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VECTOR VALUED PARANORMED $\ell(p)$ SPACES ASSOCIATED WITH MULTIPLIER SEQUENCES

Abstract. In this article we introduce the multiplier vector valued sequence space $\ell\{E_k, \Lambda, p\}$, where $\Lambda = (\lambda_k)$ is an associated multiplier sequence of non-zero complex numbers and the terms of the sequence are chosen from the seminormed spaces $E_k, k \in N$. This generalizes the scalar sequence space $\ell\{p\}$. We study some properties of this space like solidity, symmetricity, completeness, separability. Prove some inclusion results and obtain their duals.

Key words: paranormed sequence spaces, solid spaces, multiplier sequence.

AMS Mathematics Subject Classification: 40A05, 46A45.

1. Introduction

The notion of paranormed sequence space was introduced by Nakano [9] and Simons [14]. It was further investigated from sequence space point of view and linked with summability theory by Maddox [7], Lascarides [5], Nanda [10], Ratha [12], Rath and Tripathy [11], Tripathy and Sen [15] and many others.

The studies on vector valued sequence spaces was exploited by Kamthan [3], Ratha and Srivastava [13], Leonard [6], Gupta[2] and many others.

The scope for the studies on sequence spaces was extended on introducing the notion of associated multiplier sequences. Goes and Goes [1] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, with the help of multiplier sequences (k^{-1}) and (k) respectively. Kamthan [3] used the multiplier sequence $(k!)$. In this article we shall consider a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars.

2. Definitions and preliminaries

A vector valued sequence space E is called *solid* (or *normal*) if $\alpha x = (\alpha_k x_k) \in E$, whenever $x = (x_k) \in E$ and for all sequences $\alpha = (\alpha_k)$ of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space E is said to be *monotone* if E contains the canonical preimages of all its stepspaces.

A sequence space E is said to be *symmetric* if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of N.

A vector valued sequence space $Z(E_k)$ is said to be *convergence free* if $(y_k) \in Z(E_k)$ whenever $(x_k) \in Z(E_k)$ and $x_k = \theta_{E_k}$ implies $y_k = \theta_{E_k}$.

Throughout the article E_k will denote a seminormed space, seminormed by f_k for all $k \in N$, defined over C, the field of complex numbers. Throughout $p = (p_k)$ represents a sequence of strictly positive numbers and $t_k = p_k^{-1}$, for all $k \in N$.

We define the following vector valued multiplier sequence spaces:

$$\ell(E_k, \Lambda, p) = \{(x_k) : x_k \in E_k \text{ for all } k \in N \text{ and } \sum_k (f_k(\lambda_k x_k))^{p_k} < \infty\}.$$

$$\ell\{E_k, \Lambda, p\} = \{(x_k) : x_k \in E_k \text{ for all } k \in N \text{ and there exists } r > 0, \text{ such that } \sum_k (f_k(\lambda_k x_k r))^{p_k} t_k < \infty\}.$$

Two sequence spaces E and F are said to be *equivalent* if there exists a sequence $u = (u_k)$ of strictly positive numbers such that the mapping $u : E \to F$ defined by $y = ux = (u_k x_k) \in F$, whenever $(x_k) \in E$, is a one-to-one correspondence between E and F. It is denoted by $E \cong F(u)$ or simply $E \cong F$ (see for instance Nakano [9]).

It is remarked by Lascarides [5] (Remark 3) that "*If E is a sequence space paranormed* (*or normed*) *by g and $E \cong F(u)$, then F is a sequence space paranormed* (*or normed*) *by g_u defined by $g_u(y) = g(u^{-1}y), y \in F$*".

Further it is noted by Lascarides [5] that "*If* $(p_k) \in \ell_\infty$, then $c_0(p) \cong c_0 p(u)$, (*as well as* $\ell_\infty(p) \cong \ell_\infty\{p\}(u)$), *where* $u = (p_k^{t_k})$".

For E and F two sequence spaces we define $M(F, E)$ as follows:

$$M(F, E) = \{\lambda_k : (\lambda_k x_k) \in E, \quad \text{for all} \quad (x_k) \in F\},$$

where $\Lambda = (\lambda_k)$ is a multiplier sequence.

