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ON ITERATED VOLTERRA INTEGRODIFFERENTIAL EQUATION OF HIGHER ORDER

Abstract. In this paper we study the existence and other properties of solutions of a certain iterated Volterra integrodifferential equation of higher order. The tools employed in the analysis are based on application of the Leray-Schauder alternative and a certain integral inequality which provides explicit bound on the unknown function.

Key words: integrodifferential equation, higher order, Leray-Schauder alternative, integral inequality, initial value problem, global existence, completely continuous operators, fixed point, r-derivatives.

AMS Mathematics Subject Classification: 34K10, 35R10.

1. Introduction

Consider the initial value problem (IVP for short) for higher order iterated Volterra integrodifferential equation of the form

$$y^{(n)}(t) = f\left(t, y(t), y'(t), ..., y^{(n-1)}(t), K(t,y)\right), \tag{1}$$

for $t \in I = [t_0, T]$ *and* $n \geq 1$ is an arbitrary integer, with the given initial conditions

$$y^{(k)}(t_0) = c_k, \quad k = 0, 1, ..., n-1, \tag{2}$$

where

$$K(t,y) = \int_{t_0}^{t} g\left(t, \sigma, y(\sigma), y'(\sigma), ..., y^{(n-1)}(\sigma), L(t,\sigma,y)\right) d\sigma, \tag{3}$$

in which

$$L(t,\sigma,y) = \int_{t_0}^{\sigma} h\left(t, \sigma, \tau, y(\tau), y'(\tau), ..., y^{(n-1)}(\tau)\right) d\tau, \tag{4}$$

and f, g, h are the elements of R , the set of real numbers and c_k are given real constants. Let $I = [t_0, T]$ ($T > t_0 \geq 0$ is a constant) and $R_+ = [0, \infty)$ be the given subsets of R and $C(S_1, S_2)$ denotes the class of continuous functions from the set S_1 to the set S_2 . For $t_0 \leq \tau \leq \sigma \leq t \leq T$, we assume that $f \in C(I \times R^{n+1}, R)$, $g \in C(I^2 \times R^{n+1}, R)$, $h \in C(I^3 \times R^n, R)$. We define $B = C^{n-1}(I) = C^{n-1}(I, R)$ to be the Banach space of the functions u such that $u^{(n-1)}$ is continuous on I endowed with norm $\|u\| = \max_{t\in I} \{|u|_0, |u'|_0, ..., |u^{(n-1)}|_0\}$, where $|u|_0 = \max\{|u(t)| : t \in I\}$ and we also define $B_0 = C_0^{n-1}(I) = \{u \in C^{n-1}(I) : u(t_0) = 0\}$.

The problems of existence and other properties of solutions of the special versions of IVP (1)-(2) have been studied by many authors by using different techniques. In [7] Morchalo and in [9] Pachpatte studied the special versions of IVP (1)-(2) when the term $L(t, \sigma, y)$ in (3) is absent. The IVP (1)-(2) considered here is in the general spirit of the investigations in [7, 9], see also [3, 6]. In this paper, our main objective is to study the existence and other properties of solutions of IVP (1)-(2). The application of the topological transversality theorem also known as Leray-Schauder alternative and a certain integral inequality with explicit estimate are used to establish the results.

2. Global existence

Our approach and arguments are based on the formula, namely, any solution $y(t)$ of IVP (1)-(2) and its derivatives are represented by the equivalent integral equations

$$y^{(j)}(t) = \sum_{i=j}^{n-1} \frac{c_i (t-t_0)^{i-j}}{(i-j)!} + \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} f\left(s, y(s), y'(s), ..., y^{(n-1)}(s), K(s, y)\right), \tag{5}$$

for $0 \leq j \leq n-1$. In proving existence of solutions of IVP (1)-(2) we will use the following version of the topological transversality theorem given by Granas [2, p. 61].

Lemma 1. *Let B be a convex subset of a normed linear space E and assume $0 \in B$. Let $S : B \to B$ be a completely continuous operator and let $U(S) = \{y : y = \lambda Sy\}$ for $0 < \lambda < 1$. Then either $U(S)$ is unbounded or S has a fixed point.*

Now, we are able to state and prove the following theorem which deals with the global existence of solutions of IVP (1)-(2).

