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COMMON FIXED POINTS OF MAPPINGS NOT SATISFYING CONTRACTIVE CONDITION

Abstract. The aim of this paper is to consider a new approach for obtaining common fixed point theorems in metric spaces by subjecting the triangle inequality to a Lipschitz type condition. For values of the Lipschitz constant $k < 1/3$ the condition reduces to a Banach type contractive condition and we get the results known so far. However, values of $k \geq 1/3$ yield new result. It may be observed that in the setting of metric spaces $k \geq 1/3$ generally does not ensure the existence of fixed points and there is no known method for dealing these cases. In Theorem 1 and Theorem 2 we provide results under a new condition. In the last section of this paper (Theorem 3 and Theorem 4) by using the (E.A) property introduced by Aamri and Moutawakil [2] we extend the results obtained in Theorem 1 and Theorem 2.

Key words. noncompatible maps, common fixed point, contractive condition, (E.A.) property.

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1. Introduction

The study of common fixed points of compatible mappings satisfying contractive conditions emerged as an area of vigorous research activity ever since Jungck [1] introduced the notion of compatibility. However, the study of common fixed points of noncompatible mappings is also interesting. Pant [3-5] initiated work along these lines by employing the notion of pointwise R-weak commutativity. In the study of common fixed points of compatible mappings we often require assumptions on completeness of the space or continuity of the mappings involved besides some contractive condition but the study of fixed points of noncompatible mappings can be extended to the class of nonexpansive or Lipschitz type mapping pairs even without assuming continuity of the mappings involved or completeness of the space.

Two selfmaps f and g of a metric space X are called compatible (see Jungck [1]) if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X

such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$. f and g are called noncompatible if they are not compatible. It is clear from the definition above that f and g will be noncompatible if there exists at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$ but $\lim_n d(fgx_n, gfx_n)$ is either non-zero or non-existent.

Two selfmappings f and g of a metric space X are called R-weakly commuting (see Pant [4]) at a point $x \in X$ if $d(fgx, gfx) \leq Rd(fx, gx)$ for some $R > 0$. The maps f and g are called pointwise R-weakly commuting on X if given $x \in X$ there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$. It is easy to show (see Pant [3-5]) that pointwise R-weak commutativity is equivalent to commutativity at coincidence points. Commutativity at coincidence points is, in turn, equivalent to the condition that fx is a coincidence point of f and g whenever x is a coincidence point. Therefore, pointwise R-weakly commuting maps may equivalently be called as coincidence preserving maps. Compatible maps are necessarily coincidence preserving since compatible maps commute at coincidence points. However, as shown in the examples on the following pages, coincidence preserving maps need not be compatible.

In the present paper using the notion of pointwise R-weak commutativity we demonstrate that the triangle inequality can be used to establish common fixed point theorems by subjecting it to a Lipschitz type condition. While analogous results for compatible mapping pairs hold under contractive conditions and require the assumption of continuity and completeness; our results hold for mappings that may not satisfy any contractive condition and do not assume completeness of the space or continuity of the mappings. Our theorems can thus be considered as examples of a new class of common fixed point theorems.

2. Results

If f is a selfmapping of a metric space (X, d) and let $\overline{fX}$ denote the closure of the range of f.

Theorem 1. *Let f and g be noncompatible pointwise R-weakly commuting selfmappings of a metric space (X, d) satisfying*

(i) $\overline{fX} \subseteq gX$,

and

(ii) $d(fx, fy) \leq k[d(fx, gx) + d(gx, gy) + d(gy, fy)]$, $0 \leq k < 1$. *Then f and g have a unique common fixed point.*

Proof. Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but either

$\lim_n d(fgx_n, gfx_n) \neq 0$ or the limit does not exist. Then since $t \in \overline{fX}$ and $\overline{fX} \subset gX$ there exists u in X such that $t = gu$. Using (ii) we get

$$d(fx_n, fu) \leq k[d(fx_n, gx_n) + d(gx_n, gu) + d(gu, fu)]$$

On letting $n \to \infty$ this yields $d(fu, gu) \leq kd(fu, gu)$, that is, $(1-k)d(fu, gu) = 0$. Hence $fu = gu$. Pointwise R-weak Commutativity of f and g implies that $fgu = gfu$. Also $ffu = fgu = gfu = ggu$. Using (ii) again we get

$$d(fu, ffu) \leq k[d(fu, gu) + d(gu, gfu) + d(gfu, ffu)] = kd(fu, ffu),$$

that is, $(1 - k)d(fu, ffu) = 0$. This implies that $fu = ffu$ since, $1 - k \neq 0$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness of the common fixed point follows from (ii). This completes the proof of the theorem. We now give an example to illustrate the theorem. ■

Example 1. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = 2 \quad \text{if} \quad x = 2 \text{ or } > 5, \qquad fx = 6 \quad \text{if} \quad 2 < x5,$$
$$g2 = 2, \quad gx = 7 \text{ if } 2 < x5, \quad gx = (4x + 10)/15 \text{ if } x > 5.$$

