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Baburao G. Pachpatte

ON NEUTRAL TYPE HYPERBOLIC INTEGRODIFFERENTIAL EQUATION

Abstract. In this paper we study the existence, uniqueness and other properties of solutions of a certain neutral type hyperbolic integrodifferential equation in two independent variables. The Banach fixed point theorem and a certain integral inequality with explicit estimate are used to establish the results.

Key words: neutral type, hyperbolic integrodifferential equation, Banach fixed point theorem, integral inequality, Bielecki type norm, existence and uniqueness, boundedness and continuous dependence.

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1. Introduction

Let R^n denotes the real n-dimensional Eucleaden space with appropriate norm denoted by $|\cdot|$. We denote by $R_+ = [0, \infty)$ the given subset of R, the set of real numbers, $\Delta = R_+ \times R_+$, $E = R^n \times R^n$ and $D = \{(x, y, s, t) \in \Delta^2 : 0 \le s \le x < \infty,\ \ 0 \le t \le y < \infty\}$. The partial derivatives of a function $z(x, y)$ for $x, y \in R_+$ with respect to x, y and xy are denoted by $D_1 z\,(x, y)$, $D_2 z\,(x, y)$ and $D_1 D_2 z\,(x, y) = D_2 D_1 z\,(x, y)$. Let $C\,(S_1, S_2)$ denotes the class of continuous functions from the set S_1 to the set S_2.

In 1965, B. Palczewski [10] studied the uniqueness and convergence of successive approximations of Darboux problem for the equation

$$\text{(P)} \qquad u_{xy} = f\left(x, y, u, u_x, u_y, \int_0^x \int_0^y g\,(x, y, s, t, u_s\,(s,t)\,, u_t\,(s,t))\,dsdt\right),$$

see also [8]. The Darboux problem for the special versions of equation (P) when the integral term is absent have been studied by many authors under a variety of hypotheses by using different techniques, see [1,11,12] and the references cited therein. Motivated by the important results obtained in [10] for the equation (P), in the present paper we study the existence, uniqueness and other properties of solutions of the following neutral type hyperbolic integrodifferential equation

(1) $$D_2D_1u(x,y) = f(x,y,u(x,y), D_2D_1u(x,y), (Hu)(x,y)),$$

with the given initial boundary conditions

(2) $$u(x,0) = \sigma(x), \quad u(0,y) = \tau(y), \quad u(0,0) = 0,$$

for $x, y \in R_+$, where

(3) $$(Hu)(x,y) = \int_0^x \int_0^y g(x,y,m,n,u(m,n), D_2D_1u(m,n))\, dndm,$$

and $f \in C(\Delta \times E \times R^n, R^n)$, $g \in C\left(\Delta^2 \times E, R^n\right)$, $\sigma, \tau \in C(R_+, R^n)$. Obviously, $(H0)(x,y) = \int_0^x \int_0^y g(x,y,m,n,0,0)\, dndm$. The main tools employed in the analysis are based on the applications of the well known Banach fixed point theorem (see [4, p. 37]) coupled with Bielecki type norm [2] and a suitable variant of the integral inequality with explicit estimate given in [9, Theorem 2.4.1].

2. Existence and uniqueness

For a function $\phi(x,y)$ and its derivative $D_2D_1\phi(x,y)$ in $C(\Delta, R^n)$ we denote by $|\phi(x,y)|_1 = |\phi(x,y)| + |D_2D_1\phi(x,y)|$. Let S be the space of those functions $(\phi(x,y), D_2D_1\phi(x,y)) \in E$ which are continuous for $(x,y) \in \Delta$ and fulfil the condition

(4) $$|\phi(x,y)|_1 = O(\exp(\lambda(x+y))),$$

for $(x,y) \in \Delta$, where $\lambda > 0$ is a constant. In the space S we define the norm (see [2])

(5) $$|\phi|_S = \sup_{(x,y)\in\Delta} \left[|\phi(x,y)|_1 \exp(-\lambda(x+y))\right].$$

It is easy to see that S with norm defined in (5) is a Banach space. We note that the condition (4) implies that there exists a constant $N \geq 0$ such that $|\phi(x,y)|_1 \leq N \exp(\lambda(x+y))$ for $(x,y) \in \Delta$. Using this fact in (5) we observe that

