviously, $W(x, x, \lambda) = x$.

Let X be a convex metric space. A nonempty subset E of X is said to be *convex* if, $W(x, y, \lambda) \in E$ whenever $(x, y, \lambda) \in E \times E \times [0, 1]$. A subset E of a convex metric space is said to be $q-$*starshaped* or *starshaped with respect to* q, if there exist q in E such that $W(x, q, \lambda) \in E$, whenever $(x, \lambda) \in E \times [0, 1]$. Obviously q- starshaped subsets of X contain all convex subsets of X as a proper subclass. Takahashi [21] has shown that open spheres $B(x, r) = \{y \in X : d(y, x) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(y, x) \leq r\}$ are convex in a convex metric space X. A convex metric space X is said to have property (A) if: $d(W(y, x, \lambda), W(z, x, \lambda)) \leq \lambda d(y, z)$, for all $x, y, z \in X$ and $\lambda \in (0, 1)$. Property (A) is a convex metric space analogue of condition (I) for the starshaped metric spaces of Guay, Singh and Whitfield, see, Definition 3.2 [11]. Throughout this paper, a convex metric space X is assumed to have a property (A).

Also note that every normed space is a convex metric space. There are many examples of convex metric spaces which cannot be embedded in any normed space [21].

Example 1. Let $X = \{(x_1, x_2, x_3) \in R^3 : x_1, x_2, x_3 > 0\}$. For $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ in X, and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, define a mapping $W : X^3 \times [0, 1]^3 \to X$ by

$$W(x, y, z, \alpha, \beta, \gamma) = (\alpha x_1 + \beta x_2 + \gamma x_3, \alpha y_1 + \beta y_2 + \gamma y_3, \alpha z_1 + \beta z_2 + \gamma z_3),$$

and a metric $d : X \times X \to [0, \infty)$ by, $d(x, y) = |x_1y_1 + x_2y_2 + x_3y_3|$. Here X is a convex metric space but it is not a normed space.

Example 2. Let $X = \{(x_1, x_2) \in R^2 : x_1, x_2 > 0\}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X and $\alpha \in [0, 1]$. Define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \alpha) = \left(\alpha x_1 + (1 - \alpha)y_1, \frac{\alpha x_1x_2 + (1 - \alpha)y_1y_2}{\alpha x_1 + (1 - \alpha)y_1}\right),$$

and a metric $d : X \times X \to [0, \infty)$ by $d(x, y) = |x_1 - y_1| + |x_1x_2 - y_1y_2|$. It can be verified that X is a convex metric space but not a normed space.

Definition 2. *Let $T, S : X \to X$. A point $x \in X$ is called:*

(1) a fixed point of T if $T(x) = x$;

(2) a coincidence point of the pair $\{T, S\}$ if $Tx = Sx$;

(3) a common fixed point of the pair $\{T, S\}$ if $x = Tx = Sx$.

$F(T), C(T, S)$ and $F(T, S)$ denote set of all fixed points of T, the set of all coincidence points of the pair $\{T, S\}$, and the set of all common fixed points of the pair $\{T, S\}$, respectively.

Definition 3. *Let E be a $q-$starshaped subset of a convex metric space X, $q \in F(S)$, with E is both T and S invariant where, $T, S : X \to X$. Put*

$$Y_q^{Tx} = \{y_\lambda : y_\lambda = W(Tx, q, \lambda) \text{ and } \lambda \in [0, 1]\},$$

and, for each x in X, $d(Sx, Y_q^{Tx}) = \inf\{d(Sx, y_\lambda) : \lambda \in [0, 1]\}$. The map T is said to be:

(1) an $S-$ contraction if there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(Sx, Sy);$$

(2) asymptotically $S-$nonexpansive if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$ such that $d(T^nx, T^ny) \leq k_n\ d(Sx, Sy)$, for each x, y in E and $n \in N$. If $k_n = 1$, for all $n \in N$, then T is called an $S-$ nonexpansive mapping. If $S = I$ (the identity map), then T is an asymptotically nonexpansive mapping;

(3) $R-$weakly commuting if there exists a real number $R > 0$ such that

$$d(STx, TSx) \leq Rd(Tx, Sx)$$

for all x in E;

(4) $R-$subweakly commuting if there exists a real number $R > 0$ such that

$$d(TSx, STx) \leq Rd(Sx, Y_q^{Tx});$$

for all $x \in E$;

(5) uniformly R-subweakly commuting if there exists a real number $R > 0$ such that

$$d(T^nSx, ST^nx) \leq Rd(Sx, Y_q^{T^nx});$$

for all $x \in E$.