Then f and g satisfy the conditions of the above theorem and have a unique common fixed point $x = 2$. It may be verified in this example that $\overline{fX} = \{2\} \cup \{6\}$, $gX = [2, 6] \cup \{7\}$ and $\overline{fX} \subset gX$. Also, f and g are noncompatible but pointwise R-weakly commuting maps. f and g are pointwise R-weakly commuting since they commute at their coincidence point. To see that f and g are noncompatible, let us consider the sequence $\{x_n = 5 + 1/n\}$. Then $fx_n = 2$, $gx_n \to 2$, $fgx_n = 6$ and $gfx_n = 2$. Hence $\lim_n d(fgx_n, gfx_n) = 4$ and f and g are noncompatible. It can also be verified that f and g satisfy the condition

$$d(fx, fy) \leq (2/3)[d(fx, gx) + d(gx, gy) + d(gy, fy)].$$

However, f and g do not satisfy the contractive condition

$$d(fx, fy) < \max d(fx, gx), d(gx, gy), d(gy, fy)$$

nor do they satisfy the contractive condition

$$d(fx, fy) \leq \max\{d(fx, gx), d(gx, gy), d(gy, fy), [d(fx, gy) + d(fy, gx)]/2\}$$

which is one of the most general contractive conditions.

Remark 1. If we set $k = 1$ in condition (ii) of the above theorem then condition (ii) becomes the triangle inequality. In that case, since the triangle inequality always holds in a metric space, assumption (ii) with $k = 1$ does not subject to mappings to any condition and, therefore, does not ensure the existence of a fixed point. The significance of the above theorem lies in the fact that it guarantees the existence of a fixed point for each $k < 1$ and extends the known results ($k < 1/3$) to values of $k < 1$.

In the next theorem we further improve Theorem 1 by replacing condition (ii) with a much general inequality (iii). Our result demonstrates that the class of mappings considered by us ensures the existence of fixed points even under the assumption of strict triangle inequality (condition (iv) of Theorem 2). It may, however, be noted that under the relaxed conditions of Theorem 2, the common fixed point need not be unique.

Theorem 2. *Let f and g be noncompatible pointwise R-weakly commuting selfmappings of a metric space (X, d) satisfying (i)*

(iii) $d(fx, fy) \leq ad(fx, gx) + bd(gx, gy) + cd(fy, gy)$, $0 \leq a,\ c < 1,\ b \geq 0$, *and*

(iv) $d(fx, f^2x) < d(fx, gx) + d(gx, gfx) + d(gfx, f^2x)$,
whenever the right hand side is nonzero. Then f and g have a common fixed point.

Proof. Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but either $\lim_n d(fgx_n, gfx_n) \neq 0$ or the limit does not exist. Using (iii) we get

$$d(fxn, fu) \leq ad(fx_n, gx_n) + b(gx_n, gu) + cd(gu, fu)$$

On making $n \to \infty$ this yields a contradiction unless $fu = gu$. By virtue of (iv) and pointwise R-weak commutativity of f and g it follows that $fu = ffu = gfu$, that is, fu is a common fixed point of f and g. This establishes the theorem. The next example illustrates the theorem. ■

Example 2. Let $X = [0, 1]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = (\sqrt{5 - 4(2x-1)^2} - 1)/4, \quad gx = (1/3) \text{ fractional part of } (1 - x).$$

Then f and g satisfy all the conditions of Theorem 2 and have two common fixed points, $x = 0$ and $x = 1/4$. In this example, f and g are pointwise R-weakly commuting maps since they commute at their coincidence points viz, $x = 0, 1/4, 1$. Moreover if we consider the sequence $\{x_n = 1 - 1/n\}$ we get $\lim_n fx_n = \lim_n gx_n = 0$, $\lim_n fgx_n = 0$ and $lim_n gfx_n = 1/3$. Thus f and g are noncompatible. It can also be verified that f and g satisfy the condition $d(fx, fy) \leq (1/2)d(fx, gy) + 6d(gx, gy) + (1/2)d(gy, fy)$ together with the triangle inequality (iv).

Remark 2. The above example and condition (ii) and (iii) show that the above theorems are applicable in diverse settings as compared to the result on contractive mappings. This can be judged from the fact that contractive type conditions do not allow the existence of two fixed points.

In a recent work, generalizing the concept of noncompatible maps, Aamri and Moutawakil [2] have introduced a new property, namely the (E.A) property.

Definition [2]**.** *Let f and g be two selfmappings of a metric space (X, d). We say that f and g satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ such that*

$$\lim_n fx_n = \lim_n gx_n = t \text{ for some } t \in X$$

If two maps are noncompatible they satisfy the E.A property. The converse, however, is not necessarily true.

Using the notion of the E.A property we now generalize respectively the results obtained in Theorem 1 and Theorem 2 above.