For any normed space E, the set of all continuous linear functionals on E is called its continuous dual and is denoted by E^*.

If we take E_k's to be normed linear spaces, normed by $||.||_{E_k}$ for all $k \in N$, then the Köthe-Toeplitz dual of $Z(E_k)$ is defined as

$$[Z(E_k)]^\alpha = \{(y_k) : y_k \in E_k^* \quad \text{for all} \quad k \in N \quad \text{and} \quad (||x_k||_{E_k} ||y_k||_{E_k^*} \in \ell_1\}.$$

Lemma 1. [Kamthan and Gupta [4]] *A sequence space E is solid implies E is monotone.*

Lemma 2. [Maddox [8], Theorem 1.]*If $p_k > 1$, for all $k \in N$, then*

$$[\ell(p)]^\alpha = M(p) = \{(a_k) : \sum_k |a_k|^{q_k} N^{-\frac{q_k}{p_k}} < \infty, \text{ for some integer } N > 1\}.$$

3. Main results

The proof of the following result is easy, so omitted.

Theorem 1. *$\ell\{E_k, \Lambda, p\}$ is a linear space for any sequence $p = (p_k)$.*

Theorem 2. *If $p_k \geq 1$, for all $k \in N$ and each E_k is complete seminormed space, seminormed by f_k, for all $k \in N$, then $\ell\{E_k, \Lambda, p\}$ is a complete paranormed space, paranormed by*

$$h(x) = \left[\sum_{k=1}^{\infty} (f_k(r\lambda_k x_k p_k^{-t_k}))^{p_k}\right]^{\frac{1}{M}},$$

where $M = \max\{1, \sup p_k\}$.

Proof. It is clear that for any $x \in \ell\{E_k, \Lambda, p\}$, $h(x) \geq 0$, and $h(\theta) = 0$. Further for $x, y \in \ell\{E_k, \Lambda, p\}$, we have $h(x+y) \leq h(x) + h(y)$. When $x \to \theta$, we have $h(\eta x) \to 0$. Also when $\eta \to 0, h(\eta x) \to 0$ follows from the following:
Since $\eta \to 0$, without loss of generality let $|\eta| < 1$. Then

$$h(\eta x) = \left[\sum_{k=1}^{\infty} (f_k(r\eta\lambda_k x_k p_k^{-t_k}))^{p_k}\right]^{\frac{1}{M}} \leq |\eta| h(x) \to 0, \quad \text{as } \eta \to 0.$$

Hence h is a paranorm on $\ell\{E_k, \Lambda, p\}$.

Let $(x^{(i)})$ be a Cauchy sequence in $\ell\{E_k, \Lambda, p\}$. Then for a given $\varepsilon > 0$, there exists n_0 such that $h(x^i - x^j) < \varepsilon$, for all $i, j \geq n_0$.

$$\text{(1)} \qquad \Rightarrow \left[\sum_{k=1}^{\infty} (f_k(r\lambda_k(x_k^i - x_k^j) p_k^{-t_k}))^{p_k}\right]^{\frac{1}{M}} < \varepsilon, \text{ for all } i, j \geq n_0.$$
$$\Rightarrow (f_k(r\lambda_k(x_k^i - x_k^j) p_k^{-t_k})) < \varepsilon, \quad \text{for all} i, j \geq n_0.$$
$$\Rightarrow (x_k^i - x_k^j) < \varepsilon, \quad \text{for all } i, j \geq n_0, \quad \text{for all } k \in N.$$

Hence $(x_k^i)_{i=1}^{\infty}$ is a Cauchy sequence in E_k, for each $k \in N$.

Since E_k are complete for each $k \in N$, so $(x_k^i)_{i=1}^{\infty}$ converges in E_k, for each $k \in N$. Let $\lim\limits_{i\to\infty} x_k^i = x_k$, for each $k \in N$.

On taking limit as $j \to \infty$ in (1), we have

$$\left[\sum_{k=1}^{\infty} (f_k(r\lambda_k(x_k^i - x_k) p_k^{-t_k}))^{p_k}\right]^{\frac{1}{M}} < \varepsilon, \quad \text{for all } i \geq n_0.$$
$$\Rightarrow (x^{(i)} - x) \in \ell\{E_k, \Lambda, p\}.$$

Since $\ell\{E_k, \Lambda, p\}$ is a linear space, so we have $x = x^{(i)} - (x^{(i)} - x) \in \ell\{E_k, \Lambda, p\}$.

