F A S C I C U L I   M A T H E M A T I C I

Nr 40 2008

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ON THE SOLUTION OF RECURSIVE SEQUENCE OF ORDER TWO

Abstract. We obtain in this paper the solution of the following difference equation

$$x_{n+1} = \frac{x_n}{x_{n-1}(x_n \pm 1)}, \quad n = 0, 1, \ldots$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers.

Key words: difference equations, recursive sequence, periodic solution.

AMS Mathematics Subject Classification: 39A10.

1. Introduction

In this paper we obtain the solution of the following recursive sequence

$$x_{n+1} = \frac{x_n}{x_{n-1}(x_n \pm 1)}, \quad n = 0, 1, \ldots \tag{1}$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers.

Recently there has been a lot of interest in studying the global attractivity, boundedness character the periodic nature, and giving the solution of nonlinear difference equations. For some results in this area, see for example [1-11]. Since Cinar [1,2,3] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Elabbasy et al. [4] investigated the global stability, periodicity character and give the solution of special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Elabbasy et al. [5] studied the global stability, periodicity character and give the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [6] investigated the global stability, periodicity character and give the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Karatas et al. [8] gave that the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Simsek et al. [11] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, \ldots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [10].

A point $\overline{x} \in I$ is called an equilibrium point of Eq(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition. [Periodicity] *A sequence* $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -k$. ∎

2. Main results

2.1. First equation

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_n}{x_{n-1}(x_n+1)}, \quad n = 0, 1, \dots \tag{3}$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers with x_{-1}, $x_0 \notin \{0, -1\}$.

Theorem 1. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq(3). Then equation (3) have all solutions and the solutions are*

$$x_{5n-1} = k, \qquad x_{5n} = h, \qquad x_{5n+1} = \frac{h}{k\,(1+h)},$$
$$x_{5n+2} = \frac{1}{(k+h+hk)}, \qquad x_{5n+3} = \frac{k}{h\,(1+k)},$$

where $x_{-1} = k$, $x_{-0} = h$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. We shall show that the result holds for n. From our assumption for $n-1$, we have the following:

$$x_{5n-6} = k, \qquad x_{5n-5} = h, \qquad x_{5n-4} = \frac{h}{k\,(1+h)},$$
$$x_{5n-3} = \frac{1}{(k+h+hk)}, \qquad x_{5n-2} = \frac{k}{h\,(1+k)},$$

Now, it follows from Eq(3) that

$$\begin{aligned}
x_{5n-1} &= \frac{x_{5n-2}}{x_{5n-3}(1+x_{5n-2})} = \frac{k\,(k+h+hk)}{h\,(1+k)\,(1+\dfrac{k}{h\,(1+k)})} \\
&= \frac{k\,(k+h+hk)}{h\,(1+k)+k} = k. \\
x_{5n} &= \frac{x_{5n-1}}{x_{5n-2}(1+x_{5n-1})} = \frac{hk\,(1+k)}{k(1+k)} = h. \\
x_{5n+1} &= \frac{x_{5n}}{x_{5n-1}(1+x_{5n})} = \frac{h}{k(1+h)}. \\
x_{5n+2} &= \frac{x_{5n+1}}{x_{5n}(1+x_{5n+1})} = \frac{h}{k(1+h)h(1+\dfrac{h}{k(1+h)})} = \frac{1}{(k(1+h)+h)}.
\end{aligned}$$

$$\begin{aligned} x_{5n+3} &= \frac{x_{5n+2}}{x_{5n+1}(1+x_{5n+2})} = \frac{k(1+h)}{(k+kh+h)h(1+\dfrac{1}{(k(1+h)+h)})} \\ &= \frac{k(1+h)}{h(k+kh+h+1)} = \frac{k(1+h)}{h(k+1)(h+1)} = \frac{k}{h(k+1)}. \end{aligned}$$

Thus, the proof is completed. ■

Theorem 2. *Suppose that $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (3). Then all solutions of equation (3) are periodic with period five.*

Proof. From Eq(3), we see that

$$\begin{aligned} x_{n+1} &= \frac{x_n}{x_{n-1}(1+x_n)}, \\ x_{n+2} &= \frac{x_{n+1}}{x_n(1+x_{n+1})} = \frac{x_n}{x_n x_{n-1}(1+x_n)(1+\dfrac{x_n}{x_{n-1}(1+x_n)})} \\ &= \frac{1}{(x_{n-1}(1+x_n)+x_n)}. \\ x_{n+3} &= \frac{x_{n+2}}{x_{n+1}(1+x_{n+2})} \\ &= \frac{x_{n-1}(1+x_n)}{(x_{n-1}+x_n x_{n-1}+x_n)x_n(1+\dfrac{1}{(x_{n-1}+x_n x_{n-1}+x_n)})} \\ &= \frac{x_{n-1}(1+x_n)}{x_n(x_{n-1}+x_n x_{n-1}+x_n+1)} = \frac{x_{n-1}}{x_n(1+x_{n-1})}. \\ x_{n+4} &= \frac{x_{n+3}}{x_{n+2}(1+x_{n+3})} = \frac{x_{n-1}(x_{n-1}+x_n x_{n-1}+x_n)}{x_n(1+x_{n-1})(1+\dfrac{x_{n-1}}{x_n(1+x_{n-1})})} \\ &= \frac{x_{n-1}(x_{n-1}+x_n x_{n-1}+x_n)}{(x_n(1+x_{n-1})+x_{n-1})} = x_{n-1}. \\ x_{n+5} &= \frac{x_{n+4}}{x_{n+3}(1+x_{n+4})} = \frac{x_{n-1}(1+x_{n-1})x_n}{x_{n-1}(1+x_{n-1})} = x_n. \end{aligned}$$

