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A GENERALIZATION OF SOME CONVERGENCE RESULTS USING A JUNGCK-NOOR THREE-STEP ITERATION PROCESS IN ARBITRARY BANACH SPACE

Abstract. In this paper, we shall introduce a Jungck-Noor three-step iteration process to establish a strong convergence result for a pair of nonselfmappings in an arbitrary Banach space by employing a general contractive condition.
Our result is a generalization and extension of a multitude of results. In particular, it is a generalization and extension of some of the results of Kannan [11, 12], Rhoades [17, 18] and those of Berinde [4], Rafiq [16] and Olatinwo [15].

Key words: Arbitrary Banach space; strong convergence result; Jungck-Noor three-step iteration process.

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1. Introduction

Let $(E, ||.||)$ be a Banach space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{p \in E \mid Tp = p\}$ is the set of fixed points of T.

There are several iteration processes for which the fixed points of operators have been approximated over the years by various authors. In the Banach space setting, we shall state some of these iteration processes as follows:

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, \cdots, \tag{1}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$, is called the Mann iteration process (see Mann [13]). For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tz_n \\ z_n = (1 - \beta_n)x_n + \beta_n Tx_n \end{cases}, \quad n = 0, 1, \cdots, \tag{2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_{\rm n}\}_{\rm n=0}^{\infty}$ are sequences in $[0,1]$, is called the Ishikawa iteration process (see Ishikawa [8]). The three-step iteration process of Noor [14] is defined iteratively by the sequence $\{x_n\}_{n=0}^{\infty}$ as follows:

$$(3)\qquad \begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n \\ z_n = (1-\beta_n)x_n + \beta_n T y_n \\ y_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases}, \quad n = 0, 1, \cdots, \quad x_0 \in E,$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

The following is the iteration process introduced by Singh et al [22] to establish some stability results: Let S and T be operators on an arbitrary set Y with values in E such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of E. Then, for $x_0 \in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$(4)\qquad Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n T x_n, \quad n = 0, 1, \cdots,$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ is called the *Jungck-Mann* iteration process.

If $\alpha_n = 1$ and $Y = E$ in (2), then we obtain

$$(5)\qquad Sx_{n+1} = Tx_n, \quad n = 0, 1, 2, \cdots,$$

which is the *Jungck iteration.* See Jungck [9] for detail.

Berinde [4] obtained a strong convergence result in an arbitrary Banach space for the Ishikawa iteration process, while Rafiq [16] continued the study in a normed space by using the Noor three-step iteration process. Moreover, both authors employed the following contractive definition: For a mapping $T : E \to E$, there exist real numbers α, β, γ satisfying $0 \le \alpha < 1$, $0 \le \beta < \frac{1}{2}$, $0 \le \gamma < \frac{1}{2}$ respectively such that for each $x, y \in E$, at least one of the following is true:

$$(6)\qquad \begin{cases} (z_1) \;\; d(Tx,Ty) \le \alpha d(x,y), \\ (z_2) \;\; d(Tx,Ty) \le \beta\left[d(x,Tx) + d(y,Ty)\right], \\ (z_3) \;\; d(Tx,Ty) \le \gamma\left[d(x,Ty) + d(y,Tx)\right]. \end{cases}$$

(6) is called the *Zamfirescu contraction condition* which was employed by Zamfirescu [23]. Condition (6) implies

$$(7)\qquad d(Tx,Ty) \le 2\delta d(x,Tx) + \delta d(x,y), \;\; \forall\; x, y \in E,$$

where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $0 \le \delta < 1$. See Theorem 2.4 of Berinde [3], Theorem 2 of Berinde [4] and Theorem 3 of Rafiq [16] for the deduction of (7).

Condition (z_1) of (6) is known as the *Banach's contraction condition.* It is a significant condition in the Banach's fixed point theorem which is contained in Banach [1], Zeidler [24] and several other references. Any mapping satisfying condition (z_2) of (6) is called a *Kannan mapping,* while Chatterjea [6] employed condition (z_3) to establish fixed point result in 1972.

In the next section, we shall introduce a Jungck-Noor iteration process to extend the results of Berinde [4] and Rafiq [16] in an arbitrary Banach space. In establishing our result, a more general contractive condition than (6) will be considered.

2. Preliminaries

We shall introduce the following iteration process in establishing our results. Let $(E, ||.||)$ be a Banach space and Y an arbitrary set. Let $S, T : Y \to E$ be two nonselfmappings such that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of E and S is injective. Then, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty}$ iteratively by

$$(8) \qquad \begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n = (1-\beta_n)Sx_n + \beta_n Ty_n \\ Sy_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases}, \quad n = 0, 1, \cdots,$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$. The iteration process defined in (8) will be called the *Jungck-Noor* iteration process. The iteration processes (1)-(5) are special cases of (8).

(i) For instance, if in (8), S is identity operator and $Y = E$, then we obtain the Noor three-step iteration process of (3).

(ii) Since S is injective, if $\gamma_n = 0$ in (8), we obtain the Jungck-Ishikawa iteration process of Olatinwo [15].