(6) $$|\,\phi\,|_S \leq N.$$

By a solution of equation (1) with the given initial boundary conditions (2) (IBVP (1)-(2) for short), we mean a function $u(x,y)$ which satisfy the equations (1) and (2). It is easy to observe that the solution $u(x,y)$ of IBVP (1)-(2) satisfies the following integral equation

$$(7)\quad u(x,y) = \sigma(x)+\tau(y) + \int_0^x\int_0^y f(s,t,u(s,t),D_2D_1u(s,t),(Hu)(s,t))\,dtds,$$

for $(x,y)\in\Delta$.

We need the following variant of the integral inequality established by Pachpatte in [9, Theorem 2.4.1]. We shall give it in the following lemma for completeness.

Lemma. *Let* $w(x,y), a(x,y) \in C(\Delta, R_+)$, $b(x,y,s,t)$, $D_1b(x,y,s,t)$, $D_2b(x,y,s,t)$, $D_2D_1b(x,y,s,t) \in C(D,R_+)$, $e(x,y,s,t)\in C(D,R_+)$ *and* $c\geq 0$ *is a constant. If*

$$(8)\quad w(x,y) \leq c+\int_0^x\int_0^y \Big\{a(s,t)w(s,t)+b(x,y,s,t)w(s,t) + \int_0^s\int_0^t e(s,t,m,n)w(m,n)\,dndm\Big\}dtds,$$

for $(x,y)\in\Delta$, *then*

$$(9)\quad w(x,y) \leq c\exp\left(\int_0^x\int_0^y [a(s,t)+A(s,t)]\,dtds\right),$$

for $(x,y)\in\Delta$, *where*

$$(10)\quad A(x,y) = b(x,y,x,y)+\int_0^x D_1b(x,y,m,y)\,dm+\int_0^y D_2b(x,y,x,n)\,dn + \int_0^x\int_0^y D_2D_1b(x,y,m,n)\,dndm + \int_0^x\int_0^y e(x,y,m,n)\,dndm.$$

Proof. Define a function $z(x,y)$ by the right hand side of (8). Then $z(x,0)=z(0,y)=c$, $w(x,y)\leq z(x,y)$, $z(x,y)$ is nondecreasing in x and y and (see [9, p. 65])

$$(11)\quad D_2D_1z(x,y) = a(x,y)w(x,y)+b(x,y,x,y)w(x,y) + \int_0^x D_1b(x,y,m,y)w(m,y)\,dm+\int_0^y D_2b(x,y,x,n)w(x,n)\,dn$$

$$+\int_0^x\int_0^y D_2D_1b\,(x,y,m,n)\,w\,(m,n)\,dndm$$
$$+\int_0^x\int_0^y e\,(x,y,m,n)\,w\,(m,n)\,dndm$$
$$\leq\; a\,(x,y)\,z\,(x,y)+b\,(x,y,x,y)\,z\,(x,y)$$
$$+\int_0^x D_1b\,(x,y,m,y)\,z\,(m,y)\,dm+\int_0^y D_2b\,(x,y,x,n)\,z\,(x,n)\,dn$$
$$+\int_0^x\int_0^y D_2D_1b\,(x,y,m,n)\,z\,(m,n)\,dndm$$
$$+\int_0^x\int_0^y e\,(x,y,m,n)\,z\,(m,n)\,dndm$$
$$\leq\Big[a\,(x,y)+b\,(x,y,x,y)+\int_0^x D_1b\,(x,y,m,y)\,dm+\int_0^y D_2b\,(x,y,x,n)\,dn$$
$$+\int_0^x\int_0^y D_2D_1b\,(x,y,m,n)\,dndm+\int_0^x\int_0^y e\,(x,y,m,n)\,dndm\Big]\,z\,(x,y)$$
$$=[a\,(x,y)+A\,(x,y)]\,z\,(x,y)\,,$$

where $A(x,y)$ is given by (10). Now by following the proof of Theorem 4.2.1 given in [7], from (11) we get

$$z\,(x,y)\;\leq\;c\exp\left(\int_0^x\int_0^y[a\,(s,t)+A\,(s,t)]\,dtds\right).\tag{12}$$

Using (12) in $w\,(x,y)\leq z\,(x,y)$ we get the required inequality in (9). ■

Our main result in this section is given in the following theorem.