Theorem 3. *Let f and g be pointwise R-weakly commuting selfmappings of a metric space (X, d) satisfying the property (E.A) and*

(v) $\overline{fX} \subseteq gX$, where $\overline{fX}$ is the closure of the range of f,
and

(vi) $d(fx, fy) \le k[d(fx, gx) + d(gx, gy) + d(gy, fy)]$, $0 \le k < 1$. Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\} \in X$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$. Then since $t \in \overline{fX}$ and $\overline{fX} \subset gX$ there exists $u \in X$ such that $t = gu$. Using (vi) we get

$$d(fx_n, fu) \le k[d(fx_n, gx_n) + d(gx_n, gu) + d(gu, fu)]$$

On letting $n \to \infty$ this yields $d(fu, gu) \le kd(fu, gu)$, a contradiction since $k < 1$. Hence $fu = gu$. Pointwise R-weak Commutativity of f and g implies that $fgu = gfu$. Also $ffu = fgu = gfu = ggu$. Using (vi) again we get

$$d(fu, ffu) \le k[d(fu, gu) + d(gu, gfu) + d(gfu, ffu)] = kd(fu, ffu),$$

that is, $(1 - k)d(fu, ffu) = 0$. This implies that $fu = ffu$ since, $1 - k \ne 0$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness of common fixed point follows from (vi). This completes the proof of the theorem. We now give an example to illustrate the theorem. ■

Example 3. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = 2 \text{ if } x = 2 \text{ or } > 5, \quad fx = (4x - 2)/3 \text{ if } 2 < x \le 5,$$

$$g2 = 2, \quad gx = 7 \text{ if } 2 < x \leq 5, \quad gx = (4x + 10)/15 \text{ if } x > 5.$$

Then f and g satisfy the conditions of the above theorem and have a unique common fixed point $x = 2$. It may be verified in this example that $\overline{fX} = [2, 6]$, $gX = [2, 6] \cup \{7\}$ and gX. Also, f and g are noncompatible but pointwise R-weakly commuting maps. f and g are pointwise R-weakly commuting since they commute at their coincidence point. It may also be observed that f and g satisfy the property (E.A), however, f and g fail to be noncompatible. It can also be verified that f and g satisfy the condition

$$d(fx, fy) \leq (2/3)[d(fx, gx) + d(gx, gy) + d(gy, fy)]$$

Theorem 4. *Let f and g be noncompatible pointwise R-weakly commuting selfmappings of a metric space (X, d) satisfying the property (E.A), the condition (i) ibid and*

(vii) $d(fx, fy) \leq ad(fx, gx) + bd(gx, gy) + cd(fy, gy)$, $0a$, $c < 1$, $b \geq 0$, *and*

(viii) $d(fx, f^2x) < d(fx, gx) + d(gx, gfx) + d(gfx, f^2x)$,

whenever the right hand side is nonzero. Then f and g have a common fixed point.

Proof. Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\} \in X$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$. Using (vii) we get

$$d(fxn, fu) \leq ad(fx_n, gxn) + b(gx_n, gu) + cd(gu, fu).$$

On making $n \to \infty$ this yields a contradiction unless $fu = gu$. By virtue of (viii) and pointwise R-weak commutativity of f and g it follows that $fu = ffu = gfu$, that is, fu is a common fixed point of f and g. This establishes the theorem. ■

Remark 3. One important difference between noncompatible mappings and mappings satisfying (E.A) property is worth mention. In Theorem 3 both f and g can be continuous at the common fixed point. But in Theorem 1 at most one of f and g can be continuous at common fixed point $t = fu = gu$. For, if both f and g are continuous at the common fixed point $t = fu = gu$, then following the proof of Theorem 1, we have $\lim_n fgx_n = ft = ffu$ and $\lim_n gfx_n = gt = gfu = ffu$. This implies that $\lim_n d(fgx_n, gfx_n) = d(ffu, ffu) = 0$ contradicting the fact that (x_n) is a sequence for which $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent. This observation also serves to distinguish between Theorem 1 and Theorem 3.

Remark 4. Theorem 1 has an important application. It provides an answer to the question (see Rhoades [6]) on the existence of a contractive condition which generates a fixed point but does not force the mappings to be continuous at the fixed point. It is clear from the Remark 3 and Example 1 that noncompatible mappings satisfying the conditions of Theorem 1 are either both discontinuous at the fixed point or at least one of them is discontinuous at the fixed point. In Example 1, both f and g are discontinuous at the common fixed point $x = 2$. In fact, Theorem 1 answers the question of Rhoades [6] not only for contractive mappings ($k < 1/3$ in Theorem 1) but also for a more general class of mappings ($k < 1$).

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References

[1] Jungck G., Compatible Mappings and Common Fixed Points, *Internat. J. Math. Sci.,* 9(1986), 771-779.

[2] Aamri M., Moutawakil D.El, Some New Common Fixed Point Theorems Under Strict Contractive Conditions, *J. Math. Anal. Appl.,* 270(2002), 181-188.

[3] Pant R.P., Common Fixed Points of Noncommuting Mappings, *J. Math. Anal. Appl.,* 188(1994), 436-440.

[4] Pant R.P., Common Fixed Points of Contractive Maps, *J. Math. Anal. Appl.,* 226(1998), 251-258.

[5] Pant R.P., Discontinuity and Fixed Points, *J. Math. Anal. Appl.,* 240(1999), 284-289.

[6] Rhoades B.E., Contractive Definitions and Continuity, *Contemporary Math.,* 72(1998), 233-245.

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