(6) C_q- commuting if $STx = TSx$ for all $x \in C_q(S,T)$, where $C_q(S,T) = U\{C(S,T_k) : 0 \leq k \leq 1\}$, and $T_k x = W(Tx, q, k)$.
Clearly C_q- commuting maps are weakly compatible but converse is not true in general (see for example [2]).

A self mapping T on a convex metric space X is said to be

(7) affine on E if

$$T(W(x, y, \lambda)) = W(Tx, Ty, \lambda),$$

for all $x, y \in E$ and $\lambda \in (0, 1)$;

(8) uniformly asymptotically regular on E if, for each $\varepsilon > 0$, there exists a positive integer N such that $d(T^n x, T^n y) < \varepsilon$ for all $n \geq N$ and for all x in E.

Definition 4. *Let E be a q-starshaped subset of a convex metric space X, and $T, S : E \to E$ be maps with $q \in F(S)$. Then T and S are said to be uniformly C_q-commuting on E if $ST^n x = T^n Sx$ for all $x \in C_q(S,T)$ and $n \in N$.*

Clearly, uniformly C_q-commuting maps on E are C_q-commuting but not conversely in general, as the following example shows.

Example 3. Let X be set of all real numbers with usual metric and $E = [1, \infty)$. Define, $Tx = 2x - 1$ and $Sx = x^2$, for all $x \in E$. Take, $q = 1$. Then E is $q-$ starshaped with $Sq = q$ and $C_q(S,T) = \{1\}$. Note that S and T are C_q-commuting maps but not uniformly C_q-commuting, because $ST^2 1 \neq T^2 S1$.

Uniformly $R-$ subweakly commuting maps are uniformly C_q -commuting but the converse does not hold in general, for this, we consider a following example.

Example 4. Let X be set of all real numbers with usual metric, and $E = [0, \infty)$. If,

$$Sx = \begin{cases} \frac{x}{2} & \text{if} \quad 0 \leq x < 1, \\ x & \text{if} \quad x \geq 1 \end{cases}$$

and

$$Tx = \begin{cases} \frac{1}{2} & \text{if} \quad 0 \leq x < 1, \\ x^2 & \text{if} \quad x \geq 1, \end{cases}$$

then E is $1-$ starshaped with $S1 = 1$ and $C_q(S,T) = [1, \infty]$. Note that S and T are uniformly C_q- commuting but not $R-$weakly commuting for all $R > 0$. Thus S and T are neither $R-$ subweakly commuting nor uniformly $R-$ subweakly commuting maps.

2. Common fixed point results

In this section, the existence of common fixed points of uniformly C_q- commuting, C_q- commuting, and uniformly $R-$ subweakly commuting mappings is established in a convex metric space.

Theorem 1. *Let E be a nonempty $q-$ starshaped complete subset of convex metric space, and T, f and g be self mappings on X. Suppose $q \in F(f) \cap F(g)$, T is continuous, f and g are continuous and affine on E, $cl(T(E))$ is compact and $T(E) \subset f(E) = g(E)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are C_q- commuting and satisfy, for all $x, y \in E$,*

$$(1) \qquad d(Tx,Ty) \leq \max\{d(fx,gy), d(fx,Y_q^{Tx}), d(gy,Y_q^{Ty}), \frac{1}{2}[d(fx,Y_q^{Ty}) + d(gy,Y_q^{Tx})]\},$$

then T, f and g have a common fixed point in E.

Proof. Define $T_n : E \to E$ by

$$T_n x = W(Tx, q, \lambda_n),$$

where $\lambda_n \in (0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Since E is $q-$ starshaped, T_n is the self mapping on E for each $n \geq 1$. As f and T are C_q- commuting and f is affine on E with $fq = q$, if, $x \in C(f, T_n) \subset C_q(f, T)$, then

$$fT_n x = f(W(Tx, q, \lambda_n)) = W(fTx, q, \lambda_n) = W(Tfx, q, \lambda_n) = T_n fx.$$

Thus f and T_n are weakly compatible for all n. Also since g and T are C_q- commuting and g is affine on E with $gq = q$, g and T_n are weakly compatible for all n. Also,

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx,q,\lambda_n), W(Ty,q,\lambda_n)) \leq \lambda_n d(Tx,Ty) \\ &\leq \lambda_n \max\{d(fx,gy), d(fx,Y_q^{Tx}), d(gy,Y_q^{Ty}), \\ &\qquad \frac{1}{2}[d(fx,Y_q^{Ty}) + d(gy,Y_q^{Tx})]\} \\ &\leq \lambda_n \max\{d(fx,gy), d(fx,T_n x), d(gy,T_n y), \\ &\qquad \frac{1}{2}[d(fx,T_n y) + d(gy,T_n x)]\}. \end{aligned}$$