Theorem 1. *Assume that*
(*i*) *the functions f, g in equation (1) satisfy the conditions*

$$|f\,(x,y,u,v,w)-f\,(x,y,\bar{u},\bar{v},\bar{w})|\;\leq\;k\,(x,y)\,[|u-\bar{u}|+|v-\bar{v}|]+|w-\bar{w}|\,,\tag{13}$$

$$|g\,(x,y,m,n,u,v)-g\,(x,y,m,n,\bar{u},\bar{v})|\;\leq\;h\,(x,y,m,n)\times[|u-\bar{u}|+|v-\bar{v}|]\,,\tag{14}$$

where $k\,(x,y)\in C\,(\Delta,R_+)$, $h\,(x,y,m,n)\in C\,(D,R_+)$,
(*ii*) *for λ as in (4)*
(*a*) *there exists a nonnegative constant α such that $\alpha<1$ and*

$$L\,(x,y)+\int_0^x\int_0^y L\,(s,t)\,dtds\leq\alpha\exp\left(\lambda\,(x+y)\right),\tag{15}$$

for $(x,y)\in\Delta$*, where*

$$L(x,y) = k(x,y)\exp(\lambda(x+y)) + \int_0^x\int_0^y h(x,y,m,n)\exp(\lambda(m+n))\,dndm, \tag{16}$$

(b) *there exists a nonnegative constant* β *such that*

$$|\sigma(x)|+|\tau(y)|+|f(x,y,0,0,(H0)(x,y))| + \int_0^x\int_0^y |f(s,t,0,0,(H0)(s,t))|\,dtds \;\le\; \beta\exp(\lambda(x+y)). \tag{17}$$

Then the IBVP (1)-(2) has a unique solution on Δ*.*

Proof. Let $u(x,y)\in S$ and define the operator T by

$$(Tu)(x,y) = \sigma(x)+\tau(y) + \int_0^x\int_0^y f(s,t,u(s,t),D_2D_1u(s,t),(Hu)(s,t))\,dtds. \tag{18}$$

From (18) we observe that

$$D_2D_1(Tu)(x,y) = f(x,y,u(x,y),D_2D_1u(x,y),(Hu)(x,y)). \tag{19}$$

First we shall show that Tu maps S into itself. Evidently Tu is continuous on Δ and $Tu\in R^n$. We verify that (4) is fulfilled. From (18), (19), using the hypotheses and (6) we have

$$\begin{aligned}
|(Tu)(x,y)|+|D_2D_1(Tu)(x,y)| \;\le\; & |\sigma(x)|+|\tau(y)| \\
&+\int_0^x\int_0^y |f(s,t,u(s,t),D_2D_1u(s,t),(Hu)(s,t)) \\
&\qquad - f(s,t,0,0,(H0)(s,t))|\,dtds \\
&+\int_0^x\int_0^y |f(s,t,0,0,(H0)(s,t))|\,dtds \\
&+ |f(x,y,u(x,y),D_2D_1u(x,y),(Hu)(x,y)) \\
&\quad - f(x,y,0,0,(H0)(x,y))| + |f(x,y,0,0,(H0)(x,y))| \\
\le\; & \beta\exp(\lambda(x+y)) + \int_0^x\int_0^y \Big\{k(s,t)\,|u(s,t)|_1 \\
&\qquad + \int_0^s\int_0^t h(s,t,m,n)\,|u(m,n)|_1\,dndm\Big\}\,dtds \\
&+ k(x,y)\,|u(x,y)|_1 + \int_0^x\int_0^y h(x,y,m,n)\,|u(m,n)|_1\,dndm
\end{aligned} \tag{20}$$

$$\le \beta \exp(\lambda(x+y)) + |u|_S \left\{ k(x,y)\exp(\lambda(x+y)) + \int_0^x\int_0^y h(x,y,m,n)\exp(\lambda(m+n))\,dndm + \int_0^x\int_0^y \left\{ k(s,t)\exp(\lambda(s+t)) + \int_0^s\int_0^t h(s,t,m,n)\exp(\lambda(m+n))\,dndm \right\} dtds \right\}$$
$$\le \beta\exp(\lambda(x+y)) + N\left\{ L(x,y) + \int_0^x\int_0^y L(s,t)\,dtds\right\}$$
$$\le [\beta + N\alpha]\exp(\lambda(x+y)).$$

From (18) it follows that $Tu \in S$. This proves that the operator T maps S into itself.