This completes the proof. ■

Theorem 3. *Eq(3) have three equilibrium points which are* 0, $\frac{\sqrt{5}-1}{2}$, $\frac{-\sqrt{5}-1}{2}$.

Proof. For the equilibrium points of Eq(3), we can write

$$\overline{x} = \frac{\overline{x}}{\overline{x}(\overline{x}+1)}.$$

Then

$$\overline{x}^3 + \overline{x}^2 - \overline{x} = 0,$$

or

$$\overline{x}(\overline{x}^2 + \overline{x} - 1) = 0.$$

Thus the equilibrium points of Eq(3) is $\overline{x} = 0$, $\overline{x} = \frac{\sqrt{5}-1}{2}$, $\overline{x} = \frac{-\sqrt{5}-1}{2}$. ■

Remark 1. Eq(3) has no prime period two solution.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq(3).

Example 1. See Fig. 1, since $x_{-1} = 15$, $x_0 = -2$.

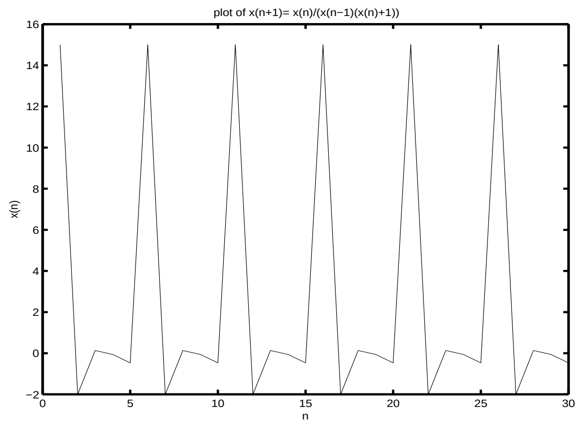

Figure 1.

Example 2. See Fig. 2, since $x_{-1} = -5$, $x_0 = 4.2$.

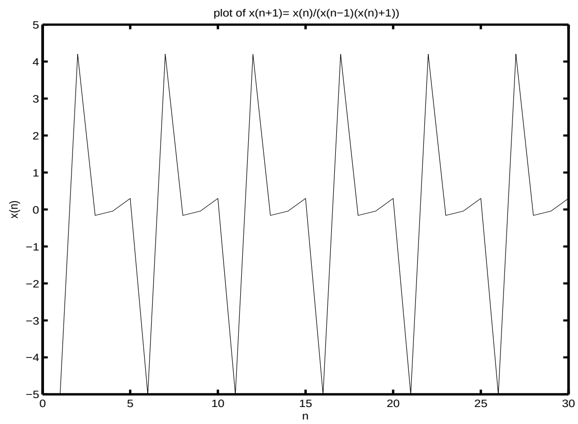

Figure 2.

2.2. Second equation

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1}=\frac{x_n}{x_{n-1}(x_n-1)},\quad n=0,1,\ldots \tag{4}$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers with $x_{-1},\ x_0\notin\{0,1\}$, $x_{-1}+x_0\neq x_0x_{-1}$.

Theorem 4. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq(4). Then equation (4) have all solutions and the solutions are*

$$x_{5n-1}=k,\qquad x_{5n}=h,\qquad x_{5n+1}=\frac{h}{k\,(h-1)},$$
$$x_{5n+2}=\frac{1}{(k+h-hk)},\qquad x_{5n+3}=\frac{k}{h\,(k-1)},$$

where $x_{-1}=k$, $x_{-0}=h$.

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. We shall show that the result holds for n. From our assumption for $n-1$, we have the following:

$$x_{5n-6}=k,\qquad x_{5n-5}=h,\qquad x_{5n-4}=\frac{h}{k\,(h-1)},$$
$$x_{5n-3}=\frac{1}{(k+h-hk)},\qquad x_{5n-2}=\frac{k}{h\,(k-1)},$$

Now, it follows from Eq(4) that

$$\begin{aligned}
x_{5n-1} &= \frac{x_{5n-2}}{x_{5n-3}(x_{5n-2}-1)}=\frac{k\,(k+h-hk)}{(k-h\,(k-1))}=k.\\
x_{5n} &= \frac{x_{5n-1}}{x_{5n-2}(x_{5n-1}-1)}=\frac{hk\,(k-1)}{k(k-1)}=h.\\
x_{5n+1} &= \frac{x_{5n}}{x_{5n-1}(x_{5n}-1)}=\frac{h}{k(h-1)}.\\
x_{5n+2} &= \frac{x_{5n+1}}{x_{5n}(x_{5n+1}-1)}=\frac{1}{(h+k-hk)}.\\
x_{5n+3} &= \frac{x_{5n+2}}{x_{5n+1}(x_{5n+2}-1)}=\frac{k}{h(k-1)}.
\end{aligned}$$