By Corollary 3.1 of [7], for each $n \geq 1$, there exist x_n in E such that x_n is a common fixed point of f, g, and T_n. The compactness of $cl(T(E))$ implies that there exists a subsequence $\{Tx_k\}$ of $\{Tx_n\}$ such that $Tx_k \to y$ as $k \to \infty$. The definitions of $T_k x_k$ and convexity structure on X give $x_k \to y$. From the continuity of T, f and g, we have $y \in F(T) \cap F(f) \cap F(g)$. ■

Corollary 1. *Let E be a nonempty $q-$ star shaped complete subset of a convex metric space X, and T, f and g be self mappings on X. Suppose that $q \in F(f) \cap F(g)$, T is continuous, f and g are continuous and affine on E, $cl(T(E))$ is compact and $T(E) \subset f(E) = g(E)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are $R-$ subweakly commuting mappings satisfying (1), then T, f and g have a common fixed point in E.*

Corollary 2. *Let E be a nonempty closed $q-$ star shaped subset of convex metric space X, and T and S be $R-$ subweakly commuting mappings on E such that $T(E) \subset S(E)$, $cl(T(E))$ is compact where $q \in F(S)$. If T is continuous $S-$ nonexpansive and S is affine on E, then $F(T) \cap F(S)$ is nonempty.*

Theorem 2 ([15])**.** *Let E be a subset of a metric space (X, d), and S and T be weakly compatible self-maps of E. Assume that $clT(E) \subset S(E)$, $clT(E)$ is complete, and T and S satisfy, for all $x, y \in E$ and $0 \leq h < 1$,*

$$\begin{aligned} d(Tx, Ty) \leq\ & h \max \{d(Sx, Sy), d(Sx, Tx), \\ & d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}. \end{aligned} \tag{2}$$

Then $E \cap F(S) \cap F(T)$ is a singleton.

Theorem 3. *Let E be a nonempty closed $q-$starshaped subset of a convex complete metric space X, and T and S be uniformly C_q- commuting mappings on $E - \{q\}$ such that $S(E) = E$ and $T(E - \{q\}) \subset S(E - \{q\})$, where $q \in F(S)$. Suppose that T is continuous asymptotically $S-$ nonexpansive with sequence $\{k_n\}$ and S is affine on E. For each $n \geq 1$, define a mapping T_n on E by $T_n x = W(T^n x, q, \alpha_n)$, where $\alpha_n = \dfrac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\lim_{n \to \infty} \lambda_n = 1$. Then for each $n \in N$, $F(T_n) \cap F(S)$ is a singleton.*

Proof. For all $x, y \in E$, we have

$$\begin{aligned} d(T_n(x), T_n(y)) &= d(W(T^n x, q, \alpha_n), W(T^n y, q, \alpha_n)) \\ &\leq \alpha_n d(T^n x, T^n y) \leq \lambda_n d(Sx, Sy). \end{aligned}$$

Moreover, as T and S are uniformly C_q- commuting and S is affine on E with $Sq = q$, for each, $x \in C(S, T_n) \subseteq C_q(S, T)$,

$$\begin{aligned} ST_n x &= S(W(T^n x, q, \lambda_n)) = W(ST^n x, q, \lambda_n) \\ &= W(T^n Sx, q, \lambda_n) = T_n Sx. \end{aligned}$$

Hence S and T_n are weakly compatible for all n. The result now follows from Theorem 2. ■

Corollary 3. *Let E be a nonempty closed $q-$starshaped subset of convex complete metric space X and T and S be C_q- commuting mappings on $E-\{q\}$ such that $S(E)=E$ and $T(E-\{q\})\subset S(E-\{q\})$, where $q\in F(S)$. Suppose that T is continuous asymptotically $S-$ nonexpansive with sequence $\{k_n\}$ and S is affine on E. For each $n\geq 1$, define a mapping T_n on E by $T_nx=W(T^nx,q,\alpha_n)$, where $\alpha_n=\dfrac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\lim\limits_{n\to\infty}\lambda_n=1$. Then for each $n\in N$, $F(T_n)\cap F(S)$ is singleton.*

Theorem 4. *Let E be a nonempty closed $q-$ starshaped subset of convex metric space X, and T and S be continuous self mappings on E such that $S(E)=E$ and $T(E-\{q\})\subset S(E-\{q\})$, $q\in F(S)$. Suppose T is uniformly asymptotically regular, asymptotically S- nonexpansive, and S is affine on E. If $cl(E-\{q\})$ is compact and S and T are uniformly C_q- commuting mappings on $E-\{q\}$. Then $F(T)\cap F(S)$ is a singleton in E.*