Theorem 1. *Suppose that the functions f, g, h in (1), (3), (4) satisfy the conditions*

$$\left|f\left(t,y(t),y'(t),...,y^{(n-1)}(t),K(t,y)\right)\right| \leq p(t)\,w_1\left(\sum_{i=0}^{n-1}\left|y^{(i)}(t)\right|\right)+|K(t,y)|, \tag{6}$$

$$\left|g\left(t,\sigma,y(\sigma),y'(\sigma),...,y^{(n-1)}(\sigma),L(t,\sigma,y)\right)\right| \leq q(t,\sigma)\,w_2\left(\sum_{i=0}^{n-1}\left|y^{(i)}(\sigma)\right|\right)+|L(t,\sigma,y)|, \tag{7}$$

$$\left|h\left(t,\sigma,\tau,y(\tau),y'(\tau),...,y^{(n-1)}(\tau)\right)\right| \leq r(t,\sigma,\tau)\,w_3\left(\sum_{i=0}^{n-1}\left|y^{(i)}(\tau)\right|\right), \tag{8}$$

for $t_0 \leq \tau \leq \sigma \leq t \leq T$, *where* $p(t) \in C(I,R_+)$, $q(t,\sigma) \in C\left(I^2,R_+\right)$, $r(t,\sigma,\tau) \in C\left(I^3,R_+\right)$, *and for* $i=1,2,3,$ $w_i : R_+ \to (0,\infty)$ *are continuous and nondecreasing functions. Let* $w(u)=\max\{w_1(u),w_2(u),w_3(u)\}$. *Then the IVP (1)-(2) has a solution* $y(t)$ *defined on* I *provided* T *satisfies*

$$N\int_{t_0}^{T}F(s)\,ds<\int_{M}^{\infty}\frac{ds}{w(s)}, \tag{9}$$

where

$$N=\sum_{j=0}^{n-1}\frac{(T-t_0)^{n-j-1}}{(n-j-1)!}, \tag{10}$$

$$M=\sum_{j=0}^{n-1}\left[\sum_{i=j}^{n-1}\frac{|c_i|\,(T-t_0)^{i-j}}{(i-j)!}\right], \tag{11}$$

and

$$F(t)=p(t)+\int_{t_0}^{t}\left\{q(t,\sigma)+\int_{t_0}^{\sigma}r(t,\sigma,\tau)\,d\tau\right\}d\sigma, \tag{12}$$

for $t \in I$.

Proof. First, we establish the priori bounds independent of λ for the solutions of the family of problems

$$y^{(n)}(t) = \lambda f\left(t, y(t), y'(t), ..., y^{(n-1)}(t), K(t,y)\right), \tag{13}$$

for $t \in I$ and $\lambda \in (0,1)$, with the given initial conditions (2). If $y(t)$ is a solution of IVP (13)-(2), then the solution $y(t)$ and its derivatives can be written as

$$y^{(j)}(t) = \sum_{i=j}^{n-1} \frac{c_i (t-t_0)^{i-j}}{(i-j)!} + \lambda \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} f\left(s, y(s), y'(s), ..., y^{(n-1)}(s), K(s,y)\right) ds, \tag{14}$$