Next, we verify that the operator T is a contraction map. Let $u(x,y)$, $v(x,y) \in S$. From (18), (19) and using the hypotheses we have

$$\text{(21)}\quad |(Tu)(x,y) - (Tv)(x,y)| + |D_2D_1(Tu)(x,y) - D_2D_1(Tv)(x,y)|$$
$$\le \int_0^x\int_0^y |f(s,t,u(s,t),D_2D_1u(s,t),(Hu)(s,t)) - f(s,t,v(s,t),D_2D_1v(s,t),(Hv)(s,t))|\,dtds$$
$$+ |f(x,y,u(x,y),D_2D_1u(x,y),(Hu)(x,y)) - f(x,y,v(x,y),D_2D_1v(x,y),(Hv)(x,y))|$$
$$\le \int_0^x\int_0^y \left\{ k(s,t)\,|u(s,t)-v(s,t)|_1 + \int_0^s\int_0^t h(s,t,m,n)\,|u(m,n)-v(m,n)|_1\,dndm \right\} dtds$$
$$+ k(x,y)\,|u(x,y)-v(x,y)|_1 + \int_0^x\int_0^y h(x,y,m,n)\,|u(m,n)-v(m,n)|_1 dndm$$
$$\le |u-v|_S \left\{ k(x,y)\exp(\lambda(x+y)) + \int_0^x\int_0^y h(x,y,m,n)\exp(\lambda(m+n))dndm + \int_0^x\int_0^y \left\{ k(s,t)\exp(\lambda(s+t)) + \int_0^s\int_0^t h(s,t,m,n)\exp(\lambda(m+n))\,dndm\right\} dtds \right\}$$

$$= |u-v|_S \left\{ L(x,y) + \int_0^x \int_0^y L(s,t)\, dtds \right\}$$
$$\leq \alpha |u-v|_S \exp(\lambda (x+y)).$$

From (21) we have

$$|Tu - Tv|_S \leq \alpha |u-v|_S .$$

Since $\alpha < 1$ it follows from Banach fixed point theorem (see [4, p. 37]) that T has a fixed point in S. The fixed point of T is however a solution of IBVP (1)-(2). The proof is complete. ■

Remark 1. We note that in 1956 A. Bielecki [2] first used the norm defined in (5) for proving global existence and uniqueness of solutions of ordinary differential equations. For the developments related to this topic, see [3].

Next, we give the following theorem concerning the uniqueness of solutions of IBVP (1)-(2) in the whole space R^n without existence part.

Theorem 2. *Assume that the functions f, g in equation (1) satisfy the conditions*

$$(22) \quad |f(x,y,u,v,w) - f(x,y,\bar{u},\bar{v},\bar{w})| \leq d\left[|u-\bar{u}| + |v-\bar{v}|\right] + |w-\bar{w}|,$$

$$(23) \quad |g(x,y,m,n,u,v) - g(x,y,m,n,\bar{u},\bar{v})| \leq p(x,y,m,n) \times \left[|u-\bar{u}| + |v-\bar{v}|\right],$$

where d is a nonnegative constant such that $d < 1$ and $p(x,y,m,n)$, $D_1 p(x,y,m,n)$, $D_2 p(x,y,m,n)$, $D_2 D_1 p(x,y,m,n) \in C(D, R_+)$. Then the IBVP (1)-(2) has at most one solution on Δ.

Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of IBVP (1)-(2) and $w(x,y) = |u_1(x,y) - u_2(x,y)| + |D_2 D_1 u_1(x,y) - D_2 D_1 u_2(x,y)|$. Then by hypotheses we have

$$(24) \quad w(x,y) \leq \int_0^x \int_0^y |f(s,t,u_1(s,t), D_2D_1u_1(s,t), (Hu_1)(s,t)) - f(s,t,u_2(s,t), D_2D_1u_2(s,t), (Hu_2)(s,t))|\, dtds$$
$$+ |f(x,y,u_1(x,y), D_2D_1u_1(x,y), (Hu_1)(x,y)) - f(x,y,u_2(x,y), D_2D_1u_2(x,y), (Hu_2)(x,y))|$$
$$\leq \int_0^x \int_0^y \Big\{ d\left[|u_1(s,t) - u_2(s,t)| + |D_2D_1u_1(s,t) - D_2D_1u_2(s,t)|\right] + \int_0^s \int_0^t p(s,t,m,n) \Big[|u_1(m,n) - u_2(m,n)| + |D_2D_1u_1(m,n) - D_2D_1u_2(m,n)|\Big] dndm \Big\} dtds$$

$$+\, d\left[\left|u_1(x,y)-u_2(x,y)\right|+\left|D_2D_1u_1(x,y)-D_2D_1u_2(x,y)\right|\right]$$
$$+\int_0^x\int_0^y p(x,y,m,n)\left[\left|u_1(m,n)-u_2(m,n)\right|\right.$$
$$\left.+\left|D_2D_1u_1(m,n)-D_2D_1u_2(m,n)\right|\right]dndm.$$

From (24) we observe that

$$w(x,y)\le\frac{1}{1-d}\int_0^x\int_0^y\left[dw(s,t)+p(x,y,s,t)\,w(s,t)\right.$$
$$\left.+\int_0^s\int_0^t p(s,t,m,n)\,w(m,n)\,dndm\right]dtds. \tag{25}$$

Now a suitable application of Lemma to (25) yields

$$\left|u_1(x,y)-u_2(x,y)\right|+\left|D_2D_1u_1(x,y)-D_2D_1u_2(x,y)\right|\le 0,$$

which implies $u_1(x,y)=u_2(x,y)$ for $(x,y)\in\Delta$. Thus there is at most one solution to the IBVP (1)-(2) on Δ. ■

3. Boundedness and continuous dependence

In this section we shall study the boundedness of solutions of IBVP (1)-(2) and the continuous dependence of solutions of equation (1) on the given initial data and the functions involved therein.

The following theorem contains the estimate on the solution of IBVP (1)-(2).

Theorem 3. *Assume that*

$$\left|f(x,y,u,v,w)\right|\le\gamma\left[|u|+|v|\right]+|w|, \tag{26}$$

$$\left|g(x,y,m,n,u,v)\right|\le q(x,y,m,n)\left[|u|+|v|\right], \tag{27}$$

$$\left|\sigma(x)\right|+\left|\tau(y)\right|\le\delta, \tag{28}$$

where γ, δ are nonnegative constants such that $\gamma<1$ and $q(x,y,m,n)$, $D_1q(x,y,m,n)$, $D_2q(x,y,m,n)$, $D_2D_1q(x,y,m,n)\in C(D,R_+)$. If $u(x,y)$ for $(x,y)\in\Delta$ is any solution of IBVP (1)-(2), then

$$|u(x,y)|+|D_2D_1u(x,y)|\le\frac{\delta}{1-\gamma}\times\exp\left(\int_0^x\int_0^y\left[\frac{\gamma}{1-\gamma}+\bar{A}(s,t)\right]dtds\right), \tag{29}$$

for $(x,y)\in\Delta$, where $\bar{A}(x,y)$ is defined by the right hand side of (10), replacing $b(x,y,m,n)$ and $e(x,y,m,n)$ by $\frac{1}{1-\gamma}q(x,y,m,n)$.