(iii) Again, since S is injective, if $Y = E$, $\alpha_n = 1$, $\beta_n = \gamma_n = 0$, then for $x_0 \in Y$, Jungck-Noor iteration process (8) reduces to the Jungck iteration process of (5). In addition to the iteration process (8), we shall employ the following contractive definition:

Definition 1. *For two nonselfmappings $S, T : Y \to E$ with $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E, there exist: real numbers $M \geq 0$, $a \in [0,1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and for all $x, y \in Y$, we have*

$$(9) \qquad ||Tx - Ty|| \leq \frac{\varphi(||Sx - Tx||) + a||Sx - Sy||}{1 + M||Sx - Tx||}.$$

If $M = 0$, $Y = E$ and S is identity operator in (9), then we obtain the contractive condition which was earlier introduced in Imoru and Olatinwo

[7] to establish some stability results. The contractive condition (9) is more general than (6) or (7) in the sense that it reduces to (7) if $Y = E$, $M = 0$, $\varphi(u) = 2\delta u$, $\forall\ u \in I\!R_+$, $a = \delta \in [0, 1)$ and S is identity operator. Condition (9) also reduces to that of Singh et al [22] if $M = 0$, and $\varphi(u) = Lu$, $L \geq 0$, $\forall\ u \in I\!R_+$.

Definition 2. *Let X and Y be two nonempty sets and $S, T : X \to Y$ two mappings. Then, an element $x^* \in X$ is a coincidence point of S and T if and only if $Sx^* = Tx^*$. Denote the set of the coincidence points of S and F by $C(S, T)$. There are several papers and monographs on the coincidence point theory. However, we refer our readers to Rus [20] and Rus et al [21] for the Definition 2 and some coincidence point results.*

In this paper, we shall employ the Jungck-Noor iteration process defined in (8) to establish a strong convergence result for a pair of nonselfmappings in an arbitrary Banach space using the contractive condition (9). Our result is a generalization and extension of some of the results of Kannan [11, 12], Rhoades [17, 18] and those of Berinde [4], Rafiq [16] and Olatinwo [15].

3. Main result

Theorem 1. *Let $(E, ||.||)$ be an arbitrary Banach space and Y is an arbitrary set. Suppose that S, $T : Y \to E$ are nonselfoperators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E and S is an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy condition (9). Let $\varphi : I\!R^+ \to I\!R^+$ be a monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Noor iteration process defined by (8), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.*

Proof. Let $C(S, T)$ be the set of the coincidence points of S and T. It has been shown in Olatinwo [15] that S and T satisfying condition (9) have a unique coincidence point $z \in C(S, T)$.

We now prove that $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p (where $Sz = Tz = p$) using condition (9). Therefore,

$$
\begin{aligned}
(10)\quad ||Sx_{n+1} - p|| &\leq (1 - \alpha_n)||Sx_n - p|| + \alpha_n||Tz - Tz_n|| \\
&\leq (1 - \alpha_n)||Sx_n - p|| + \alpha_n \left[\frac{\varphi(||Sz - Tz||) + a||Sz - Sz_n||}{1 + L||Sz - Tz||}\right] \\
&= (1 - \alpha_n)||Sx_n - p|| + a\alpha_n||p - Sz_n||.
\end{aligned}
$$

Now, we have that

$$\begin{aligned} (11) \qquad ||p - Sz_n|| &= ||(1-\beta_n)(p - Sx_n) + \beta_n(p - Ty_n)|| \\ &\leq (1-\beta_n)||Sx_n - p|| + a\beta_n||p - Sy_n||. \end{aligned}$$

Using (11) in (10) yields

$$(12) \quad ||Sx_{n+1} - p|| \leq [1 - (1-a)\alpha_n - a\alpha_n\beta_n]||Sx_n - p|| + a^2\alpha_n\beta_n||p - Sy_n||.$$

Furthermore, we have

$$\begin{aligned} (13) \qquad ||p - Sy_n|| &\leq (1-\gamma_n)||Sx_n - p|| + \gamma_n||p - Tx_n|| \\ &\leq (1 - \gamma_n + a\gamma_n)||p - Sx_n||. \end{aligned}$$

Using (13) in (12) yields

$$\begin{aligned} (14) \qquad ||Sx_{n+1} - p|| &\leq [1 - (1-a)\alpha_n - (1-a)a\alpha_n\beta_n \\ &\quad - (1-a)a^2\alpha_n\beta_n\gamma_n]||Sx_n - p|| \\ &\leq [1 - (1-a)\alpha_n]||Sx_n - p|| \\ &\leq \Pi_{j=0}^{n}[1 - (1-a)\alpha_j]||Sx_0 - p|| \\ &\leq e^{-(1-a)\sum_{j=0}^{n}\alpha_j}||Sx_0 - p|| \to 0 \text{ as n} \to \infty. \end{aligned}$$

Therefore, we obtain from (14) that $\lim_{n\to\infty} ||Sx_{n+1} - p|| = 0$, i.e. $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. ■

Remark 2. Theorem 1 is a generalization and extension of a multitude of results. In particular, Theorem 1 is a generalization and extension of both Theorem 1 and Theorem 2 of Berinde [4], Theorem 3 of Rafiq [16], Theorem 2 and Theorem 3 of Kannan [11], Theorem 3 of Kannan [12], Theorem 4 of Rhoades [17] (which is Theorem 4.10 of Berinde [3]) as well as Theorem 8 of Rhoades [18] (which is Theorem 5.6 of Berinde [3]).

Corollary 1 (Olatinwo [15]). *Let$(E, ||.||)$ be an arbitrary Banach space and Y is an arbitrary set. Suppose that S, $T : Y \to E$ are nonselfoperators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E, and S is an injective operator. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy condition (9). Let $\varphi : I\!R^+ \to I\!R^+$ be a monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Ishikawa iteration process defined by*

$$(15) \qquad \begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n = (1-\beta_n)Sx_n + \beta_n Tx_n, \quad n = 0, 1, \cdots, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

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