for $0 \le j \le n-1$. From (14) and using the hypotheses (6)-(8) we obtain

$$\begin{aligned} \sum_{j=0}^{n-1} \left| y^{(j)}(t) \right| &\le \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{|c_i| (t-t_0)^{i-j}}{(i-j)!} \right] \\ &+ \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \left| f\left(s, y(s), y'(s), ..., y^{(n-1)}(s), K(s,y)\right) \right| ds \\ &\le \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \frac{|c_i| (T-t_0)^{i-j}}{(i-j)!} \\ &+ \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(T-t_0)^{n-j-1}}{(n-j-1)!} \left| f\left(s, y(s), y'(s), ..., y^{(n-1)}(s), K(s,y)\right) \right| ds \\ &\le M + N \int_{t_0}^{t} \left[p(s) w_1 \left(\sum_{j=0}^{n-1} \left| y^{(j)}(s) \right| \right) \right. \\ &+ \int_{t_0}^{s} \left\{ q(s,\sigma) w_2 \left(\sum_{j=0}^{n-1} \left| y^{(j)}(\sigma) \right| \right) \right. \\ &+ \left. \left. \int_{t_0}^{\sigma} r(s,\sigma,\tau) w_3 \left(\sum_{j=0}^{n-1} \left| y^{(j)}(\tau) \right| \right) d\tau \right\} d\sigma \right] ds. \end{aligned} \tag{15}$$

Define a function $u(t)$ by the right hand side of (15), then we have

$$\sum_{j=0}^{n-1} \left| y^{(j)}(t) \right| \le u(t), \quad u(t_0) = M,$$

and

$$
\begin{aligned}
u'(t) = N & \left[p(t) w_1 \left(\sum_{j=0}^{n-1} \left| y^{(j)}(t) \right| \right) + \int_{t_0}^{t} \left\{ q(t,\sigma) w_2 \left(\sum_{j=0}^{n-1} \left| y^{(j)}(\sigma) \right| \right) \right.\right. \\
& \left.\left. + \int_{t_0}^{\sigma} r(t,\sigma,\tau) w_3 \left(\sum_{j=0}^{n-1} \left| y^{(j)}(\tau) \right| \right) d\tau \right\} d\sigma \right] \\
\leq \ N & \left[p(t) w_1 (u(t)) + \int_{t_0}^{t} \left\{ q(t,\sigma) w_2 (u(\sigma)) \right.\right. \\
& \left.\left. + \int_{t_0}^{\sigma} r(t,\sigma,\tau) w_3 (u(\tau)) d\tau \right\} d\sigma \right] \\
\leq \ & N w(u(t)) \left[p(t) + \int_{t_0}^{t} \left\{ q(t,\sigma) + \int_{t_0}^{\sigma} r(t,\sigma,\tau) d\tau \right\} d\sigma \right] \\
= \ & N F(t) w(u(t)),
\end{aligned}
$$

i.e.,

$$
\frac{u'(t)}{w(u(t))} \leq N F(t). \tag{16}
$$

Integration of (16) from t_0 to $t \in I$ and the use of the change of variable and the condition (9) gives

$$
\int_{M}^{u(t)} \frac{ds}{w(s)} \leq N \int_{t_0}^{t} F(s) ds \leq N \int_{t_0}^{T} F(s) ds < \int_{M}^{\infty} \frac{ds}{w(s)}. \tag{17}
$$

From (17) we conclude that there is a constant Q independent of $\lambda \in (0,1)$ such that $u(t) \leq Q$ for $t \in I$ and hence $\sum_{j=0}^{n-1} \left| y^{(j)}(t) \right| \leq Q$ for $t \in I$. Thus we have $\left| y^{(j)}(t) \right| \leq Q$, $t \in I$ for $0 \leq j \leq n-1$ and consequently $\|y\| \leq Q$.

In the next step we rewrite the IVP (1)-(2) as follows. If $y(t) = e(t) + z(t)$, where $e(t) = \sum_{i=0}^{n-1} \frac{c_i (t-t_0)^i}{i!}$, $t \in I$, then it is easy to see that $z(t)$ satisfies

$$
z(t_0) \ = \ 0,
$$

$$
z(t) \ = \ \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f^* (z(s)) ds, \tag{18}
$$

if and only if $y(t)$ satisfies IVP (1)-(2) or its equivalent integral equation

$$y(t) = \sum_{i=0}^{n-1} \frac{c_i (t-t_0)^i}{i!} + \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f\left(s, y(s), y'(s), \ldots, y^{(n-1)}(s), K(s,y)\right) ds. \tag{19}$$

In (18) for convenience we have set

$$f^*(z(s)) = f\left(s, e(s)+z(s), e'(s)+z'(s), \ldots, e^{(n-1)}(s)+z^{(n-1)}(s), K(s, e+z)\right). \tag{20}$$

Define $S : B_0 \to B_0$ by

$$Sz(t) = \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f^*(z(s))\, ds, \tag{21}$$

for $t \in I$. Then S is clearly continuous. Now we shall show that S is completely continuous.