Proof. From Theorem 3, for each $n\in N$, $F(T_n)\cap F(S)$ is singleton in E. Thus,

$$Sx_n=x_n=W(T^nx_n,q,\alpha_n).$$

Also,

$$\begin{aligned} d(x_n,T^nx_n) &= d(W(T^nx_n,q,\alpha_n),T^nx_n)) \\ &\leq (1-\alpha_n)d(q,T^nx_n)\leq(1-\alpha_n)d(q,T^nx_n). \end{aligned}$$

Since $T(E-\{q\})$ is bounded, $d(x_n,T^nx_n)\to 0$ as $n\to\infty$. Now,

$$\begin{aligned} d(x_n,Tx_n) &\leq d(x_n,T^nx_n)+d(T^nx_n,T^{n+1}x_n)+d(T^{n+1}x_n,Tx_n) \\ &\leq d(x_n,T^nx_n)+d(T^nx_n,T^{n+1}x_n)+k_1d(ST^nx_n,Sx_n) \\ &\leq d(x_n,T^nx_n)+d(T^nx_n,T^{n+1}x_n)+k_1d(ST^nx_n,SW(T^nx_n,q,\alpha_n)) \\ &\leq d(x_n,T^nx_n)+d(T^nx_n,T^{n+1}x_n)+k_1d(ST^nx_n,W(ST^nx_n,q,\alpha_n)) \\ &\leq d(x_n,T^nx_n)+d(T^nx_n,T^{n+1}x_n)+k_1(1-\alpha_n)d(ST^nx_n,Sq) \\ &\leq d(x_n,T^nx_n)+d(T^nx_n,T^{n+1}x_n)+k_1(1-\alpha_n)d(ST^nx_n,Sq), \end{aligned}$$

which implies that, $d(x_n,Tx_n)\to 0$, as $n\to\infty$. As $cl(E-\{q\})$ is compact and E is closed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\to x_0\in E$ as $i\to\infty$. The continuity of T implies that $T(x_0)=x_0$. Since $T(E-\{q\})\subset S(E-\{q\})$, it follows that $x_0=T(x_0)=Sy$ for some $y\in E$.
Moreover,

$$d(Tx_{n_i},Ty)\leq k_1d(Sx_{n_i},Sy)=k_1d(x_{n_i},x_0).$$

Taking the limit as $i\to\infty$, we get $Tx_0=Ty$. Thus $Tx_0=Sy=Ty=x_0$. Since S and T are uniformly C_q- commuting on $E-\{q\}$, and $y\in C(S,T)$,

$$d(Tx_0,Sx_0)=d(TSy,STy)=0.$$

Hence the result follows. ■

Corollary 4. *Let E be a nonempty closed q-starshaped subset of a convex metric space X, and T and S be continuous self mappings on E such that $S(E) = E$ and $T(E - \{q\}) \subset S(E - \{q\})$, $q \in F(S)$. Suppose that T is uniformly asymptotically regular, asymptotically S- nonexpansive and S is affine on E. If $cl(E - \{q\})$ is compact and S and T are C_q- commuting mappings on $E - \{q\}$, then $F(T) \cap F(S)$ is a singleton in E.*

3. Invariant approximation results

Meinardus [16] was the first to employ a fixed point theorem to prove the existence of an invariant approximation in Banach spaces. Subsequently, several interesting and valuable results have appeared in the literature of approximation theory ([1], [18] and [20]). In this section we obtain results on best approximation as a fixed point of uniformly C_q- commuting mappings and C_q- commuting in a convex metric space.

Definition 5. *Let X be a metric space and M be a closed subset of X. If there exists a $y_0 \in M$ such that $d(x, y_0) = d(x, M) = inf\{d(x, y) : y \in M\}$, then y_0 is called a best approximation to x out of M. We denote by $P_M(x)$, the set of all best approximations to x out of M.*

Remark 1. Let M be a closed convex subset of a convex metric space. As, $W(u, v, \lambda) \in M$ for $(u, v, \lambda) \in M \times M \times [0, 1]$, the definition of convexity structure on X further implies that $W(u, v, \lambda) \in P_M(x)$. Hence $P_M(x)$ is a convex subset of X. Also, $P_M(x)$ is a closed subset of X. Moreover, it can also be shown that $P_M(x) \subset \partial M$, where ∂M stands for the boundary of M.