Thus $\ell\{E_k, \Lambda, p\}$ is a complete paranormed space.

This completes the proof of the Theorem. ■

Proposition 1. *The space $\ell\{E_k, \Lambda, p\}$ is normal.*

Proof. Let $x = (x_k) \in \ell\{E_k, \Lambda, p\}$ and $|\alpha_k| \le 1$, for all $k \in N$.
Since $|\alpha_k|^{p_k} \le \max\,(1, |\alpha_k|^H) \le 1$, for all $k \in N$, so

$$\sum_k (f_k(\lambda_k(\alpha_k x_k)r))^{p_k} t_k \le \sum_k (f_k(\lambda_k x_k r))^{p_k} t_k.$$

Thus $x \in \ell\{E_k, \Lambda, p\}$ and $|\alpha_k| \le 1$ for all $k \in N$ implies $\alpha x \in \ell\{E_k, \Lambda, p\}$.
Hence $\ell\{E_k, \Lambda, p\}$ is a normal space.

The next result follows immediately from Lemma 1 and Proposition 1. ■

Proposition 2. *The space $\ell\{E_k, \Lambda, p\}$ is monotone.*

Note: The symmetric property of the space $\ell(E_k, \Lambda, p)$ depends on the sequence (p_k). If $p_k = p$, for all $k \in N$, one can easily verify that $\ell(E_k, \Lambda, p) = \ell_p(E_k)$ if and only if

$$0 < \inf_k |\lambda_k| \le \sup_k |\lambda_k| < \infty.$$

In this case the space $\ell(E_k, \Lambda, p)$ is symmetric, since $\ell(E_k, \Lambda, p)$ is symmetric.

But if (p_k) is not a constant sequence, then $\ell(E_k, \Lambda, p)$ is not symmetric in general. This follows from the following example.

Example 1. Let $E_k = C$, for all $k \in N$, $\lambda_k = 1$, for all $k \in N$, $p_k = 1$, for k odd and $p_k = 2$, for k even.

Consider the sequence (x_k) defined by

$$x_k = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ k^{-1}, & \text{if } k \text{ is even,} \end{cases}$$

Then $(x_k) \in \ell(E_k, \Lambda, p)$.

Let (y_k) be a rearrangement of (x_k) defined as

$$y_k = \begin{cases} (k+1)^{-1}, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

Then $(y_k) \notin \ell(E_k, \Lambda, p)$.

Hence $\ell(E_k, \Lambda, p)$ is not symmetric.

Following the similar arguments we can easily get the next result.

Theorem 3. *(i) The space $\ell\{E_k, \Lambda, p\}$ is symmetric if and only if (p_k) is a constant sequence and $0 < \inf_k |\lambda_k| \leq \sup_k |\lambda_k| < \infty$.*

(ii) If (p_k) is not a constant sequence, then $\ell\{E_k, \Lambda, p\}$ is not symmetric in general.

Proposition 3. *The spaces $\ell(E_k, \Lambda, p)$ and $\ell\{E_k, \Lambda, p\}$ are not convergence free.*

Proof. The result follows from the following example. ■

Example 2. Let $E_k = C$, for all $k \in N, \lambda_k = 1$, for all $k \in N$, $p_k = 2$, for k odd and $p_k = 1$, for k even. Consider the sequence (x_k) defined by

$$x_k = \begin{cases} k^{-1}, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

Then $(x_k) \in \ell(E_k, \Lambda, p)$.
Consider the sequence (y_k) defined by

$$y_k = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

Then $(y_k) \notin \ell(E_k, \Lambda, p)$.
Hence $\ell(E_k, \Lambda, p)$ is not convergence free. Similarly we can show that $\ell\{E_k, \Lambda, p\}$ is not convergence free.