Let $\{a_k\}$ be a bounded sequence in B_0, i.e., $\|a_k\| \le b$ for all k, where b is a positive constant. Using the hypotheses (6)-(8), letting $\bar{F} = \sup\{F(t) : t \in I\}$ and $\bar{e} = \sup\left\{e^{(j)}(t) : t \in I, 0 \le j \le n-1\right\}$, from (20) we obtain

$$\begin{aligned} |f^*(a_k(s))| &\le p(s)\, w_1\left(\sum_{j=0}^{n-1} \left\{\left|e^{(j)}(s)\right| + \left|a_k^{(j)}(s)\right|\right\}\right) \\ &\quad + \int_{t_0}^{s} \left\{ q(s,\sigma)\, w_2\left(\sum_{j=0}^{n-1} \left\{\left|e^{(j)}(\sigma)\right| + \left|a_k^{(j)}(\sigma)\right|\right\}\right) \right. \\ &\quad \left. + \int_{t_0}^{\sigma} r(s,\sigma,\tau)\, w_2\left(\sum_{j=0}^{n-1} \left\{\left|e^{(j)}(\tau)\right| + \left|a_k^{(j)}(\tau)\right|\right\}\right) d\tau \right\} d\sigma \\ &\le p(s)\, w_1(n\{\bar{e}+b\}) + \int_{t_0}^{s} \left\{ q(s,\sigma)\, w_2(n\{\bar{e}+b\}) \right. \\ &\quad \left. + \int_{t_0}^{\sigma} r(s,\sigma,\tau)\, w_3(n\{\bar{e}+b\})\, d\tau \right\} d\sigma \le F(s)\, w(n\{\bar{e}+b\}). \end{aligned} \tag{22}$$

Now from (21) and (22) we observe that

$$(23)\qquad \left|(Sa_k(t))^{(j)}\right| \le \frac{1}{(n-j-1)!}\int_{t_0}^{t}(t-s)^{n-j-1}\left|f^*(a_k(s))\right|ds$$
$$\le \frac{(T-t_0)^{n-j-1}}{(n-j-1)!}w(n\{\bar{e}+b\})\int_{t_0}^{T}F(s)\,ds$$
$$\le \frac{(T-t_0)^{n-j}}{(n-j-1)!}w(n\{\bar{e}+b\})\bar{F}=N_j,$$

for $0\le j\le n-1$. Hence from (23) we obtain $\|Sa_k\|\le\bar{N}$, where $\bar{N}=\max\{N_j:0\le j\le n-1\}$. This means that $\{Sa_k\}$ is uniformly bounded.

Now we shall show that the sequence $\{Sa_k\}$ is equicontinuous. Let $t_0\le t_1\le t_2\le T$. Then from (21) and using the hypotheses (6)-(8), the elementary inequality (see [4, p. 39]) $x^r-y^r\le rx^{r-1}(x-y)$ for $r\ge 1$ and x,y nonnegative reals, (22) and letting $\{a_k\}$, $\bar{F},\bar{e}$ as defined above, we observe the following cases.