Thus, the proof is completed. ■

Theorem 5. *Suppose that* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of equation (4). Then all solutions of equation (4) are periodic with period five.*

Proof. From Eq(4), we see that

$$\begin{aligned} x_{n+1} &= \frac{x_n}{x_{n-1}(x_n-1)}, \\ x_{n+2} &= \frac{x_{n+1}}{x_n(x_{n+1}-1)} = \frac{x_n}{x_{n-1}x_n(x_n-1)(\frac{x_n}{x_{n-1}(x_n-1)}-1)} \\ &= \frac{1}{(x_n-x_{n-1}x_n+x_{n-1})}, \\ x_{n+3} &= \frac{x_{n+2}}{x_{n+1}(x_{n+2}-1)} \\ &= \frac{x_{n-1}(x_n-1)}{(x_n-x_{n-1}x_n+x_{n-1})x_n(\frac{1}{(x_n-x_{n-1}x_n+x_{n-1})}-1)} \\ &= \frac{x_{n-1}(x_n-1)}{x_n(1-x_n+x_{n-1}x_n-x_{n-1})} \\ &= \frac{x_{n-1}(x_n-1)}{x_n(x_n-1)(x_{n-1}-1)} = \frac{x_{n-1}}{x_n(x_{n-1}-1)}. \end{aligned}$$

$$\begin{aligned} x_{n+4} &= \frac{x_{n+3}}{x_{n+2}(x_{n+3}-1)} = \frac{(x_n-x_{n-1}x_n+x_{n-1})x_{n-1}}{x_n(x_{n-1}-1)(\frac{x_{n-1}}{x_n(x_{n-1}-1)}-1)} = x_{n-1} \\ x_{n+5} &= \frac{x_{n+4}}{x_{n+3}(x_{n+4}-1)} = \frac{x_{n-1}(x_{n-1}-1)x_n}{x_{n-1}(x_{n-1}-1)} = x_n. \end{aligned}$$

This completes the proof. ■

Theorem 6. *Eq(4) have three equilibrium points which are* 0, $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$.

Proof. For the equilibrium points of Eq(4), we can write

$$\overline{x} = \frac{\overline{x}}{\overline{x}(\overline{x}-1)}.$$

Then

$$\overline{x}^3 - \overline{x}^2 - \overline{x} = 0,$$

or

$$\overline{x}(\overline{x}^2 - \overline{x} - 1) = 0.$$

Thus the equilibrium points of Eq(4) is $\overline{x} = 0$, $\overline{x} = \frac{1+\sqrt{5}}{2}$, $\overline{x} = \frac{1-\sqrt{5}}{2}$. ■

Remark 2. Eq(4) has no prime period two solution.

Numerical examples

Example 3. Consider $x_{-1} = 7$, $x_0 = 3$. See Fig. 3.

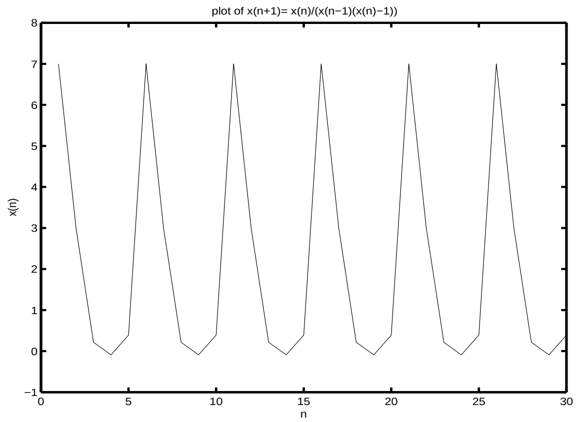

Figure 3.

Example 4. See Fig. 4, since $x_{-1} = -6$, $x_0 = -8$.

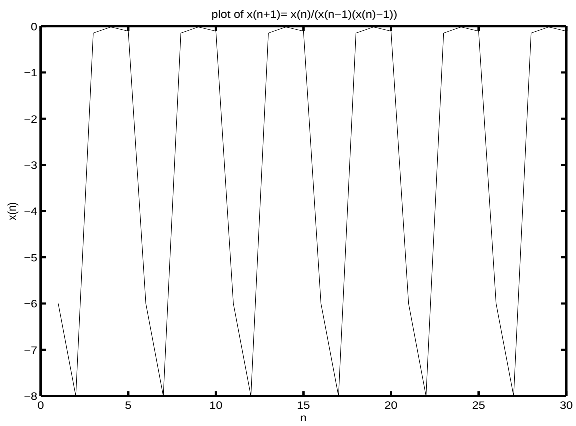

Figure 4.

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Received on 24.11.2007 and, in revised form, on 19.04.2008.