Theorem 4. *If $0 < p_k \leq q_k \leq sup q_k$, then $\ell\{E_k, \Lambda, p\} \subseteq \ell\{E_k, \Lambda, q\}$.*

Proof. Let $x \in \ell\{E_k, \Lambda, p\}$. Then there exists $r > 0$ such that

$$\sum_{k=1}^{\infty} (f_k(\lambda_k x_k r))^{p_k} t_k < \infty$$
$$\Rightarrow (f_k(\lambda_k x_k r))^{p_k} t_k \to 0, \quad \text{as } k \to \infty$$
$$\Rightarrow \text{the exists } k_0 \in N \text{ such that}$$

$$(f_k(\lambda_k x_k r))^{p_k} t_k < H^{-1}, \text{ for all } k \geq k_0, (H = \sup_k p_k).$$
$$\Rightarrow (f_k(\lambda_k x_k r))^{p_k} < 1, \quad \text{for all } k \geq k_0$$

$$\Rightarrow (f_k(\lambda_k x_k r))^{q_k} \leq (f_k(\lambda_k x_k r))^{p_k}, \quad \text{for all } k \geq k_0. \tag{2}$$

Also

$$0 < p_k \leq q_k \Rightarrow \frac{1}{q_k} \leq \frac{1}{p_k}. \tag{3}$$

Thus $(f_k(\lambda_k x_k r))^{q_k} q_k^{-1} \leq (f_k(\lambda_k x_k r))^{p_k} t_k$, for all sufficiently large k, by (2) and (3). So, $\sum\limits_{k=1}^{\infty} (f_k(\lambda_k x_k r))^{q_k} q_k^{-1} < \infty \Rightarrow (x_k) \in \ell\{E_k, \Lambda, q\}$.

Hence the result. ■

Proposition 4. *Let (p_k) be a given sequence of strictly positive real numbers. Then $(\lambda_k) \in M(E, E)$ if and only if $((\lambda_k)^{p_k}) \in \ell_\infty$, where $E = \ell(E_k, p)$ or $\ell\{E_k, p\}$.*

Corollary 1. *$M(E, E) = \ell_\infty$, for $E = \ell(E_k, p)$ or $\ell\{E_k, p\}$ if and only if $h = inf p_k > 0$ and $H = sup p_k < \infty$.*

Proposition 5. *Let $h = \inf p_k$ and $H = \sup p_k$. Then the following are equivalent:*

(i) $H < \infty$ and $h > 0$.

(ii) $\ell\{E_k, \Lambda, p\} = \ell(E_k, \Lambda, p)$.

Proof. Suppose (i) holds. Then for any $r > 0$, we have

$$\min(1, r^H) \leq r^{p_k} \leq \max(1, r^H), \quad \text{for all } k \in N. \tag{4}$$

From (4) we have

$$\sum_{k=1}^{\infty} (f_k(\lambda_k x_k r))^{p_k} t_k \{\max(1, r^H)\}^{-1} h \leq \sum_{k=1}^{\infty} (f_k(\lambda_k x_k))^{p_k} \leq \sum_{k=1}^{\infty} (f_k(\lambda_k x_k r))^{p_k} t_k \{\min(1, r^H)\}^{-1} H. \tag{5}$$

From (5) we get $\ell\{E_k, \Lambda, p\} = \ell(E_k, \Lambda, p)$.

Conversely let (ii) holds. Then $H < \infty$. Consider the sequence (x_k), defined as

$$x_k = |\lambda_k|^{-1} I_k, \quad \text{for all } k \in N,$$

where I_k is the identity element of E_k, for all $k \in N$.

Then $(x_k) \in \ell\{E_k, \Lambda, p\}$. So there exists $r > 0$ and $M > 0$ such that $r^{p_k} \leq M p_k$, for all $k \in N$. Then by inequality (4) we have $h > 0$.

Hence the result. ■

Theorem 5. *Let $p_k \geq 1$, for all $k = 1, 2, 3, \ldots$. Then $\ell\{E_k, \Lambda, p\}$ is separable if E_k is separable for all $k = 1, 2, 3, \ldots$.*

Proof. Let $E_k, k = 1, 2, 3, \ldots$ be separable. Then there exists countable dense subsets $H_k \subset E_k$, $k = 1, 2, 3, \ldots$. We show that $\ell\{H_k, \Lambda, p\}$ is a countable dense subset of $\ell\{E_k, \Lambda, p\}$. Since H_k is countable, so $\ell\{H_k, \Lambda, p\}$ is also countable.