Theorem 5. *Let M be a nonempty subset of a convex metric space X, T, f and g be self maps on X such that u is common fixed point of f, g and T and $T(\partial M \cap M) \subset M$. Suppose that f and g are affine and continuous on $P_M(u)$ with $P_M(u)$ $q-$ starshaped, $f(P_M(u)) = P_M(u) = g(P_M(u))$ and, $q \in F(f) \cap F(g)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are C_q- commuting and satisfy,*

$$d(Tx, Ty) \leq \begin{cases} d(fx, gu) & \text{if} \quad y = u, \\ \max\{d(fx, gy), d(fx, Y_q^{Tx}), & \\ \quad d(gy, Y_q^{Ty}), \frac{1}{2}[d(fx, Y_q^{Ty}) + d(gy, Y_q^{Tx})]\} & \text{if} \quad y \in P_M(u) \end{cases}$$

for all $x \in P_M(u) \cup \{u\}$, and if $cl(P_M(u))$ is compact and $P_M(u)$ is complete, then $P_M(u) \cap F(T) \cap F(f) \cap F(g)$ is nonempty.

Proof. Let $x \in P_M(u)$, then $d(x,u) = d(x,M)$. Note that for any $\lambda \in (0,1)$

$$d(y_\lambda, u) = d(W(x,u,\lambda),u) \leq \lambda d(x,u) < d(x,u) = d(x,M),$$

which shows that, $Y_u^x = \{y_\lambda : y_\lambda = W(x,u,\lambda)\} \cap M$ is empty so $x \in \partial M \cap M$ and $Tx \in M$. Since $fx \in P_M(u)$, u is common fixed point of f, g and T, from the given contractive condition we obtain

$$d(Tx,u) = d(Tx,Tu) \leq d(fx,gu) = d(fx,u) = d(u,M).$$

Thus $P_M(u)$ is T- invariant. Also,

$$T(P_M(u)) \subset P_M(u) = f(P_M(u)) = g(P_M(u)),$$

and result follows from Theorem 1. ■

Theorem 6. *Let M be a nonempty subset of a convex metric space X, and T, S be two continuous self mappings on X such that and $T(\partial M \cap M) \subset M$, $u \in F(S) \cap F(T)$ for some u in X. Suppose that T is uniformly asymptotically regular, asymptotically $S-$ nonexpansive and S is affine on $P_M(u)$ with $S(P_M(u)) = P_M(u)$, $q \in F(S)$ and $P_M(u)$ is $q-$ starshaped. If $cl(P_M(u))$ is compact, $P_M(u)$ is complete and the pair $\{S, T\}$ is uniformly C_q- commuting on $P_M(u) \cup \{u\}$ satisfying $d(Tx,Tu) \leq d(Sx,Su)$, then $P_M(u) \cap F(T) \cap F(S) \neq \phi$.*

Proof. Let $x \in P_M(u)$, then $d(x,u) = d(x,M)$. Note that for any $\lambda \in (0,1)$,

$$d(y_\lambda, u) = d(W(x,u,\lambda),u) \leq \lambda d(x,u) < d(x,u) = d(x,M),$$

which shows that, $Y_u^x = \{y_\lambda : y_\lambda = W(x,u,\lambda)\} \cap M$ is empty so $x \in \partial M \cap M$ and $Tx \in M$. Since $Sx \in P_M(u)$, u is common fixed point of S and T, and therefore, by given contractive condition, we obtain

$$d(Tx,u) = d(Tx,Tu) \leq d(Sx,Su) = d(Sx,u) = d(u,M).$$

Thus $P_M(u)$ is T- invariant. Also,

$$T(P_M(u)) \subset P_M(u) = S(P_M(u)).$$

The result now follows from Theorem 4. ■

Corollary 5. *Let M be a nonempty subset of convex metric space X, and T and S be two continuous self mappings on X such that and $T(\partial M \cap M) \subset M$, $u \in F(S) \cap F(T)$ for some u in X. Suppose that T is uniformly asymptotically regular, asymptotically S- nonexpansive, S is affine on $P_M(u)$ with $S(P_M(u)) = P_M(u)$, $q \in F(S)$ and, $P_M(u)$ is $q-$ starshaped. If $cl(P_M(u))$ is compact, $P_M(u)$ is complete and pair $\{S, T\}$ is uniformly $R-$subweakly commuting on $P_M(u) \cup \{u\}$ satisfying $d(Tx, Tu) \leq d(Sx, Su)$, then $P_M(u) \cap F(T) \cap F(S) \neq \phi$.*

Remark 2. Theorem 4 extends and improves Theorem 6 of [3] to convex metric spaces. The results of this paper generalize the comparable results of [5] along with the reference in [5].

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