Proof. Using the fact that $u(x,y)$ is a solution of IBVP (1)-(2) and the hypotheses we have

$$(30)\quad \begin{aligned} |u(x,y)| + |D_2D_1u(x,y)| &\le |\sigma(x)| + |\tau(y)| \\ &+ \int_0^x\int_0^y |f(s,t,u(s,t), D_2D_1u(s,t), (Hu)(s,t))| dtds \\ &+ |f(x,y,u(x,y), D_2D_1u(x,y), (Hu)(x,y))| \\ \le\ & \delta + \int_0^x\int_0^y \Big\{ \gamma\left[|u(s,t)| + |D_2D_1u(s,t)|\right] \\ &+ \int_0^s\int_0^t q(s,t,m,n)\left[|u(m,n)| + |D_2D_1u(m,n)|\right] dndm \Big\} dtds \\ &+ \gamma\left[|u(x,y)| + |D_2D_1u(x,y)|\right] \\ &+ \int_0^x\int_0^y q(x,y,m,n)\left[|u(m,n)| + |D_2D_1u(m,n)|\right] dndm. \end{aligned}$$

From (30) we observe that

$$(31)\quad \begin{aligned} |u(x,y)| &+ |D_2D_1u(x,y)| \\ &\le \frac{\delta}{1-\gamma} + \frac{1}{1-\gamma}\int_0^x\int_0^y \Big\{ \gamma\left[|u(s,t)| + |D_2D_1u(s,t)|\right] \\ &\qquad + q(x,y,s,t)\left[|u(s,t)| + |D_2D_1u(s,t)|\right] \\ &\qquad + \int_0^s\int_0^t q(s,t,m,n)\left[|u(m,n)| + |D_2D_1u(m,n)|\right] dndm \Big\} dtds. \end{aligned}$$

Now a suitable application of Lemma to (31) yields (29). ■

Remark 2. We note that, if the estimate obtained in (29) is bounded, then the solution $u(x,y)$ of IBVP (1)-(2) and also $D_2D_1u(x,y)$ is bounded on Δ.

The next result deals with the continuous dependence of solutions of equation (1) on given initial boundary values.

Theorem 4. *Assume that the functions f, g in equation (1) satisfy the conditions (22), (23). Let $u_1(x,y)$ and $u_2(x,y)$ be the solutions of equation (1) with the given initial boundary conditions*

$$(32)\qquad u_1(x,0) = \sigma_1(x), \quad u_1(0,y) = \tau_1(y), \quad u_1(0,0) = 0,$$

and

$$(33)\qquad u_2(x,0) = \sigma_2(x), \quad u_2(0,y) = \tau_2(y), \quad u_2(0,0) = 0,$$

respectively, where $\sigma_1, \sigma_2, \tau_1, \tau_2 \in C(R_+, R^n)$ *and*

$$|\sigma_1(x) + \tau_1(y) - \sigma_2(x) - \tau_2(y)| \leq \mu, \tag{34}$$

where $\mu \geq 0$ *is a constant. Then*

$$\begin{aligned} &|u_1(x,y) - u_2(x,y)| + |D_2D_1u_1(x,y) - D_2D_1u_2(x,y)| \\ &\qquad \leq \frac{\mu}{1-d}\exp\left(\int_0^x\int_0^y \left[\frac{d}{1-d} + B(s,t)\right] dtds\right), \end{aligned} \tag{35}$$

for $(x,y) \in \Delta$*, where* $B(x,y)$ *is defined by the right hand side of (10), replacing* $b(x,y,m,n)$ *and* $e(x,y,m,n)$ *by* $\frac{1}{1-d}p(x,y,m,n)$*.*

Proof. Let $w(x,y) = |u_1(x,y) - u_2(x,y)| + |D_2D_1u_1(x,y) - D_2D_1u_2(x,y)|$ for $(x,y) \in \Delta$. From the hypotheses we have