Case I. If $j=0,1,2,...,n-2$, then $n-j-1\ge 1$, and

$$\left|(Sa_k(t_2))^{(j)}-(Sa_k(t_1))^{(j)}\right| = \frac{1}{(n-j-1)!}\left|\int_{t_1}^{t_2}(t_2-s)^{n-j-1}f^*(a_k(s))\,ds\right.$$
$$\left.+\int_{t_0}^{t_1}\left[(t_2-s)^{n-j-1}-(t_1-s)^{n-j-1}\right]f^*(a_k(s))\,ds\right|$$
$$\le\frac{1}{(n-j-1)!}\left[\int_{t_1}^{t_2}(t_2-s)^{n-j-1}\left|f^*(a_k(s))\right|ds\right.$$
$$\left.+\int_{t_0}^{t_1}(n-j-1)(t_2-s)^{n-j-2}(t_2-t_1)\left|f^*(a_k(s))\right|ds\right]$$
$$\le\frac{1}{(n-j-1)!}\left[(T-t_0)^{n-j-1}\int_{t_1}^{t_2}F(s)\,w(n\{\bar{e}+b\})\,ds\right.$$
$$\left.+(n-j-1)(T-t_0)^{n-j-2}(t_2-t_1)\int_{t_0}^{t_1}F(s)\,w(n\{\bar{e}+b\})\,ds\right]$$
$$\le\frac{1}{(n-j-1)!}\left[\int_{t_1}^{t_2}(T-t_0)^{n-j-1}\bar{F}w(n\{\bar{e}+b\})\,ds\right.$$

$$+ (n-j-1)(T-t_0)^{n-j-2}(t_2-t_1)\int_{t_0}^{T}\bar{F}w(n\{\bar{e}+b\})\,ds\Bigg].$$

Case II. If $j = n-1$, then $n-j-1 = 0$ and

$$\left|(Sa_k(t_2))^{(n-1)} - (Sa_k(t_1))^{(n-1)}\right| = \left|\int_{t_1}^{t_2} f^*(a_k(s))\,ds\right|$$
$$\le \int_{t_1}^{t_2} |f^*(a_k(s))|\,ds \le \int_{t_1}^{t_2} F(s)\,w(n\{\bar{e}+b\})\,ds$$
$$\le \int_{t_1}^{t_2} \bar{F}w(n\{\bar{e}+b\})\,ds.$$

From the above estimates we conclude that $\{Sa_k\}$ is equicontinuous and hence by the Arzela-Ascoli theorem the operator S is completely continuous.

Moreover, the set $U(S) = \{z \in B_0 : z = \lambda S z,\ \lambda \in (0,1)\}$ is bounded, since for every z in $U(S)$ the function $y(t) = e(t) + z(t)$ is a solution of IVP (13)-(2), for which we have proved that $\|y\| \le Q$ and hence $\|z\| \le \bar{e} + Q$. By applying Lemma 1, the IVP (1)-(2) has a solution $y(t)$ on I .

The proof is complete. ∎

Remark 1. We note that our Theorem 1 extends the well known theorem of Wintner [16] on the global existence of solution of Cauchy problem for first order differential equation to the IVP (1)-(2). If we choose $NF(t) = 1$ in (9) and the integral on the right hand side of (9) is assumed to diverge, then the solution of IVP (1)-(2) exists for every $T < \infty$, that is, on the entire interval R_+. Further, we note that our Theorem 1 contains in the special cases the global existence of solutions of the equations studied in [1, 7, 9, 10]. For the detailed account on the applications of the topological transversality method, see [5, 8].

3. Properties of solutions

In this section we study the uniqueness, boundedness and continuous dependence of solutions of IVP (1)-(2) under some suitable conditions on the functions involved in (1), (3), (4). The following inequality due to Bykov and Salpagarov (see [14, Theorem 1.4.2, p. 32]) is crucial in the analysis which follows. For detailed account on such inequalities, see [11, 14].

Lemma 2. *Let* $u(t), p(t) \in C(R_+, R_+)$ *and for* $0 \le \tau \le \sigma \le t < \infty$, $q(t,\sigma) \in C\left(R_+^2, R_+\right)$, $r(t,\sigma,\tau) \in C\left(R_+^3, R_+\right)$. *If*

$$u(t) \le k + \int_0^t \left[p(s)\, u(s) + \int_0^s \left\{ q(s,\sigma)\, u(\sigma) + \int_0^\sigma r(s,\sigma,\tau)\, u(\tau) d\tau \right\} d\sigma \right] ds,$$

for $t \in R_+$, *where* $k \ge 0$ *is a constant, then*

$$u(t) \le k \exp\left(\int_0^t F(s)\, ds \right),$$

for $t \in R_+$, *where*

$$F(t) = p(t) + \int_0^t \left\{ q(t,\sigma) + \int_0^\sigma r(t,\sigma,\tau)\, d\tau \right\} d\sigma,$$

for $t \in R_+$

First, we shall give the following theorem which deals with the uniqueness of solutions of IVP (1)-(2).