Let x be a limit point of $\ell\{H_k, \Lambda, p\}$. Then there exists a sequence $(x^{(n)})$ in $\ell\{H_k, \Lambda, p\}$ such that
$x^{(n)} \to x$ in the seminorm of $\ell\{E_k, \Lambda, p\}$.

$\Rightarrow h(x^{(n)} - x) \to 0$, as $n \to \infty$.

$\Rightarrow \sum_{k=1}^{\infty} (f_k(\lambda_k r^{-1}(x_k^{(n)} - x_k)p_k^{-t_k}))^{p_k} \to 0$, as $n \to \infty$, for some $r > 0$.

$\Rightarrow$ Given $\varepsilon > 0$, there exists $n_0 \in N$, such that

$\sum_{k=1}^{\infty} (f_k(\lambda_k r^{-1}(x_k^{(n)} - x_k)p_k^{-t_k}))^{p_k} < \varepsilon$, for all $n \geq n_0$, for some $r > 0$.

$\Rightarrow (x_k^{(n)} - x_k)_{k \in N} \in \ell\{E_k, \Lambda, p\}$, for all $n \geq n_0$.

Since $\ell\{E_k, \Lambda, p\}$ is linear, so $x \in \ell\{E_k, \Lambda, p\}$.

Hence $\overline{\ell\{H_k, \Lambda, p\}} \subseteq \ell\{E_k, \Lambda, p\}$.

Conversely let $x \in \ell\{E_k, \Lambda, p\} - \ell\{H_k, \Lambda, p\}$. Since $x_k \in E_k$ and H_k is dense in E_k, so for a given $\varepsilon > 0$, we can choose $x_k^{\varepsilon} \in H_k$ such that

$$f_k(x_k^{\varepsilon} - x_k) < r|\lambda_k|^{-1} \left[\frac{\varepsilon^M p_k}{2^k}\right]^{\frac{1}{k}}.$$

Then $h(x_k^{\varepsilon}) - (x_k) = \sum_{k=1}^{\infty} (f_k(\lambda_k r^{-1}(x_k^{\varepsilon} - x_k)))^{p_k} t_k < \varepsilon$.
Hence $x \in \overline{\ell\{H_k, \Lambda, p\}}$.
Thus $\overline{\ell\{H_k, \Lambda, p\}} = \ell\{E_k, \Lambda, p\}$.
This completes the proof of the theorem. ■

For the next result we will take E_k's to be normed linear spaces, normed by $||.||_{E_k}$ for all $k \in N$.

Theorem 6. *If $p_k > 1$, for all $k \in N$, then*

(a) $[\ell\{E_k, \Lambda, p\}]^{\alpha} = \{(a_k) : a_k \in E_k^*$, *for all* $k \in N$ *and*
$$\sum_{k=1}^{\infty} ||\lambda_k^{-1} a_k||_{E_k}^{q_k} N^{-\frac{q_k}{p_k}} < \infty \text{ for some integer } N > 1\}.$$

(b) $[\ell\{E_k, \Lambda, p\}]^{\alpha} = \{(a_k) : a_k \in E_k^*$, *for all* $k \in N$ *and*
$\sum_{k=1}^{\infty} ||\lambda_k^{-1} r^{-1} p_k^{t_k} a_k||_{E_k}^{q_k} N^{-\frac{q_k}{p_k}} < \infty$, *for some integer* $N > 1$ *and for* $r > 0\}$,

where $\frac{1}{p_k} + \frac{1}{q_k} = 1$, *for all* $k \in N$.

Proof. We have the following well known inequality

$$|a_k y_k| \le |a_k|^{q_k} + |y_k|^{p_k}, \qquad \text{for all } \ k \in N.$$

The proof follows from the above inequality, Lemma 3 and the following expression

$$\begin{aligned}\sum_{k=1}^{\infty} ||a_k||_{E_k^*} ||x_k||_{E_k} &= \sum_{k+1}^{\infty} ||\lambda_k^{-1} a_k||_{E_k^*} ||\lambda_k x_k||_{E_k} \\ &= \sum_{k=1}^{\infty} ||r^{-1}\lambda_k^{-1} p_k^{t_k} a_k||_{E_k^*} ||r\lambda_k t_k^{t_k} x_k||_{E_k}.\end{aligned}$$

■

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