$$\begin{aligned} w(x,y) &\leq |\sigma_1(x) + \tau_1(y) - \sigma_2(x) - \tau_2(y)| \\ &\quad + \int_0^x\int_0^y |f(s,t,u_1(s,t), D_2D_1u_1(s,t), H(u_1)(s,t)) \\ &\qquad - f(s,t,u_2(s,t), D_2D_1u_2(s,t), H(u_2)(s,t))| \, dtds \\ &\quad + |f(x,y,u_1(x,y), D_2D_1u_1(x,y), H(u_1)(x,y)) \\ &\qquad - f(x,y,u_2(x,y), D_2D_1u_2(x,y), H(u_2)(x,y))| \\ &\leq \mu + \int_0^x\int_0^y \Big\{ d\Big[|u_1(s,t) - u_2(s,t)| \\ &\qquad + |D_2D_1u_1(s,t) - D_2D_1u_2(s,t)|\Big] \\ &\qquad + \int_0^s\int_0^t p(s,t,m,n)\,[|u_1(m,n) - u_2(m,n)| \\ &\qquad + |D_2D_1u_1(m,n) - D_2D_1u_2(m,n)|]\, dndm \Big\} dtds \\ &\quad + d\,[|u_1(x,y) - u_2(x,y)| + |D_2D_1u_1(x,y) - D_2D_1u_2(x,y)|] \\ &\quad + \int_0^x\int_0^y p(x,y,m,n)\,[|u_1(m,n) - u_2(m,n)| \\ &\qquad + |D_2D_1u_1(m,n) - D_2D_1u_2(m,n)|]\, dndm. \end{aligned} \tag{36}$$

From (36) we observe that

$$\begin{aligned} w(x,y) \leq \frac{\mu}{1-d} + \frac{1}{1-d}\int_0^x\int_0^y \Big\{ &dw(s,t) + p(x,y,s,t)\,w(s,t) \\ &+ \int_0^s\int_0^t p(s,t,m,n)\,w(m,n)\,dndm \Big\} dtds. \end{aligned} \tag{37}$$

Now a suitable application of Lemma to (37) yields the bound in (35), which shows the dependency of solutions of equation (1) on given initial boundary conditions.

Next, we consider the IBVP (1)-(2) and the corresponding IBVP

$$D_2 D_1 v(x,y) = F\left(x, y, v(x,y), D_2 D_1 v(x,y), \left(\bar{H}(v)(x,y)\right)\right), \tag{38}$$

with the given initial boundary conditions

$$v(x,0) = \bar{\sigma}(x), \quad v(0,y) = \bar{\tau}(y), \quad v(0,0) = 0, \tag{39}$$

for $x, y \in R_+$, where

$$\left(\bar{H}v\right)(x,y) = \int_0^x \int_0^y G(x, y, m, n, v(m,n), D_2 D_1 v(m,n))\, dndm,$$

and $F \in C(\Delta \times E \times R^n, R^n)$, $G \in C\left(\Delta^2 \times E, R^n\right)$, $\bar{\sigma}, \bar{\tau} \in C(R_+, R^n)$. ■

Finally, we present the following theorem which deals with the continuous dependence of solutions of IBVP (1)-(2) on the functions involved therein.

Theorem 5. *Assume that the functions f, g in equation (1) satisfy the conditions (22), (23) and*

$$\begin{aligned} &|\sigma(x) + \tau(y) - \bar{\sigma}(x) - \bar{\tau}(y)| \\ &\quad + \int_0^x \int_0^y |f(s,t,v(s,t), D_2 D_1 v(s,t), (Hv)(s,t)) \\ &\qquad - F\left(s,t,v(s,t), D_2 D_1 v(s,t), \left(\bar{H}v\right)(s,t)\right)\Big|\, dtds \\ &\quad + |f(x,y,v(x,y), D_2 D_1 v(x,y), (Hv)(x,y)) \\ &\qquad - F\left(x,y,v(x,y), D_2 D_1 v(x,y), \left(\bar{H}v\right)(x,y)\right)\Big| \le \varepsilon, \end{aligned} \tag{40}$$

where f, g, σ, τ and F, G, $\bar{\sigma}$, $\bar{\tau}$ are as in IBVP (1)-(2) and IBVP (38)-(39), $\varepsilon \ge 0$ is a constant and $v(x,y)$ is a solution of IBVP (38)-(39). Then the solution $u(x,y)$ of IBVP (1)-(2) depends continuously on the functions involved therein.