Theorem 2. *Suppose that the functions* f, g, h *in (1), (3), (4) satisfy the conditions*

$$\begin{aligned} &\left| f\left(t, y(t), y'(t), ..., y^{(n-1)}(t), K(t,y)\right) \right. \\ &\qquad \left. - f\left(t, z(t), z'(t), ..., z^{(n-1)}(t), K(t,z)\right) \right| \\ &\qquad \le p(t) \sum_{i=0}^{n-1} \left| y^{(i)}(t) - z^{(i)}(t) \right| + \left| K(t,y) - K(t,z) \right|, \end{aligned} \tag{24}$$

$$\begin{aligned} &\left| g\left(t, \sigma, y(\sigma), y'(\sigma), ..., y^{(n-1)}(\sigma), L(t,\sigma,y)\right) \right. \\ &\qquad \left. - g\left(t, \sigma, z(\sigma), z'(\sigma), ..., z^{(n-1)}(\sigma), L(t,\sigma,z)\right) \right| \\ &\qquad \le q(t,\sigma) \sum_{i=0}^{n-1} \left| y^{(i)}(t) - z^{(i)}(t) \right| + \left| L(t,\sigma,y) - L(t,\sigma,z) \right|, \end{aligned} \tag{25}$$

$$\left| h\left(t, \sigma, \tau, y(\tau), y'(\tau), ..., y^{(n-1)}(\tau)\right) \right. \tag{26}$$

$$- \ h\left(t, \sigma, \tau, z(\tau), z'(\tau), ..., z^{(n-1)}(\tau)\right)\Big|$$
$$\leq \ r(t, \sigma, \tau) \sum_{i=0}^{n-1} \left| y^{(i)}(\tau) - z^{(i)}(\tau) \right|,$$

where p, q, r are as defined in Theorem 1 and

$$\int_{t_0}^{T} F(s)\, ds < \infty, \tag{27}$$

in which $F(t)$ is given by (12). Then IVP (1)-(2) has at most one solution on I .

Proof. Let $y_1(t)$ and $y_2(t)$ for $t \in I$ be two solutions of IVP (1)-(2). Then from (5) we have

$$\begin{aligned} y_1^{(j)}(t) - y_2^{(j)}(t) = & \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \\ & \times \left\{ f\left(s, y_1(s), y_1'(s), ..., y_1^{(n-1)}(s), K(s, y_1)\right) \right. \\ & \left. - f\left(s, y_2(s), y_2'(s), ..., y_2^{(n-1)}(s), K(s, y_2)\right) \right\} ds, \end{aligned} \tag{28}$$

for $0 \leq j \leq n-1$. From (28) and using the hypotheses (24)-(26) we have

$$\begin{aligned} & \sum_{j=0}^{n-1} \left| y_1^{(j)}(t) - y_2^{(j)}(t) \right| \\ & \leq \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \Big| f\left(s, y_1(s), y_1'(s), ..., y_1^{(n-1)}(s), K(s, y_1)\right) \\ & \quad - f\left(s, y_2(s), y_2'(s), ..., y_2^{(n-1)}(s), K(s, y_2)\right) \Big| ds \\ & \leq \int_{t_0}^{t} N \left[p(s) \sum_{j=0}^{n-1} \left| y_1^{(j)}(s) - y_2^{(j)}(s) \right| \right. \\ & \quad + \int_{t_0}^{s} \left\{ q(s, \sigma) \sum_{j=0}^{n-1} \left| y_1^{(j)}(\sigma) - y_2^{(j)}(\sigma) \right| \right. \\ & \quad \left. \left. + \int_{t_0}^{\sigma} r(s, \sigma, \tau) \sum_{j=0}^{n-1} \left| y_1^{(j)}(\tau) - y_2^{(j)}(\tau) \right| d\tau \right\} d\sigma \right] ds, \end{aligned} \tag{29}$$

where N is given by (10). Now a suitable application of Lemma 2 (when $k = 0$) to (29) yields

$$\sum_{j=0}^{n-1}\left|y_1^{(j)}(t)-y_2^{(j)}(t)\right|\le 0,$$

which implies $y_1(t)=y_2(t)$, that is, the IVP (1)-(2) has at most one solution on I. ■

The next theorem deals with the boundedness of solutions of IVP (1)-(2).