Proof. Let $w(x,y) = |u(x,y) - v(x,y)| + |D_2 D_1 u(x,y) - D_2 D_1 v(x,y)|$ for $(x,y) \in \Delta$. From the hypotheses we have

$$\begin{aligned} w(x,y) \le\ &|\sigma(x) + \tau(y) - \bar{\sigma}(x) - \bar{\tau}(y)| \\ &+ \int_0^x \int_0^y |f(s,t,u(s,t), D_2 D_1 u(s,t), (Hu)(s,t)) \\ &\qquad - f(s,t,v(s,t), D_2 D_1 v(s,t), (Hv)(s,t))|\, dtds \\ &+ \int_0^x \int_0^y |f(s,t,v(s,t), D_2 D_1 v(s,t), (Hv)(s,t)) \\ &\qquad - F\left(s,t,v(s,t), D_2 D_1 v(s,t), \left(\bar{H}v\right)(s,t)\right)\Big|\, dtds \\ &+ |f(x,y,u(x,y), D_2 D_1 u(x,y), (Hu)(x,y)) \end{aligned} \tag{41}$$

$$
\begin{aligned}
&\qquad - f\left(x,y,v\left(x,y\right),D_2D_1v\left(x,y\right),\left(Hv\right)\left(x,y\right)\right)| \\
&\quad + \left|f\left(x,y,v\left(x,y\right),D_2D_1v\left(x,y\right),\left(Hv\right)\left(x,y\right)\right)\right. \\
&\qquad \left. - F\left(x,y,v\left(x,y\right),D_2D_1v\left(x,y\right),\left(\bar{H}v\right)\left(x,y\right)\right)\right| \\
\le\ & \varepsilon + \int_0^x\int_0^y \Big\{ d\left[\left|u\left(s,t\right)-v\left(s,t\right)\right| + \left|D_2D_1u\left(s,t\right)-D_2D_1v\left(s,t\right)\right|\right] \\
&\quad + \int_0^s\int_0^t p\left(s,t,m,n\right)\Big[\left|u\left(m,n\right)-v\left(m,n\right)\right| \\
&\qquad + \left|D_2D_1u\left(m,n\right)-D_2D_1v\left(m,n\right)\right|\Big]\,dndm\Big\}\,dtds \\
&+ d\left[\left|u\left(x,y\right)-v\left(x,y\right)\right| + \left|D_2D_1u\left(x,y\right)-D_2D_1v\left(x,y\right)\right|\right] \\
&+ \int_0^x\int_0^y p\left(x,y,m,n\right)\left[\left|u\left(m,n\right)-v\left(m,n\right)\right|\right. \\
&\qquad \left. + \left|D_2D_1u\left(m,n\right)-D_2D_1v\left(m,n\right)\right|\right]dndm.
\end{aligned}
$$

From (41) we observe that

$$
\text{(42)}\quad w\left(x,y\right) \le \frac{\varepsilon}{1-d} + \frac{1}{1-d}\int_0^x\int_0^y \Big\{ dw\left(s,t\right) + p\left(x,y,s,t\right)w\left(s,t\right) + \int_0^s\int_0^t p\left(s,t,m,n\right)w\left(m,n\right)dndm\Big\}\,dtds.
$$

Now a suitable application of Lemma to (42) yields

$$
\text{(43)}\qquad w\left(x,y\right) \le \frac{\varepsilon}{1-d}\exp\left(\int_0^x\int_0^y\left[\frac{d}{1-d} + B\left(s,t\right)\right]dtds\right),
$$

for $(x,y)\in\Delta$, where $B(x,y)$ is defined as in Theorem 4. From (43) it follows that the solutions of IBVP (1)-(2) depends continuously on the functions involved therein. ■

Remark 3. We note that in the literature there are many papers dealing with the existence, uniqueness and other properties of solutions for equations of the form (P) and even for more general equations, see for instence [5,6,12]. Usually, the equations are discussed when the functions f, g in (P) are independent of the term D_2D_1u. Here, it is to be noted that our approach to the study of IBVP (1)-(2) is different from those of used in [10] and we believe that the results given above are of independent interest.

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$$u_{xy} = f\left(x, y, u, u_x, u_y, \int_0^x \int_0^y g\left(x, y, s, t, u\left(s, t\right), u_s\left(s, t\right), u_t\left(s, t\right)\right) dsdt\right),$$

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Baburao G. Pachpatte
57 Shri Niketan Colony, Near Abhinay Talkies
Aurangabad 431 001 (Maharashtra) India
e-mail: bgpachpatte@gmail.com

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