Theorem 3. *Suppose that the functions f, g, h in (1), (3), (4) satisfy the conditions*

$$\left|f\left(t,y(t),y'(t),...,y^{(n-1)}(t),K(t,y)\right)\right| \le p(t)\sum_{i=0}^{n-1}\left|y^{(i)}(t)\right|+\left|K(t,y)\right|, \tag{30}$$

$$\left|g\left(t,\sigma,y(\sigma),y'(\sigma),...,y^{(n-1)}(\sigma),L(t,\sigma,y)\right)\right| \le q(t,\sigma)\sum_{i=0}^{n-1}\left|y^{(i)}(\sigma)\right|+\left|L(t,\sigma,y)\right|, \tag{31}$$

$$\left|h\left(t,\sigma,\tau,y(\tau),y'(\tau),...,y^{(n-1)}(\tau)\right)\right| \le r(t,\sigma,\tau)\sum_{i=0}^{n-1}\left|y^{(i)}(\tau)\right|, \tag{32}$$

where p,q,r are as in Theorem 1 and the condition (27) holds. Then all solutions of IVP (1)-(2) are bounded on I .

Proof. Any solution $y(t)$ of IVP (1)-(2) and its derivatives are represented by (5). From (5) and using the hypotheses (30)-(32) we have

$$\sum_{j=0}^{n-1}\left|y^{(i)}(t)\right|\le\sum_{j=0}^{n-1}\left[\sum_{i=j}^{n-1}\frac{|c_i|(t-t_0)^{i-j}}{(i-j)!}\right] + \sum_{j=0}^{n-1}\int_{t_0}^{t}\frac{(t-s)^{n-j-1}}{(n-j-1)!}\left|f\left(s,y(s),y'(s),...,y^{(n-1)}(s),K(s,y)\right)\right|ds \tag{33}$$

$$\le M+\int_{t_0}^{t}N\left[p(s)\sum_{j=0}^{n-1}\left|y^{(j)}(s)\right|+\int_{t_0}^{s}\left\{q(s,\sigma)\sum_{j=0}^{n-1}\left|y^{(j)}(\sigma)\right| + \int_{t_0}^{\sigma}r(s,\sigma,\tau)\sum_{j=0}^{n-1}\left|y^{(j)}(\tau)\right|d\tau\right\}d\sigma\right]ds,$$

where N, M are given by (10), (11). Now a suitable application of Lemma 2 to (33) yields

$$\sum_{j=0}^{n-1} \left| y^{(j)}(t) \right| \leq M \exp\left(N \int_{t_0}^{t} F(s)\, ds \right), \tag{34}$$

where $F(t)$ is given by (31). The estimation (34) in view of the assumption (27) implies the boundedness of all solutions of IVP (1)-(2) on I . ■

The following theorem deals with the dependency of solutions of equation (1) on given initial values.

Theorem 4. *Suppose that the functions f, g, h in (1),(3),(4) satisfy the conditions (24)-(27). Let $y(t)$ and $z(t)$ be the solutions of equation (1) with the given initial conditions*

$$y^{(k)}(t_0) = c_k, \quad k = 0, 1, ...n-1, \tag{35}$$

and

$$z^{(k)}(t_0) = d_k, \quad k = 0, 1, ...n-1, \tag{36}$$

where c_k and d_k are given constants. Then

$$\sum_{j=0}^{n-1} \left| y^{(j)}(t) - z^{(j)}(t) \right| \leq \bar{M} \exp\left(N \int_{t_0}^{t} F(s)\, ds \right), \tag{37}$$

for $t \in I$, where

$$\bar{M} = \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{(T - t_0)^{i-j}}{(i-j)!} |c_i - d_i| \right],$$

and N and $F(t)$ are given by (10) and (12).

Proof. Since $y(t)$ and $z(t)$ are the solutions of IVP (1)-(35) and IVP (1)-(36) we have

$$y^{(j)}(t) - z^{(j)}(t) = \sum_{i=j}^{n-1} \frac{(t-t_0)^{i-j}}{(i-j)!} (c_i - d_i) \tag{38}$$

$$+ \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \left\{ f\left(s, y(s), y'(s), ..., y^{(n-1)}(s), K(s,y)\right) \right.$$

$$\left. - f\left(s, z(s), z'(s), ..., z^{(n-1)}(s), K(s,z)\right) \right\} ds,$$

for $0 \leq j \leq n-1$. From (38) and using the hypotheses (24)-(26) we have

$$\sum_{j=0}^{n-1} \left| y^{(j)}(t) - z^{(j)}(t) \right| \leq \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{(t-t_0)^{i-j}}{(i-j)!} |c_i - d_i| \right] \tag{39}$$

$$+ \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \left| f\left(s, y(s), y'(s), ..., y^{(n-1)}(s), K(s,y)\right) \right.$$

$$\left. - f\left(s, z(s), z'(s), ..., z^{(n-1)}(s), K(s,z)\right) \right| ds$$

$$\leq \bar{M} + \int_{t_0}^{t} N \left[p(s) \sum_{j=0}^{n-1} \left| y^{(j)}(s) - z^{(j)}(s) \right| \right.$$

$$+ \int_{t_0}^{s} \left\{ q(s,\sigma) \sum_{j=0}^{n-1} \left| y^{(j)}(\sigma) - z^{(j)}(\sigma) \right| \right.$$

$$\left. \left. + \int_{t_0}^{\sigma} r(s,\sigma,\tau) \sum_{j=0}^{n-1} \left| y^{(j)}(\tau) - z^{(j)}(\tau) \right| d\tau \right\} d\sigma \right] ds.$$

Now a suitable application of Lemma 2 to (39) yields the estimate (37), which shows the dependency of solutions of equation (1) on given initial values. ■

Remark 2. We note that the results obtained in this paper can be extended to the integrodifferential equation of the form

$$D_r^{(n)} y(t) = f\left(t, D_r^{(0)} y(t), D_r^{(1)} y(t), ..., D_r^{(n-1)} y(t), \bar{K}\left(t, D_r^{(0)} y\right)\right), \tag{40}$$

for $t \in I$ and $n > 1$, with the given initial conditions

$$D_r^{(m)} y(t_0) = c_m, m = 0, 1, ..., n-1, \tag{41}$$

where

$$\bar{K}\left(t, D_r^{(0)} y\right) \tag{42}$$

$$= \int_{t_0}^{t} g\left(t, \sigma, D_r^{(0)} y(\sigma), D_r^{(1)} y(\sigma), ..., D_r^{(n-1)} y(\sigma), \bar{L}\left(t, \sigma, D_r^{(0)} y\right)\right) d\sigma,$$

in which

$$\bar{L}\left(t, \sigma, D_r^{(0)} y\right) \tag{43}$$

$$= \int_{t_0}^{\sigma} h\left(t, \sigma, \tau, D_r^{(0)} y(\tau), D_r^{(1)} y(\tau), ..., D_r^{(n-1)} y(\tau)\right) d\tau.$$

In (40)-(43) for sufficiently smooth functions $r_i(t) > 0$, $i = 1, ..., n-1$ and $y(t)$ defined on I, the r-derivatives of a function $y(t)$ are defined by (see [13, p. 312])

$$D_r^{(0)} y = y,$$
$$D_r^{(k)} y = r_k \left(D_r^{(k-1)} y \right), \quad k = 1, ..., n-1, \left(' = \frac{d}{dt} = D \right),$$
$$D_r^{(n)} y = \left(D_r^{(n-1)} y \right)',$$

and c_m are given real constants. Naturally, these considerations will make the analysis more complicated, here we do not discuss the details. For the study of special version of IVP (40)-(41), see [12].

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